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EMBEDDING OF LINEAR HAMILTONIAN SYSTEMS ON SMALL TIME SCALES

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ABSTRACT. If a time scale $\mathbb{T}$ is in a sense small, then for any linear Hamiltonian dynamic system $x^\Delta = S(t)x$ on $\mathbb{T}$ its translation operator $e_{S(\cdot)(t,s)}$ is a restriction of a translation operator generated by a linear Hamiltonian ordinary differential system $\dot{x} = H(t)x$.

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1. INTRODUCTION

The development of the theory of time scales was initiated by Hilger in his PhD thesis in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. In the last decade we have been witnesses of great efforts especially in the theory of dynamic equations of time scales. Nevertheless there are considerable difficulties, mainly due to the fact that at the present time the kernel of the theory– the time scale calculus is still no match for the traditional calculus of Newton and Leibniz (as we can see on difficulties with the chain rule).

One of possibilities how to overcome such technical difficulties is to use an idea of *embedding*, more precisely the idea that solutions of dynamic equations on a time scale are projections of solutions of suitable ordinary differential equations to the time scale.

This beautiful idea is well known in the geometric and topological dynamics, where usually some kind of flow interpolation method is used. This approach was applied to dynamic equations on time scales by Garay and Hilger [6] who used a flow interpolation based on a parametrized Hermite interpolation. Of course the resulting ordinary differential equations are not uniquely determined.

In our paper [2] we proposed an entirely different, much more analytical approach. Our method was limited to linear systems and we had to resort to the Carathéodory theory of ordinary differential equations whereas the method of Garay and Hilger

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(GHM) guarantees continuity of the right-hand side of the resulting ordinary differential equation. Surprisingly even our "clumsy" approach has some advantages, e.g., proofs of some classical assertions are relatively straightforward [1].

In this paper we will demonstrate another advantage of our approach. We will prove that if the original dynamic equation is linear and Hamiltonian ([4, 5]) then so is the resulting ordinary differential equation constructed via our method.

2. HYPOTHESES AND AUXILIARY RESULTS

Throughout this paper we will use mostly the standard notation widely used in the theory of time scales (e.g. [4] or [6]) and ordinary differential equations (e.g. [7]). A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. The most prominent examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, but of course even the middle third Cantor set is a time scale.

For any time scale $\mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and the *graininess* function $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$, so if $\mathbb{T} = \mathbb{R}$, then $\mu(t) \equiv 0$ and if $\mathbb{T} = \mathbb{Z}$, then $\mu(t) \equiv 1$.
If $f : \mathbb{T} \to \mathbb{R}$ is a function, then by f^σ we will denote the composition $f \circ \sigma$.

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is (Δ) *differentiable* at $t \in \mathbb{T}$, if there exists a real number, denoted as $f^\Delta(t)$, called the (Δ) *derivative* of f at t, such that for all $\varepsilon > 0$ in a neighbourhood $\Omega(t)$ of t

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \qquad \text{for } s \in \Omega(t).$$

For any time scale $\|\mathbb{T}\|$, we can define its *norm* as $\|\mathbb{T}\| := \sup\{\mu(t) : t \in \mathbb{T}\}$. Clearly $\|\mathbb{T}\| \in [0, +\infty]$, e.g. $\|\mathbb{R}\| = 0$, $\|\mathbb{Z}\| = 1$, if $a, b > 0$, then $\|\bigcup_{k=1}^\infty [k(a+b), k(a+b)+a]\| = b$, and $\|\{n^2 : n \in \mathbb{N}\}\| = +\infty$.
Any interval on a time scale $\mathbb{T}$ will be denoted by the subscript $\mathbb{T}$, so e.g. $[a, b]_\mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\}$. If it is clear that for $\mathbb{T} = \mathbb{R}$, we omit the subscript $\mathbb{R}$.

Moreover we will use $\mathrm{Mat}(d \times d)$ for the set of all (real) $d \times d$ matrices equipped with the spectral norm $\|A\|$. That is $\|A\|$ is the square root of the largest eigenvalue of A^*A, where A^* is the conjugate transpose of A. By I_d we will denote the d-dimensional identity matrix and by J the standard $2d$-dimensional symplectic matrix, so

$$J := \begin{bmatrix} 0, & I_d \\ -I_d, & 0 \end{bmatrix},$$

where 0 is the d-dimensional zero matrix.

Finally we will need the following result from [2].

Theorem 2.1. *Let $\mathbb{T}$ be a time scale, $t_0, T \in \mathbb{T}$, $t_0 \leq T < \infty$. Let $A(t)$ be a $d \times d$ matrix function continuous on $[t_0, T]_\mathbb{R}$. If $\|\mathbb{T}\| < \|A(t)\|^{-1}$ on $[t_0, T]_\mathbb{T}$, then the* *translation operator* $e_{A(\cdot)}(t, t_0)$ *of the matrix equation*

$$X^{\Delta} = A(t)X, \qquad t \in [t_0, T]_{\mathbb{T}}, \tag{2.1}$$

is the restriction of the translation operator $Y(t, t_0)$ *of the matrix equation*

$$\dot{X} = H(t, A(\bar{\rho}(t)))X, \qquad t \in [t_0, T]_{\mathbb{R}}, \tag{2.2}$$

on the time scale $\mathbb{T}$*, where the matrix function* $H : [t_0, T]_{\mathbb{R}} \times \mathrm{Mat}(d \times d) \to \mathrm{Mat}(d \times d)$ *is defined by*

$$H(t, B) := B - \frac{m(t)}{2}B^2 + \frac{m^2(t)}{3}B^3 - \cdots + (-1)^n \frac{m^n(t)}{n+1}B^{n+1} + \cdots,$$

$\bar{\rho} : [t_0, T]_{\mathbb{R}} \to \mathbb{T}$ *is defined by* $\bar{\rho}(t) := \sup\{s \in \mathbb{T} : s \leq t\}$*, and* $m : [t_0, T]_{\mathbb{R}} \to \mathbb{R}$ *is defined by* $m(t) := (\mu \circ \bar{\rho})(t)$*.*

3. MAIN RESULT

We will start with a traditional definition of a $2d$-dimensional linear Hamiltonian dynamic system on a time scale $\mathbb{T}$

$$\begin{bmatrix} p \\ q \end{bmatrix}^{\Delta} = \begin{bmatrix} A(t), & B(t) \\ C(t), & -A^*(t) \end{bmatrix} \begin{bmatrix} p^{\sigma} \\ q \end{bmatrix}, \qquad t \in \mathbb{T}. \tag{3.1}$$

The right-hand side matrix is continuous and Hamiltonian, so $A(t)$, $B(t)$, $C(t)$ are continuous $d \times d$ matrix functions, $B(t) = B^*(t)$, $C(t) = C^*(t)$ and the matrix $I_{2d} - \mu(t)A(t)$ is invertible (the "strange condition" [8, p. 216]). On the very first view (3.1) is a nicely looking system, but its right hand side depends on the state $x = [p, q]^*$ not only at the time t, but also at the time $t + \mu(t)$. This makes impossible to apply any of the known theories of embedding.

Therefore we rewrite (3.1) as

$$\begin{bmatrix} p \\ q \end{bmatrix}^{\Delta} = \begin{bmatrix} A(t)(I_d - \mu(t)A(t))^{-1}, & (I_d - \mu(t)A(t))^{-1}B(t) \\ C(t)(I_d - \mu(t)A(t))^{-1}, & -A^*(t) + \mu(t)C(t)(I_d - \mu(t)A(t))^{-1}B(t) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{3.2}$$

for $t \in \mathbb{T}$, or shortly as

$$x^{\Delta} = S(t)x \qquad \text{for } t \in \mathbb{T}, \tag{3.3}$$

where $S(t)$ is the matrix on the right hand side of (3.2). Even this system is called Hamiltonian ([3, 8]) and it is well known ([3, Proposition 1.2]), that $S(t)$ is $\mathbb{T}$-symplectic, which means that

$$S^*(t)J + JS(t) + \mu(t)S^*(t)JS(t) = 0$$

for $t \in \mathbb{T}$. The $\mathbb{T}$-symplecticity of $S(t)$ implies that

$$(I_{2d} + \mu(t)S(t))^* J (I_{2d} + \mu(t)S(t)) = J \tag{3.4}$$

for $t \in \mathbb{T}$.

Clearly $S(t)$ is continuous, but we will need more – namely some kind of proportional smallness of $S(t)$ *and* $\mathbb{T}$. The usual approach here is to put an upper limit on the term $\|S(t)\| \cdot \|\mathbb{T}\|$. (For example, in [6], Garay and Hilger use the condition $31\|S(t)\| \cdot \|\mathbb{T}\| + 4(\|S(t)\| \cdot \|\mathbb{T}\|)^2 < 27$.) In our case it would be sufficient to suppose $\|S(t)\| \cdot \|\mathbb{T}\| < 1$, but because our resulting ordinary differential equation is linear we prefer to put limits on $\mathbb{T}$ so we will write this condition as

$$\|\mathbb{T}\| < \|S(t)\|^{-1}, \tag{3.5}$$

and we will say that the time scale $\mathbb{T}$ is *sufficiently small* with respect to the system (3.2) if (3.5) is fulfilled for all considered t. Of course if $\|S(t)\| = 0$, this condition holds trivially.

The time scale $\mathbb{T}$ is a closed subset of the real line $\mathbb{R}$, therefore for $t_0, T \in \mathbb{T}$, $t_0 < T$, we have

$$[t_0, T]_{\mathbb{R}} \setminus \mathbb{T} = \bigcup_{i=1}^{\omega} (a_i, b_i),$$

where $\omega \in \mathbb{N} \cup \{0, \infty\}$, $a_i, b_i \in \mathbb{R}$, $a_i < b_i$. If $\omega > 0$, then the *complementary intervals* (a_i, b_i) are pairwise disjoint and uniquely determined with the obvious exception of their ordering.

If $t \in \mathbb{T}$, then t is either right dense or right scattered. In the first case $\mu(t) = 0$, so (3.2) is reduced to the linear system

$$\frac{d}{dt}\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} A(t), & B(t) \\ C(t), & -A^*(t) \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}, \tag{3.6}$$

which is clearly Hamiltonian. If t is right scattered, then t is left boundary point of some complementary interval, hence we can restrict our attention to "time-gaps" $[a_i, b_i)$. Let one of them be chosen fixed, say the one given by boundary points $a, b \in \mathbb{T}$, $\mu(a) = b - a > 0$. Then for $t \in [a, b)$ Theorem 2.1 gives the corresponding ordinary differential equation (2.2) on $[a, b)$ as

$$\dot{x} = N(t)x, \qquad t \in [a, b), \tag{3.7}$$

where

$$N(t) := S(a) - \frac{\mu(a)}{2} S^2(a) + \frac{\mu^2(a)}{3} S^3(a) - \cdots = \frac{1}{\mu(a)} \log\left(I_{2d} + \mu(a)S(a)\right)$$

is a $2d \times 2d$ constant matrix. Because $\mathbb{T}$ is small with respect to (3.2), the spectral radius

$$\varrho(\mu(a)S(a)) \le \|\mu(a)S(a)\| = \mu(a)\|S(a)\| \le \|\mathbb{T}\| \cdot \|S(a)\| < 1,$$

and any eigenvalue λ of the matrix $I_{2d} + \mu S(\mu)$ fulfills

$$\Re\lambda(Id + \mu(a)S(\mu(a))) = 1 + \Re\lambda(\mu(a)S(\mu(a))) \ge 1 - \varrho(\mu(a)S(\mu(a))) > 0.$$

Moreover, due to (3.4) the matrix $I_{2d} + \mu(a)S(\mu(a))$ is symplectic in the usual sense. Finally, we will need the following Lemma ([9, Chap.II, Appendix Lemma 1.]).

Lemma 3.1. *Let L be a real symplectic matrix which does not have -1 as an eigenvalue. Then there exists a real, Hamiltonian matrix H such that $L = \exp(H)$.*

By the Lemma the matrix $N(t)$ is Hamiltonian for $t \in [a, b)$.

We can do this for any time-gap $[a_i, b_i)$ and combine (3.6) and (3.7) into one system

$$\dot{x} = H(t)x, \qquad t \in \mathbb{R}, \tag{3.8}$$

which is Hamiltonian and its matrix $H(t)$ is Lebesgue measurable in $t \in \mathbb{R}$. Summing this up we have proved the following theorem.

Theorem 3.2. *Let a time scale $\mathbb{T}$ be sufficiently small with respect to the linear Hamiltonian dynamic system* (3.2)*. Then there exists a Lebesgue measurable Hamiltonian matrix function $H : \mathbb{R} \to \mathrm{Mat}(d \times d)$ such that the translation operator $e_{A(\cdot)}(t, s)$ of* (3.1) *is exactly the restriction of the unique translation operator $X(t, s)$ of the linear Hamiltonian ordinary differential equation* (3.8) *to $\mathbb{T}$.*

Remark 3.3. It is interesting to note that GHM cannot guarantee that the resulting ordinary differential system will be Hamiltonian again. Indeed let us consider the following 2-dimensional Hamiltonian system

$$\begin{bmatrix} p \\ q \end{bmatrix}^{\Delta} = \begin{bmatrix} 0, & 1 \\ -1, & 0 \end{bmatrix} \begin{bmatrix} p^{\sigma} \\ q \end{bmatrix},$$

on a time gap $[a, b)$, $a, b \in \mathbb{T}$, $0 < \mu(a) = b - a$, and $\mathbb{T}$ is such that b is a right dense point, that is $\mu(b) = 0$. The right hand side matrix in (3.2) is

$$S(t) = \begin{bmatrix} 0, & 1 \\ -1, & -\mu(t) \end{bmatrix},$$

and GHM defines the right hand side of the corresponding differential equation on $[a, b)$ as

$$F_{[a,b)}(t, x) = S(a)x + \{S(a)x - S(b)\,[x + (b - a)S(a)x]\}\, P(t),$$

where $x = [p, q]^*$ and $P(t) := 2\frac{t-a}{b-a} - 3\frac{(t-a)^2}{(b-a)^2}$. Whence $F_{[a,b)}(t, x) = F(t)x$ is linear in x, nevertheless the matrix

$$F = S(a) + \{S(a) - S(b) - \mu(a)S(b)S(a)\}\, P(t) = \begin{bmatrix} \mu(a)P(t), & 1 + \mu^2(a)P(t) \\ -1, & -\mu(a) \end{bmatrix}$$

is not Hamiltonian.

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