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GENERALIZED QUASILINEARIZATION AND HIGHER ORDER OF CONVERGENCE FOR FIRST ORDER INITIAL VALUE PROBLEMS

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ABSTRACT. In this paper the method of generalized quasilinearization for initial value problems has been extended when the forcing function is the sum of hyperconvex and hyperconcave functions of order m, $m \geq 0$. The cases when m is even and m is odd has been discussed separately.

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1. INTRODUCTION

The method of quazilinearization introduced by Bellman and Kalaba [1, 2] yields iterates which are lower bounds to the solutions of the nonlinear problem when the forcing function f is convex. Furthermore, this monotone sequence of approximate solutions converges uniformly, monotonically, and quadratically to the unique solution of the nonlinear problem on the interval of existence. However, if f is concave a dual result can be developed which yields upper bounds to the solution of the nonlinear problem. Recently, the method of quasilinearization combined with the method of upper and lower solutions has been extended, generalized, and refined so as to include the cases when the forcing function is the sum of convex and concave functions. See [4] for details. The method is extremely useful in scientific computations due to its accelerated rate of convergence as in [5, 6].

In [3] Cabada and Nieto have obtained a higher order of convergence (an order more than 2). The idea used is on the same lines as monotone method, which requires the nonlinearity of the iterates to be the same as that of the order of convergence. However, in [7] they have extended the quasilinearization method of Bellman and Kalaba to obtain a higher order of convergence when the forcing function is either hyperconvex or hyperconcave. Furthermore, the nonlinearity of the iterates are one less than that of the iterates in [3].

In this paper we consider the situation when the forcing function is the sum of a hyperconvex and a hyperconcave function of the same order. Further, we consider all

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possible coupled upper and lower solutions based on the hyperconvex and hyperconcave functions. The results of [7] can be obtain as special cases of our main results. For this purpose, consider the initial value problem (IVP for short)

$$u' = f(t,u) + g(t,u), \qquad u(0) = u_0, \qquad t \in J \equiv [0,T], \tag{1.1}$$

where $f, g \in C[J \times R, R]$ such that $f(t,u)$ is convex in u and $g(t,u)$ is concave in u. The IVP (1.1) leads to the possibility of the following four types of upper and lower solutions:

Definition 1.1. The functions $\alpha_0, \beta_0 \in C^1[J, R]$ are said to be (A1) natural lower and upper solutions if

$$\begin{array}{rcllcll} \alpha_0' & \leq & f(t,\alpha_0) + g(t,\alpha_0), & \alpha_0(0) & \leq & u_0 & \text{on} \quad J \\ \beta_0' & \geq & f(t,\beta_0) + g(t,\beta_0), & \beta_0(0) & \geq & u_0 & \text{on} \quad J; \end{array}$$

(A2) coupled lower and upper solutions of type I if

$$\begin{array}{rcllcll} \alpha_0' & \leq & f(t,\alpha_0) + g(t,\beta_0), & \alpha_0(0) & \leq & u_0 & \text{on} \quad J \\ \beta_0' & \geq & f(t,\beta_0) + g(t,\alpha_0), & \beta_0(0) & \geq & u_0 & \text{on} \quad J; \end{array}$$

(A3) coupled lower and upper solutions of type II if

$$\begin{array}{rcllcll} \alpha_0' & \leq & f(t,\beta_0) + g(t,\alpha_0), & \alpha_0(0) & \leq & u_0 & \text{on} \quad J \\ \beta_0' & \geq & f(t,\alpha_0) + g(t,\beta_0), & \beta_0(0) & \geq & u_0 & \text{on} \quad J; \end{array}$$

(A4) coupled lower and upper solutions of type III if

$$\begin{array}{rcllcll} \alpha_0' & \leq & f(t,\beta_0) + g(t,\beta_0), & \alpha_0(0) & \leq & u_0 & \text{on} \quad J \\ \beta_0' & \geq & f(t,\alpha_0) + g(t,\alpha_0), & \beta_0(0) & \geq & u_0 & \text{on} \quad J. \end{array}$$

In order to facilitate later explanations, we shall need the following definition:

Definition 1.2. A function $h : A \to B,\ A, B \subset R$ is called m-hyperconvex , $m \geq 0$, if $h \in C^{m+1}[A, B]$ and $d^{m+1}h/du^{m+1} \geq 0$ for $u \in A$; h is called m-hyperconcave if the inequality is reversed.

In this paper we use the maximum norm of $u(t)$ over J, i.e.

$$\| u \| = \max_{t \in J} | u(t) | .$$

In view of the above four types of coupled upper and lower solutions of (1.1), we shall develop results when f is hyperconvex and g is hyperconcave of order $m-1$. The cases when m is even and m is odd have been discussed separately since the iterates will be different depending on whether m is odd or even. Furthermore, we show that these iterates converge uniformly and monotonically to the unique solution of (1.1), and the convergence is of order m.

2. PRELIMINARIES

In this section we recall some known existence and comparison theorems which we need in our main results and which are related to the system of IVPs

$$u' = H(t,u), \qquad u(0) = u_0, \qquad t \in J \equiv [0,T], \tag{2.1}$$

where $H \in C[J \times R^n, R^n]$.

The first comparison result for $n = 1$ in (2.1) is:

Theorem 2.1. *Let $\alpha_0, \beta_0 \in C^1[J,R]$ be lower and upper solutions respectively and suppose that*

$$H(t,x) - H(t,y) \leq L(x-y),$$

whenever $x \geq y$ for some $L > 0$. Then $\alpha_0(0) \leq \beta_0(0)$ implies $\alpha_0(t) \leq \beta_0(t)$ on J.

Let us consider equation (2.1) with $n = 2$. That is, we also need the following comparison results of two systems.

Theorem 2.2. *Let $\alpha_0, \beta_0 \in C^1[J,R]$ and $H \in C[J \times R^2, R]$. Suppose further that any of the following conditions hold:*

(H1) $\alpha_0' \leq H(t,\alpha_0,\alpha_0)$, $\beta_0' \geq H(t,\beta_0,\beta_0)$,

$$H(t,x_1,y_1) - H(t,x_2,y_2) \leq L[(x_1-x_2)+(y_1-y_2)], L \geq 0$$

whenever $x_1 \geq x_2$, $y_1 \geq y_2$;

(H2) $\alpha_0' \leq H(t,\alpha_0,\beta_0)$, $\beta_0' \geq H(t,\beta_0,\alpha_0)$,

$$\begin{aligned} H(t,x_1,y) - H(t,x_2,y) &\leq L(x_1-x_2), L \geq 0 \\ H(t,x,y_1) - H(t,x,y_2) &\geq -L(y_1-y_2) \end{aligned}$$

whenever $x_1 \geq x_2$, $y_1 \geq y_2$;

(H3) $\alpha_0' \leq H(t,\beta_0,\alpha_0)$, $\beta_0' \geq H(t,\alpha_0,\beta_0)$,

$$\begin{aligned} H(t,x,y_1) - H(t,x,y_2) &\leq L(y_1-y_2), L \geq 0 \\ H(t,x_1,y) - H(t,x_2,y) &\geq -L(x_1-x_2) \end{aligned}$$

whenever $x_1 \geq x_2$, $y_1 \geq y_2$;

(H4) $\alpha_0' \leq H(t,\beta_0,\beta_0)$, $\beta_0' \geq H(t,\alpha_0,\alpha_0)$,

$$H(t,x_1,y_1) - H(t,x_2,y_2) \geq -L[(x_1-x_2)+(y_1-y_2)], L \geq 0$$

whenever $x_1 \geq x_2$, $y_1 \geq y_2$.

Then $\alpha_0(0) \leq \beta_0(0)$ implies $\alpha_0(t) \leq \beta_0(t)$ on J.

For details of proof of Theorems 2.1 and 2.2 see [4].

Next we present an existence result relative to the equation (2.1) which we need in our main result. For that purpose, we split $u = (u_i, [u]_{p_i}, [u]_{q_i})$ and define coupled lower and upper solutions of (2.1) in componentwise form as

$$\begin{aligned} \alpha_{0_i}' &\leq H_i(t, \alpha_{0_i}, [\alpha_0]_{p_i}, [\beta_0]_{q_i}) \\ \beta_{0_i}' &\geq H_i(t, \beta_{0_i}, [\beta_0]_{p_i}, [\alpha_0]_{q_i}), \end{aligned}$$

where $p_i + q_i = n-1, p_i, q_i \geq 0$. Also $\alpha_0, \beta_0 \in C^1[J, R^n]$ such that $\alpha_0(t) \leq \beta_0(t)$ on J.

Definition 2.3. The function $H(t,u)$, is said to possess a mixed quasimonotone property if for each i, $1 \leq i \leq n$, $H_i(t, u_i, [u]_{p_i}, [u]_{q_i})$ is monotone nondecreasing in $[u]_{p_i}$ and monotone nonincreasing in $[u]_{q_i}$.

Theorem 2.4. *Let $\alpha_0, \beta_0 \in C^1[J, R^n]$ be coupled lower and upper solutions of (2.1) respectively. If $H(t,u)$ possesses a mixed quasimonotone property, then there exists a solution $u(t)$ of (2.1) such that $\alpha_0(t) \leq u(t) \leq \beta_0(t)$ on J.*

The next result (see [4]) will be useful to prove the order of convergence of the iterates.

Corollary 2.5. *Let $v \in C^1[J, R^n]$ and $v' \leq Av + \sigma$, where $A = (a_{ij})$ is an $n \times n$ matrix satisfying $a_{ij} \geq 0$, $i \neq j$ and $\sigma \in C[J, R^n]$. Then we have*

$$v(t) \leq v(0)e^{At} + \int_0^t e^{A(t-s)}\sigma(s)ds, \qquad t \in J.$$

3. MAIN RESULTS

In this section we consider the IVP

$$u' = f(t,u) + g(t,u), \qquad u(0) = u_0, \qquad t \in J \equiv [0, T], \tag{3.1}$$

where $f, g \in C[\Omega, R]$, $\Omega = [(t,u) : \alpha_0(t) \leq u(t) \leq \beta_0(t), t \in J]$ and $\alpha_0, \beta_0 \in C^1[J, R]$ with $\alpha_0(t) \leq \beta_0(t)$ on J. Here, we state the inequalities to recall them in the proof of our main results. We note that the iterates will be different based on the hyperconvexity and hyperconcavity of even and odd orders.

Suppose that $f(t,u)$ is hyperconvex in u of order $m-1$, then we have the following inequalities depending on whether m is even or odd. (i) m=2k

$$f(t,\eta) \geq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!} \tag{3.2}$$

$$f(t,\eta) \leq \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{f^{(2k-1)}(t,\eta)(\eta-\xi)^{2k-1}}{(2k-1)!}; \tag{3.3}$$

(ii) m=2k+1

$$f(t,\eta) \geq \sum_{i=0}^{2k} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!}, \eta \geq \xi \tag{3.4}$$

$$f(t,\eta) \leq \sum_{i=0}^{2k} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!}, \eta \leq \xi \tag{3.5}$$

$$f(t,\eta) \leq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{f^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \eta \geq \xi \tag{3.6}$$

$$f(t,\eta) \geq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{f^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \eta \leq \xi. \tag{3.7}$$

Similarly, when $g(t,u)$ is hyperconcave in u of order $m-1$, we have the following inequalities depending on whether m is even or odd: (i) m=2k

$$g(t,\eta) \leq \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} \tag{3.8}$$

$$g(t,\eta) \geq \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2k-1)}(t,\eta)(\eta-\xi)^{2k-1}}{(2k-1)!}; \tag{3.9}$$

(ii) m=2k+1

$$g(t,\eta) \leq \sum_{i=0}^{2k} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!}, \eta \geq \xi \tag{3.10}$$

$$g(t,\eta) \geq \sum_{i=0}^{2k} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!}, \eta \leq \xi \tag{3.11}$$

$$g(t,\eta) \geq \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \eta \geq \xi \tag{3.12}$$

$$g(t,\eta) \leq \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\xi)(\eta-\xi)^i}{i!} + \frac{g^{(2k)}(t,\eta)(\eta-\xi)^{2k}}{(2k)!}, \eta \leq \xi. \tag{3.13}$$

Based on these inequalities, relative to each of the four coupled upper and lower solutions $(A1)$, $(A2)$, $(A3)$ and $(A4)$, we have eight theorems depending on whether m is even or odd. In all our results, we develop two monotone sequences which converge uniformly and monotonically to the unique solution of (3.1). Further, the order of convergence depends on the order of hyperconvexity and hyperconcavity of f and g in (3.1).

The first two results in this direction are relative to natural upper and lower solutions (case $(A1)$) when m is even and odd respectively.

Theorem 3.1. *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J, R]$ *are natural lower and upper solutions (case* $(A1)$*) with* $\alpha_0(t) \leq \beta_0(t)$ *on* J.

(ii) $f, g \in C^{0,2k}[\Omega, R]$ *such that* $f(t,u)$ *is* $(2k-1)$*-hyperconvex,* $k \geq 1$ *in* u *and* $g(t,u)$ *is* $(2k-1)$*-hyperconcave,* $k \geq 1$ *in* u *on* J *[i.e.* $f^{(2k)}(t,u) \geq 0$, $g^{(2k)}(t,u) \leq 0$, *for* $(t,u) \in \Omega$ *].*

Then there exist monotone sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$, $n \geq 0$ *which converge uniformly and monotonically to the unique solution of (3.1) and the convergence is of order* $2k$.

Proof. In order to construct monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, $n \geq 0$ which converge uniformly and monotonically to the unique solution of (3.1), we need to consider the following IVPs for $n = 1, 2, \dots$ together with the inequalities (3.2), (3.3), (3.8) and (3.9) :

$$\begin{aligned}
\alpha_n' &= F(t, \alpha_{n-1}, \beta_{n-1}; \alpha_n) = \\
&= \sum_{i=0}^{2k-1} \frac{f^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} + \sum_{i=0}^{2k-2} \frac{g^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} \\
&+ \frac{g^{(2k-1)}(t, \beta_{n-1})(\alpha_n - \alpha_{n-1})^{2k-1}}{(2k-1)!}, \qquad\qquad \alpha_n(0) = u_0,
\end{aligned}$$

$$\begin{aligned}
\beta_n' &= G(t, \alpha_{n-1}, \beta_{n-1}; \beta_n) = \\
&= \sum_{i=0}^{2k-2} \frac{f^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} + \frac{f^{(2k-1)}(t, \alpha_{n-1})(\beta_n - \beta_{n-1})^{2k-1}}{(2k-1)!} \\
&+ \sum_{i=0}^{2k-1} \frac{g^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!}, \qquad\qquad \beta_n(0) = u_0.
\end{aligned}$$

Since this theorem is a combination of Theorem 2.1 and Theorem 2.3 in [7] we omit the details of the proof. □

Theorem 3.2. *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J, R]$ *are natural lower and upper solutions (case* $(A1)$*) with* $\alpha_0(t) \leq \beta_0(t)$ *on* J.

(ii) $f, g \in C^{0,2k+1}[\Omega, R]$ *such that* $f(t,u)$ *is* $(2k)$*-hyperconvex,* $k \geq 1$ *in* u *and* $g(t,u)$ *is* $(2k)$*-hyperconcave,* $k \geq 1$ *in* u *on* J *[i.e.* $f^{(2k+1)}(t,u) \geq 0$, $g^{(2k+1)}(t,u) \leq 0$, *for* $(t,u) \in \Omega$ *].*

Then there exist monotone sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$, $n \geq 0$ *which converge uniformly and monotonically to the unique solution of (3.1) and the convergence is of order* $2k+1$.

Proof. Let us consider the following IVPs for $n = 1, 2, \dots$ together with the inequalities (3.4), (3.5), (3.12), and (3.13):

$$\begin{aligned}
\alpha_n' &= F(t, \alpha_{n-1}, \beta_{n-1}; \alpha_n) = \\
&= \sum_{i=0}^{2k} \frac{f^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} + \sum_{i=0}^{2k-1} \frac{g^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} \\
&+ \frac{g^{(2k)}(t, \beta_{n-1})(\alpha_n - \alpha_{n-1})^{2k}}{(2k)!}, \qquad \alpha_n(0) = u_0,
\end{aligned}$$

$$\begin{aligned}
\beta_n' &= G(t, \alpha_{n-1}, \beta_{n-1}; \beta_n) = \\
&= \sum_{i=0}^{2k} \frac{f^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} + \sum_{i=0}^{2k-1} \frac{g^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} \\
&+ \frac{g^{(2k)}(t, \alpha_{n-1})(\beta_n - \beta_{n-1})^{2k}}{(2k)!}, \qquad \beta_n(0) = u_0.
\end{aligned}$$

To prove this theorem we can refer to Theorem 2.2 and Theorem 2.4 in [7]. □

The next two Theorems are relative to the coupled upper and lower solutions of type III *case*$(A4)$ when m is even and odd respectively.

Theorem 3.3. *Assume that*

(*i*) $\alpha_0, \beta_0 \in C^1[J, R]$ *are coupled lower and upper solutions of type III (case (A4)) with* $\alpha_0(t) \le \beta_0(t)$ *on* J.

(*ii*) $f, g \in C^{0,2k}[\Omega, R]$ *such that* $f(t, u)$ *is* $(2k-1)$*-hyperconvex,* $k \ge 1$ *in* u *and* $g(t, u)$ *is* $(2k-1)$*-hyperconcave,* $k \ge 1$ *in* u *on* J *[i.e.* $f^{(2k)}(t, u) \ge 0$, $g^{(2k)}(t, u) \le 0$, *for* $(t, u) \in \Omega$ *].*

(*iii*)

$$f_u(t, u) \le -\max_{\Omega}[f^{(2k)}(t, u)] \frac{(\beta_0 - \alpha_0)^{2k-1}}{(2k-2)!} \le 0 \quad \text{on } \Omega$$

$$g_u(t, u) \le \min_{\Omega}[g^{(2k)}(t, u)] \frac{(\beta_0 - \alpha_0)^{2k-1}}{(2k-2)!} \le 0 \quad \text{on } \Omega.$$

Then there exist monotone sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$, $n \ge 0$ *which converge uniformly and monotonically to the unique solution of (3.1) and the convergence is of order* $2k$.

Proof. The assumptions $f^{(2k)}(t, u) \ge 0$, $g^{(2k)}(t, u) \le 0$ yield the inequalities (3.2), (3.3), (3.8) and (3.9) whenever $\alpha_0 \le \eta, \xi \le \beta_0$. Let us first consider the following IVPs:

$$\begin{aligned}
w' &= F(t, \alpha, \beta; v) = \\
&= \sum_{i=0}^{2k-1} \frac{f^{(i)}(t, \beta)(v - \beta)^i}{i!} + \sum_{i=0}^{2k-2} \frac{g^{(i)}(t, \beta)(v - \beta)^i}{i!} \\
&+ \frac{g^{(2k-1)}(t, \alpha)(v - \beta)^{2k-1}}{(2k-1)!}, \qquad w(0) = u_0,
\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
v' &= G(t,\alpha,\beta;w) = \\
&= \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\alpha)(w-\alpha)^i}{i!} + \frac{f^{(2k-1)}(t,\beta)(w-\alpha)^{2k-1}}{(2k-1)!} \\
&+ \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\alpha)(w-\alpha)^i}{i!}, \qquad v(0) = u_0,
\end{aligned} \tag{3.15}
$$

where $\alpha_0(0) \leq u_0 \leq \beta_0(0)$ and $t \in J$.

We develop the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ using the above IVPs (3.14) and (3.15) respectively. Initially, we prove (α_0, β_0) are coupled lower and upper solutions of (3.14) and (3.15) respectively. The inequalities (3.2) and (3.9), and (i) imply

$$
\alpha_0' \leq f(t,\beta_0) + g(t,\beta_0) = F(t,\alpha_0,\beta_0;\beta_0),\ \alpha_0(0) \leq u_0, \tag{3.16}
$$

$$
\begin{aligned}
\beta_0' &\geq f(t,\alpha_0) + g(t,\alpha_0) \qquad \geq \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\ \beta_0)(\alpha_0-\beta_0)^i}{i!} \\
&+ \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\ \beta_0)(\alpha_0-\beta_0)^i}{i!} + \frac{g^{(2k-1)}(t,\ \alpha_0)(\alpha_0-\beta_0)^{2k-1}}{(2k-1)!} \\
&= F(t,\alpha_0,\beta_0;\alpha_0), \qquad \beta_0(0) \geq u_0.
\end{aligned} \tag{3.17}
$$

Using Taylor series expansion with Lagrange remainder for $F(t,\alpha_0,\beta_0;v)$ and (iii), we get

$$
\begin{aligned}
F_v(t,\alpha_0,\beta_0;v) &= f_v(t,v) - \frac{f^{(2k)}(t,\xi_1)(v-\beta_0)^{2k-1}}{(2k-1)!} \\
&+ g_v(t,v) - \frac{g^{(2k)}(t,\xi_3)(v-\beta_0)^{2k-2}(\xi_2-\alpha_0)}{(2k-2)!} \leq 0,
\end{aligned}
$$

where $\alpha_0 \leq v \leq \xi_1, \xi_2 \leq \beta_0$, $\alpha_0 \leq \xi_3 \leq \xi_2$. Hence $F(t,\alpha_0,\beta_0;v)$ is nonincreasing in v and we can apply Theorem 2.4 together with (3.16) and (3.17) and conclude that there exists a solution $\alpha_1(t)$ of (3.14) with $\alpha = \alpha_0$ and $\beta = \beta_0$ such that $\alpha_0 \leq \alpha_1 \leq \beta_0$ on J. Since F_v exists on Ω, we can easily show that $F(t,\alpha_0,\beta_0;v)$ satisfies the one-side Lipschitz condition with respect to v for $(t,v) \in \Omega$ and $\alpha_0(t) \leq v(t) \leq \beta_0(t)$ for $t \in J$. Theorem 2.1 guarantees that the solution $\alpha_1(t)$ of (3.14) with $\alpha = \alpha_0$ and $\beta = \beta_0$ is unique.

Similarly, we can show that there exists a unique solution $\beta_1(t)$ of (3.15) on J. The inequalities (3.3) and (3.8), and (i) imply

$$
\beta_0' \geq f(t,\alpha_0) + g(t,\alpha_0) = G(t,\alpha_0,\beta_0;\alpha_0),\ \beta_0(0) \geq u_0, \tag{3.18}
$$

$$
\begin{aligned}
\alpha_0' &\le f(t,\beta_0)+g(t,\beta_0) &&\le \sum_{i=0}^{2k-2}\frac{f^{(i)}(t,\ \alpha_0)(\beta_0-\alpha_0)^i}{i!}\\
&+\frac{f^{(2k-1)}(t,\ \beta_0)(\beta_0-\alpha_0)^{2k-1}}{(2k-1)!} &&+\sum_{i=0}^{2k-1}\frac{g^{(i)}(t,\ \alpha_0)(\beta_0-\alpha_0)^i}{i!}\\
&=G(t,\alpha_0,\beta_0;\beta_0), && \alpha_0(0)\le u_0.
\end{aligned}
\tag{3.19}
$$

Using Taylor series expansion with Lagrange remainder for $G(t,\alpha_0,\beta_0;w)$ and (iii), we get

$$
\begin{aligned}
G_w(t,\alpha_0,\beta_0;w) &= f_w(t,w)-\frac{f^{(2k)}(t,\eta_3)(w-\alpha_0)^{2k-2}(\eta_1-\beta_0)}{(2k-2)!}\\
&+g_w(t,w)-\frac{g^{(2k)}(t,\eta_2)(w-\alpha_0)^{2k-1}}{(2k-1)!}\le 0,
\end{aligned}
$$

where $\alpha_0\le\eta_1,\eta_2\le w$, $\alpha_0\le\eta_1\le\eta_3\le\beta_0$. Hence $G(t,\alpha_0,\beta_0;w)$ is nonincreasing in w and we can apply Theorem 2.4 together with (3.18) and (3.19) and conclude that there exists a solution $\beta_1(t)$ of (3.15) with $\alpha=\alpha_0$ and $\beta=\beta_0$ such that $\alpha_0\le\beta_1\le\beta_0$ on J. One can easily show that $G(t,\alpha_0,\beta_0;w)$ satisfies the one-side Lipschitz condition with respect to w for $(t,w)\in\Omega$ and $\alpha_0(t)\le w(t)\le\beta_0(t)$, $t\in J$. Theorem 2.1 guarantees that the solution $\beta_1(t)$ of (3.15) with $\alpha=\alpha_0$ and $\beta=\beta_0$ is unique.

Furthermore, by (3.2) and (3.9) with $\alpha_0\le\beta_1\le\beta_0$ and $g^{(2k-1)}(t,u)$ nonincreasing in u on Ω, we have

$$
\begin{aligned}
\alpha_1' &= F(t,\alpha_0,\beta_0;\beta_1) &&= \sum_{i=0}^{2k-1}\frac{f^{(i)}(t,\ \beta_0)(\beta_1-\beta_0)^i}{i!}\\
&+\sum_{i=0}^{2k-2}\frac{g^{(i)}(t,\ \beta_0)(\beta_1-\beta_0)^i}{i!} &&+\frac{g^{(2k-1)}(t,\ \alpha_0)(\beta_1-\beta_0)^{2k-1}}{(2k-1)!}\\
&\le f(t,\beta_1)+g(t,\beta_1), && \alpha_1(0)=u_0.
\end{aligned}
\tag{3.20}
$$

Using (3.3) and (3.8) with $\alpha_0\le\alpha_1\le\beta_0$ and $f^{(2k-1)}(t,u)$ nondecreasing in u on Ω, we get

$$
\begin{aligned}
\beta_1' &= G(t,\alpha_0,\beta_0;\alpha_1) &&= \sum_{i=0}^{2k-2}\frac{f^{(i)}(t,\ \alpha_0)(\alpha_1-\alpha_0)^i}{i!}\\
&+\frac{f^{(2k-1)}(t,\ \beta_0)(\alpha_1-\alpha_0)^{2k-1}}{(2k-1)!} &&+\sum_{i=0}^{2k-1}\frac{g^{(i)}(t,\ \alpha_0)(\alpha_1-\alpha_0)^i}{i!}\\
&\ge f(t,\alpha_1)+g(t,\alpha_1), && \beta_1(0)=u_0.
\end{aligned}
\tag{3.21}
$$

Hence $\alpha_1\le\beta_1$ by (3.20), (3.21), and Theorem 2.2 with $(H4)$. Thus we get $\alpha_0\le\alpha_1\le\beta_1\le\beta_0$ on J.

Assume now that α_n and β_n are the solutions of IVPs (3.14) and (3.15) respectively with $\alpha=\alpha_{n-1}$ and $\beta=\beta_{n-1}$ such that $\alpha_{n-1}\le\alpha_n\le\beta_n\le\beta_{n-1}$ on J and

$$
\begin{aligned}
\alpha_n' &\le f(t,\beta_n)+g(t,\beta_n),\\
\beta_n' &\ge f(t,\alpha_n)+g(t,\alpha_n).
\end{aligned}
\tag{3.22}
$$

Certainly this is true for $n = 1$.

We need to show that $\alpha_n \le \alpha_{n+1} \le \beta_{n+1} \le \beta_n$ on J, where α_{n+1} and β_{n+1} are the solutions of IVPs (3.14) and (3.15) respectively with $\alpha = \alpha_n$ and $\beta = \beta_n$.

The inequalities (3.2) and (3.9) and (3.22) imply

$$\alpha_n' \le f(t,\beta_n) + g(t,\beta_n) = F(t,\alpha_n,\beta_n;\beta_n),\ \ \alpha_n(0) \le u_0, \tag{3.23}$$

$$\begin{aligned} \beta_n' &\ge f(t,\alpha_n) + g(t,\alpha_n) &&\ge \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\ \beta_n)(\alpha_n-\beta_n)^i}{i!} \\ &+ \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\ \beta_n)(\alpha_n-\beta_n)^i}{i!} &&+ \frac{g^{(2k-1)}(t,\ \alpha_n)(\alpha_n-\beta_n)^{2k-1}}{(2k-1)!} \\ &= F(t,\alpha_n,\beta_n;\alpha_n), && \qquad\qquad \beta_n(0) \ge u_0. \end{aligned} \tag{3.24}$$

This proves that α_n, β_n are coupled lower and upper solutions of (3.14) and (3.15) with $\alpha = \alpha_n$ and $\beta = \beta_n$. Hence using (3.23), (3.24), the fact that F is nondecreasing in u, Theorem 2.4, and Theorem 2.2 with $(H4)$, we can conclude that there exists a unique solution $\alpha_{n+1}(t)$ of (3.14) with $\alpha = \alpha_n$ and $\beta = \beta_n$ such that $\alpha_n \le \alpha_{n+1} \le \beta_n$ on J.

The inequalities (3.3) and (3.8), and (3.22) imply

$$\beta_n' \ge f(t,\alpha_n) + g(t,\alpha_n) = G(t,\alpha_n,\beta_n;\alpha_n),\ \ \beta_n(0) \ge u_0, \tag{3.25}$$

$$\begin{aligned} \alpha_n' &\le f(t,\beta_n) + g(t,\beta_n) &&\le \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\ \alpha_n)(\beta_n-\alpha_n)^i}{i!} \\ &+ \frac{f^{(2k-1)}(t,\ \beta_n)(\beta_n-\alpha_n)^{2k-1}}{(2k-1)!} &&+ \textstyle\sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\ \alpha_n)(\beta_n-\beta_n)^i}{i!} \\ &= G(t,\alpha_n,\beta_n;\beta_n), && \qquad\qquad \alpha_n(0) \le u_0. \end{aligned} \tag{3.26}$$

This proves that α_n, β_n are coupled lower and upper solutions of (3.14) and (3.15) with $\alpha = \alpha_n$ and $\beta = \beta_n$. Hence using (3.25), (3.26), the fact that G is nonincreasing in u, Theorem 2.4, and Theorem 2.2 with $(H4)$, we can conclude that there exists a unique solution $\beta_{n+1}(t)$ of (3.15) with $\alpha = \alpha_n$ and $\beta = \beta_n$ such that $\alpha_n \le \beta_{n+1} \le \beta_n$ on J.

Furthermore, by (3.2), (3.9) with $\alpha_n \le \beta_{n+1} \le \beta_n$ and $g^{(2k-1)}(t,u)$ nonincreasing in u, we have

$$\begin{aligned} \alpha_{n+1}' &= F(t,\alpha_n,\beta_n;\beta_{n+1}) &&= \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\ \beta_n)(\beta_{n+1}-\beta_n)^i}{i!} \\ &+ \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\ \beta_n)(\beta_{n+1}-\beta_n)^i}{i!} &&+ \frac{g^{(2k-1)}(t,\ \alpha_n)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!} \\ &\le f(t,\beta_{n+1}) + g(t,\beta_{n+1}), && \qquad\qquad \alpha_{n+1}(0) = u_0. \end{aligned} \tag{3.27}$$

Using (3.3) and (3.8) with $\alpha_n \leq \alpha_{n+1} \leq \beta_n$ and $f^{(2k-1)}(t,u)$ nondecreasing in u, we get

$$\begin{aligned}\beta'_{n+1} &= G(t,\alpha_n,\beta_n;\alpha_{n+1}) = \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\ \alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!} \\ &+ \frac{f^{(2k-1)}(t,\ \beta_n)(\alpha_{n+1}-\alpha_n)^{2k-1}}{(2k-1)!} + \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\ \alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!} \\ &\geq f(t,\alpha_{n+1}) + g(t,\alpha_{n+1}), \qquad \beta_{n+1}(0) = u_0.\end{aligned} \tag{3.28}$$

Thus we get $\alpha_{n+1} \leq \beta_{n+1}$ using (3.27), (3.28), and Theorem 2.2 with $(H4)$. This proves $\alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n$ on J. Thus by induction, we have

$$\alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_1 \leq \beta_0.$$

Let u be any solution such that $\alpha_0 \leq u \leq \beta_0$ with $\alpha_0(0) \leq u_0 \leq \beta_0(0)$ on J. Suppose for some u , we have $\alpha_n \leq u \leq \beta_n$ on J. Set $\Phi_1 = u - \alpha_{n+1}$, $\Phi_2 = \beta_{n+1} - u$ and use (3.2) and (3.9) so that

$$\begin{aligned}\Phi'_1 &= u' - \alpha'_{n+1} = f(t,u) + g(t,u) \\ &- [\ \sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\ \beta_n)(\beta_{n+1}-\beta_n)^i}{i!} \\ &+ \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\ \beta_n)(\beta_{n+1}-\beta_n)^i}{i!} + \frac{g^{(2k-1)}(t,\ \alpha_n)(\beta_{n+1}-\beta_n)^{2k-1}}{(2k-1)!}\ ] \\ &\geq f(t,u) + g(t,u) - f(t,\beta_{n+1}) - g(t,\beta_{n+1}) \\ &= \ [\ -f_u(t,\xi_1) - g_u(t,\xi_2)\ ]\ \Phi_2, \qquad \Phi_1(0) = 0,\end{aligned}$$

where $\xi_1,\ \xi_2$ are between u and β_{n+1}.

Now use (3.3) and (3.8) so that

$$\begin{aligned}\Phi'_2 &= \beta'_{n+1} - u' = \sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\ \alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!} \\ &+ \frac{f^{(2k-1)}(t,\ \beta_n)(\alpha_{n+1}-\alpha_n)^{2k-1}}{(2k-1)!} + \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\ \alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!} \\ &- f(t,u) - g(t,u) \geq f(t,\alpha_{n+1}) + g(t,\alpha_{n+1}) - f(t,u) - g(t,u) \\ &= \ [\ -f_u(t,\eta_1) - g_u(t,\eta_2)\ ]\ \Phi_1, \qquad \Phi_2(0) = 0,\end{aligned}$$

where $\eta_1,\ \eta_2$ are between u and α_{n+1}. From above we have

$$r' \geq Ar,\ r(0) = 0,$$

where

$$r = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & k_1 \\ k_1 & 0 \end{pmatrix}, \qquad |f_u(t,u) + g_u(t,u)| \leq k_1.$$

It follows that $r \geq 0$ or

$$\alpha_{n+1} \leq u, \tag{3.29}$$

$$\beta_{n+1} \geq u. \tag{3.30}$$

From (3.29) and (3.30) it is clear that $\alpha_{n+1} \leq u \leq \beta_{n+1}$. Since $\alpha_0 \leq u \leq \beta_0$, this proves by induction that $\alpha_n \leq u \leq \beta_n$ on J for all n. From this we can conclude

$$\alpha_0 \leq \alpha_1 \leq ... \leq \alpha_n \leq u \leq \beta_n \leq ... \leq \beta_1 \leq \beta_0.$$

Now one can show easily that the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ are equicontinuous and uniformly bounded. Hence applying Ascoli-Arzela's Theorem, we can conclude that there exist subsequences $\{\alpha_{n,j}(t)\}$, $\{\beta_{n,j}(t)\}$ such that $\alpha_{n,j}(t) \to \rho(t)$ and $\beta_{n,j}(t) \to r(t)$ with $\rho(t) \leq u \leq r(t)$ on J. Since the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ are monotone, we can conclude that $\alpha_n(t) \to \rho(t)$ and $\beta_n(t) \to r(t)$. Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \alpha_n(t) = \rho(t) \leq u \leq r(t) = \lim_{n\to\infty} \beta_n(t).$$

Next we show that $\rho(t) \geq r(t)$. From IVPs (3.14) and (3.15) we get

$$\begin{aligned} \rho\,'(t) &= f(t,r) + g(t,r),\ \rho(0) = 0, \\ r\,'(t) &= f(t,\rho) + g(t,\rho),\ r(0) = 0. \end{aligned}$$

Setting now $\Phi = \rho(t) - r(t)$, using f_u, g_u exist, we get

$$\begin{aligned} \Phi' &= \rho' - r' = f(t,r) + g(t,r) - f(t,\rho) - g(t,\rho) \\ &\geq -L(r-\rho) \geq L(\rho - r),\ L \geq 0,\ \Phi(0) = 0. \end{aligned}$$

From this we can conclude that $r(t) \leq \rho(t)$ on J. This proves $r(t) = \rho(t) = u(t)$ is the unique solution of (3.1). Hence $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converge uniformly and monotonically to the unique solution of (3.1).

Let us consider the order of convergence of $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ to the unique solution $u(t)$ of (3.1). To do this, set

$$\begin{aligned} p_n(t) &= u(t) - \alpha_n(t) \geq 0 \\ q_n(t) &= \beta_n(t) - u(t) \geq 0, \end{aligned}$$

for $t \in J$ with $p_n(0) = q_n(0) = 0$. Using the definitions of α_n, β_n, the Taylor series expansion with Lagrange remainder, and the mean value theorem together with (ii), we obtain

$$\begin{aligned} p'_{n+1} = u' - \alpha'_{n+1} = f(t,u) + g(t,u) - \ \Big[&\sum_{i=0}^{2k-1} \frac{f^{(i)}(t,\ \beta_n)(\beta_{n+1} - \beta_n)^i}{i!} \\ + \ \sum_{i=0}^{2k-2} \frac{g^{(i)}(t,\ \beta_n)(\beta_{n+1} - \beta_n)^i}{i!} &+ \ \frac{g^{(2k-1)}(t,\ \alpha_n)(\beta_{n+1} - \beta_n)^{2k-1}}{(2k-1)!} \Big] \\ = f(t,u) + g(t,u) - \ \Big[\ f(t,\beta_{n+1}) &- \frac{f^{(2k)}(t,\ \xi_1)(\beta_{n+1} - \beta_n)^{2k}}{(2k)!} \\ + \ g(t,\beta_{n+1}) - \frac{g^{(2k-1)}(t,\xi_2)(\beta_{n+1} - \beta_n)^{2k-1}}{(2k-1)!} &+ \ \frac{g^{(2k-1)}(t,\ \alpha_n)(\beta_{n+1} - \beta_n)^{2k-1}}{(2k-1)!} \Big] \end{aligned}$$

$$
\begin{aligned}
&= f_u(t,\eta_1)(u-\beta_{n+1}) + g_u(t,\eta_2)(u-\beta_{n+1})\\
&+ \ \frac{f^{(2k)}(t,\ \xi_1)(\beta_{n+1}-\beta_n)^{2k}}{(2k)!} + \frac{g^{(2k)}(t,\eta_3)(\beta_{n+1}-\beta_n)^{2k-1}(\xi_2-\alpha_n)}{(2k-1)!}\\
&\le \ [f_u(t,\eta_1)+g_u(t,\eta_2)](u-\beta_{n+1}) + \frac{f^{(2k)}(t,\ \xi_1)(u-\beta_n)^{2k}}{(2k)!}\\
&+ \ \frac{g^{(2k)}(t,\eta_3)(u-\beta_n)^{2k-1}[(\beta_n-u)+(u-\alpha_n)]}{(2k-1)!}\\
&= \ [-f_u(t,\eta_1)-g_u(t,\eta_2)]q_{n+1} + \frac{f^{(2k)}(t,\ \xi_1)q_n^{2k}}{(2k)!} - \ \frac{g^{(2k)}(t,\eta_3)q_n^{2k-1}(p_n+q_n)}{(2k-1)!}\ ,
\end{aligned}
$$

where $\beta_{n+1} \le \xi_1, \xi_2 \le \beta_n$, $u \le \eta_1, \eta_2 \le \beta_{n+1}$ and $\alpha_n \le \eta_3 \le \xi_2$. Let k_1, k_2 be positive constants such that

$$
|f_u(t,u)+g_u(t,u)| \le k_1, \qquad k_2 = max(k_3,k_4),
$$

where

$$
\left|\frac{f^{(2k)}(t,u)}{(2k)!} - \ \frac{g^{(2k)}(t,u)}{(2k-1)!}\right| \le k_3 \qquad \left|\frac{g^{(2k)}(t,u)}{(2k-1)!}\right| \le k_4.
$$

This proves

$$
p'_{n+1} \le k_1 q_{n+1} + k_2 q_n^{2k-1}(p_n+q_n), \qquad p_{n+1}(0) = 0.
$$

Similarly,

$$
\begin{aligned}
q'_{n+1} = \beta'_{n+1} - u' = \ &\sum_{i=0}^{2k-2} \frac{f^{(i)}(t,\ \alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!}\\
&+ \ \frac{f^{(2k-1)}(t,\ \beta_n)(\alpha_{n+1}-\alpha_n)^{2k-1}}{(2k-1)!} + \ \sum_{i=0}^{2k-1} \frac{g^{(i)}(t,\ \alpha_n)(\alpha_{n+1}-\alpha_n)^i}{i!}\\
&- \ f(t,u) - g(t,u) = \ f(t,\alpha_{n+1}) - \frac{f^{(2k-1)}(t,\ \xi_1)(\alpha_{n+1}-\alpha_n)^{2k-1}}{(2k-1)!}\\
&+ \frac{f^{(2k-1)}(t,\ \beta_n)(\alpha_{n+1}-\alpha_n)^{2k-1}}{(2k-1)!} + \ g(t,\alpha_{n+1}) - \frac{g^{(2k)}(t,\xi_2)(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!}\\
&- \ f(t,u) - g(t,u) = f_u(t,\eta_1)(\alpha_{n+1}-u) + g_u(t,\eta_2)(\alpha_{n+1}-u)\\
&+ \ \frac{f^{(2k)}(t,\ \eta 3)(\alpha_{n+1}-\alpha_n)^{2k-1}(\beta_n-\xi_1)}{(2k-1)!} - \frac{g^{(2k)}(t,\xi_2)(\alpha_{n+1}-\alpha_n)^{2k}}{(2k)!}\\
&\le \ [f_u(t,\eta_1)+g_u(t,\eta_2)](\alpha_{n+1}-u)\\
&+ \ \frac{f^{(2k)}(t,\ \eta 3)(\alpha_n-u)^{2k-1}[(u-\beta_n)+(\alpha_n-u)]}{(2k-1)!} - \frac{g^{(2k)}(t,\xi_2)(\alpha_n-u)^{2k}}{(2k)!}\\
&= \ [-f_u(t,\eta_1)-g_u(t,\eta_2)]p_{n+1} + \frac{f^{(2k)}(t,\ \eta_3)p_n^{2k-1}(p_n+q_n)}{(2k-1)!} - \ \frac{g^{(2k)}(t,\xi_2)p_n^{2k}}{(2k)!}\,,
\end{aligned}
$$

where $\alpha_n \leq \xi_1, \xi_2 \leq \alpha_{n+1}$, $\alpha_{n+1} \leq \eta_1, \eta_2 \leq u$ and $\xi_1 \leq \eta_3 \leq \beta_n$. Let $k_5 = max(k_6, k_7)$ be a positive constant where

$$\left| \frac{f^{(2k)}(t,u)}{(2k-1)!} - \frac{g^{(2k)}(t,u)}{(2k)!} \right| \leq k_6 \qquad \left| \frac{f^{(2k)}(t,u)}{(2k-1)!} \right| \leq k_7.$$

Then we have

$$q_{n+1}^{'} \leq k_1 p_{n+1} + k_5 p_n^{2k-1}(p_n + q_n), \qquad q_{n+1}(0) = 0.$$

Hence we have the following system with $k = max(k_2, k_5)$:

$$p_{n+1}^{'} \leq k_1 q_{n+1} + k q_n^{2k-1}(p_n + q_n), \qquad p_{n+1}(0) = 0,$$
$$q_{n+1}^{'} \leq k_1 p_{n+1} + k p_n^{2k-1}(p_n + q_n), \qquad q_{n+1}(0) = 0.$$

We can write this as the following vectorial inequality:

$$r_{n+1}^{\prime} \leq A r_{n+1} + \sigma_n,$$

where

$$r_{n+1} = \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix}, \; A = \begin{pmatrix} 0 & k_1 \\ k_1 & 0 \end{pmatrix}, \; \sigma_n = \begin{pmatrix} k \; q_n^{2k-1} \; (p_n + q_n) \\ k \; p_n^{2k-1} \; (p_n + q_n) \end{pmatrix}.$$

Applying Corollary 2.5, treating σ_n as a forcing term, we get

$$0 \leq r_{n+1}(t) \leq \int_0^t e^{A(t-s)} \; \sigma_n(s) \; ds,$$

which, in turn, yields

$$\|r_{n+1}(t)\| \leq A^{-1} e^{AT} \|\sigma_n\|,$$

or

$$\|u(t) - \alpha_{n+1}(t)\| \leq C_1 \|\beta_n(t) - u(t)\|^{2k-1}[\|u(t) - \alpha_n(t)\| + \|\beta_n(t) - u(t)\|] \tag{3.31}$$

and

$$\|\beta_{n+1}(t) - u(t)\| \leq C_2 \|u(t) - \alpha_n(t)|^{2k-1}[\|u(t) - \alpha_n(t)\| + \|\beta_n(t) - u(t)\|], \tag{3.32}$$

where $C_{1,2} = C_{1,2}(k, k_1, T, \alpha_0, \beta_0, f, g)$. Using (3.31) and (3.32), we obtain

$$\max_{t \in J}[|u(t) - \alpha_{n+1}(t)| + |\beta_{n+1}(t) - u(t)|] \leq C \max_{t \in J}[|u(t) - \alpha_n(t)| + |\beta_n(t) - u(t)|]^{2k}.$$

This completes the proof. □

Theorem 3.4. *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J, R]$ *are coupled lower and upper solutions of type III (case (A4)) with* $\alpha_0(t) \leq \beta_0(t)$ *on* J.

(ii) $f, g \in C^{0,2k+1}[\Omega, R]$ *such that* $f(t,u)$ *is* $(2k)$*-hyperconvex,* $k \geq 1$ *in* u *and* $g(t,u)$ *is* $(2k)$*-hyperconcave,* $k \geq 1$ *in* u *on* J *[i.e.* $f^{(2k+1)}(t,u) \geq 0$, $g^{(2k+1)}(t,u) \leq 0$, *for* $(t,u) \in \Omega$ *].*

(*iii*)

$$f_u(t,u) \leq -\max_{\Omega}[f^{(2k+1)}(t,u)]\frac{(\beta_0-\alpha_0)^{2k}}{(2k-1)!} \leq 0 \qquad on\ \Omega$$

$$g_u(t,u) \leq \quad \min_{\Omega}[g^{(2k+1)}(t,u)]\frac{(\beta_0-\alpha_0)^{2k}}{(2k-1)!} \leq 0 \qquad on\ \Omega.$$

Then there exist monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, $n \geq 0$ which converge uniformly and monotonically to the unique solution of (3.1) and the convergence is of order $2k+1$.

Proof. We can get monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, $n \geq 0$ which converge uniformly and monotonically to the unique solution of (3.1), using the following IVPs for $n = 1, 2, \ldots$ together with the inequalities (3.6), (3.7), (3.10), and (3.11):

$$\begin{aligned}\alpha_n' &= F(t,\alpha_{n-1},\beta_{n-1};\beta_n) = \\ &= \sum_{i=0}^{2k-1}\frac{f^{(i)}(t,\beta_{n-1})(\beta_n-\beta_{n-1})^i}{i!} + \frac{f^{(2k)}(t,\alpha_{n-1})(\beta_n-\beta_{n-1})^{2k}}{(2k)!} \\ &+ \sum_{i=0}^{2k}\frac{g^{(i)}(t,\beta_{n-1})(\beta_n-\beta_{n-1})^i}{i!}, \qquad \alpha_n(0) = u_0,\end{aligned}$$

$$\begin{aligned}\beta_n' &= G(t,\alpha_{n-1},\beta_{n-1};\alpha_n) = \\ &= \sum_{i=0}^{2k-1}\frac{f^{(i)}(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1})^i}{i!} + \frac{f^{(2k)}(t,\beta_{n-1})(\alpha_n-\alpha_{n-1})^{2k}}{(2k)!} \\ &+ \sum_{i=0}^{2k}\frac{g^{(i)}(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1})^i}{i!}, \qquad \beta_n(0) = u_0.\end{aligned}$$

The proof is similar to that of Theorem 3.3 with appropriate modifications. We omit the details. □

Since next four theorems are a combination of previous four theorems with appropriate conditions $f(t,u) \equiv 0$ and/or $g(t,u) \equiv 0$, we merely indicate the iterates which enable us to develop the required sequences.

The next two results are relative to the coupled upper and lower solutions of type I *case*$(A2)$ when m is even and odd respectively.

Theorem 3.5. *Assume that*

(*i*) *$\alpha_0, \beta_0 \in C^1[J,R]$ are coupled lower and upper solutions of type I (case $(A2)$) with $\alpha_0(t) \leq \beta_0(t)$ on J.*

(*ii*) *$f, g \in C^{0,2k}[\Omega,R]$ such that $f(t,u)$ is $(2k-1)$-hyperconvex, $k \geq 1$ in u and $g(t,u)$ is $(2k-1)$-hyperconcave, $k \geq 1$ in u on J [i.e. $f^{(2k)}(t,u) \geq 0$, $g^{(2k)}(t,u) \leq 0$, for $(t,u) \in \Omega$].*

(iii)

$$g_u(t,u) \leq \min_{\Omega}[g^{(2k)}(t,u)]\frac{(\beta_0-\alpha_0)^{2k-1}}{(2k-2)!} \leq 0 \qquad on\ \Omega.$$

Then there exist monotone sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$, $n \geq 0$ *which converge uniformly and monotonically to the unique solution of (3.1) and the convergence is of order* $2k$.

Proof. In order to construct monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, $n \geq 0$ which converge uniformly and monotonically to the unique solution of (3.1), we need to consider the following IVPs for $n = 1, 2, \ldots$ together with the inequalities (3.2), (3.3), (3.8), and (3.9):

$$\begin{aligned}\alpha_n' &= F(t,\alpha_{n-1},\beta_{n-1};\alpha_n,\beta_n)\\ &= \sum_{i=0}^{2k-1}\frac{f^{(i)}(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1})^i}{i!} + \sum_{i=0}^{2k-2}\frac{g^{(i)}(t,\beta_{n-1})(\beta_n-\beta_{n-1})^i}{i!}\\ &+ \frac{g^{(2k-1)}(t,\alpha_{n-1})(\beta_n-\beta_{n-1})^{2k-1}}{(2k-1)!}, \qquad \alpha_n(0) = u_0,\end{aligned}$$

$$\begin{aligned}\beta_n' &= G(t,\alpha_{n-1},\beta_{n-1};\beta_n,\alpha_n) =\\ &= \sum_{i=0}^{2k-2}\frac{f^{(i)}(t,\beta_{n-1})(\beta_n-\beta_{n-1})^i}{i!} + \frac{f^{(2k-1)}(t,\alpha_{n-1})(\beta_n-\beta_{n-1})^{2k-1}}{(2k-1)!}\\ &+ \sum_{i=0}^{2k-1}\frac{g^{(i)}(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1})^i}{i!}. \qquad \beta_n(0) = u_0,\end{aligned}$$

□

Theorem 3.6. *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J,R]$ *are coupled lower and upper solutions of type I (case (A2)) with* $\alpha_0(t) \leq \beta_0(t)$ *on* J.
(ii) $f, g \in C^{0,2k+1}[\Omega, R]$ *such that* $f(t,u)$ *is* $(2k)$*-hyperconvex,* $k \geq 1$ *in* u *and* $g(t,u)$ *is* $(2k)$*-hyperconcave,* $k \geq 1$ *in* u *on* J *[i.e.* $f^{(2k+1)}(t,u) \geq 0$, $g^{(2k+1)}(t,u) \leq 0$, *for* $(t,u) \in \Omega$ *].*
(iii)

$$g_u(t,u) \leq \min_{\Omega}[g^{(2k+1)}(t,u)]\frac{(\beta_0-\alpha_0)^{2k}}{(2k-1)!} \leq 0 \qquad on\ \Omega.$$

Then there exist monotone sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$, $n \geq 0$ *which converge uniformly and monotonically to the unique solution of (3.1) and the convergence is of order* $2k+1$.

Proof. Let us consider the following IVPs for $n = 1, 2, \ldots$ together with the inequalities (3.4), (3.5), (3.10), and (3.11):

$$\begin{aligned} \alpha_n' &= F(t, \alpha_{n-1}, \beta_{n-1}; \alpha_n, \beta_n) && = \sum_{i=0}^{2k} \frac{f^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} \\ &+ \sum_{i=0}^{2k} \frac{g^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!}, && \alpha_n(0) = u_0, \\ \beta_n' &= G(t, \alpha_{n-1}, \beta_{n-1}; \beta_n, \alpha_n) && = \sum_{i=0}^{2k} \frac{f^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} \\ &+ \sum_{i=0}^{2k} \frac{g^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!}. && \beta_n(0) = u_0, \end{aligned}$$

□

The last two Theorems are relative to the coupled upper and lower solutions of type II $case(A3)$ when m is even and odd respectively.

Theorem 3.7. *Assume that*

(*i*) $\alpha_0, \beta_0 \in C^1[J, R]$ *are coupled lower and upper solutions of type II (case* $(A3)$*) with* $\alpha_0(t) \leq \beta_0(t)$ *on* J.

(*ii*) $f, g \in C^{0,2k}[\Omega, R]$ *such that* $f(t, u)$ *is* $(2k-1)$*-hyperconvex,* $k \geq 1$ *in* u *and* $g(t, u)$ *is* $(2k-1)$*-hyperconcave,* $k \geq 1$ *in* u *on* J *[i.e.* $f^{(2k)}(t, u) \geq 0$, $g^{(2k)}(t, u) \leq 0$, *for* $(t, u) \in \Omega$ *].*

(*iii*)

$$f_u(t, u) \leq -\max_{\Omega}[f^{(2k)}(t, u)] \frac{(\beta_0 - \alpha_0)^{2k-1}}{(2k-2)!} \leq 0 \quad \text{on } \Omega.$$

Then there exist monotone sequences $\{\alpha_n(t)\}$, *and* $\{\beta_n(t)\}$, $n \geq 0$ *which converge uniformly and monotonically to the unique solution of (3.1) and the convergence is of order* $2k$.

Proof. We can get monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, $n \geq 0$ which converge uniformly and monotonically to the unique solution of (3.1), using the following IVPs for $n = 1, 2, \ldots$ together with the inequalities (3.2), (3.3), (3.8), and (3.9):

$$\begin{aligned} \alpha_n' &= F(t, \alpha_{n-1}, \beta_{n-1}; \alpha_n, \beta_n) \\ &= \sum_{i=0}^{2k-1} \frac{f^{(i)}(t, \beta_{n-1})(\beta_n - \beta_{n-1})^i}{i!} + \sum_{i=0}^{2k-2} \frac{g^{(i)}(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1})^i}{i!} \\ &+ \frac{g^{(2k-1)}(t, \beta_{n-1})(\alpha_n - \alpha_{n-1})^{2k-1}}{(2k-1)!}, \qquad \alpha_n(0) = u_0, \end{aligned}$$

$$\begin{aligned}\beta_n' &= G(t,\alpha_{n-1},\beta_{n-1};\beta_n,\alpha_n) = \\ &= \sum_{i=0}^{2k-2}\frac{f^{(i)}(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1})^i}{i!} + \frac{f^{(2k-1)}(t,\beta_{n-1})(\alpha_n-\alpha_{n-1})^{2k-1}}{(2k-1)!} \\ &+ \sum_{i=0}^{2k-1}\frac{g^{(i)}(t,\beta_{n-1})(\beta_n-\beta_{n-1})^i}{i!}, \qquad \beta_n(0)=u_0.\end{aligned}$$

□

Theorem 3.8. *Assume that*

(*i*) $\alpha_0,\beta_0 \in C^1[J,R]$ *are coupled lower and upper solutions of type II (case (A3)) with* $\alpha_0(t)\le\beta_0(t)$ *on* J.

(*ii*) $f,g\in C^{0,2k+1}[\Omega,R]$ *such that* $f(t,u)$ *is* $(2k)$*-hyperconvex,* $k\ge 1$ *in* u *and* $g(t,u)$ *is* $(2k)$*-hyperconcave,* $k\ge1$ *in* u *on* J *[i.e.* $f^{(2k+1)}(t,u)\ge 0$, $g^{(2k+1)}(t,u)\le 0$, *for* $(t,u)\in\Omega$ *].*

(*iii*)

$$f_u(t,u)\le -\max_{\Omega}[f^{(2k+1)}(t,u)]\frac{(\beta_0-\alpha_0)^{2k}}{(2k-1)!}\le 0 \quad \text{on } \Omega.$$

Then there exist monotone sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$, $n\ge0$ *which converge uniformly and monotonically to the unique solution of (3.1) and the convergence is of order* $2k+1$.

Proof. Considering the following IVPs for $n=1,2,\ldots$ together with the inequalities (3.6), (3.7), (3.12), and (3.13), we can get:

$$\begin{aligned}\alpha_n' &= F(t,\alpha_{n-1},\beta_{n-1};\alpha_n,\beta_n) &&= \sum_{i=0}^{2k-1}\frac{f^{(i)}(t,\beta_{n-1})(\beta_n-\beta_{n-1})^i}{i!} \\ &+ \frac{f^{(2k)}(t,\alpha_{n-1})(\beta_n-\beta_{n-1})^{2k}}{(2k)!} &&+\sum_{i=0}^{2k-1}\frac{g^{(i)}(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1})^i}{i!} \\ &+ \frac{g^{(2k)}(t,\beta_{n-1})(\alpha_n-\alpha_{n-1})^{2k}}{(2k)!}, &&\qquad \alpha_n(0)=u_0,\end{aligned}$$

$$\begin{aligned}\beta_n' &= G(t,\alpha_{n-1},\beta_{n-1};\beta_n,\alpha_n) &&= \sum_{i=0}^{2k-1}\frac{f^{(i)}(t,\alpha_{n-1})(\alpha_n-\alpha_{n-1})^i}{i!} \\ &+ \frac{f^{(2k)}(t,\beta_{n-1})(\alpha_n-\alpha_{n-1})^{2k}}{(2k)!} &&+\sum_{i=0}^{2k-1}\frac{g^{(i)}(t,\beta_{n-1})(\beta_n-\beta_{n-1})^i}{i!} \\ &+ \frac{g^{(2k)}(t,\alpha_{n-1})(\beta_n-\beta_{n-1})^{2k}}{(2k)!}, &&\qquad \beta_n(0)=u_0.\end{aligned}$$

□

Remark 3.9 Results of Theorems 1.3.1 to 1.3.4 and its corollaries of [4] can be considered as special cases of our results for $m=2$. Further, we have obtained the same order of convergence as in [3] when the nonlinearity of the iterates we have developed is one less than that of [3]. Our results yield the results of [7] as a special cases.

Conclusion: We have used iterates of nonlinearity of order $m-1$ when the forcing function is the sum of hyperconvex and hyperconcave of order $m-1$. Observe that when $m \geq 3$, we have nonlinear iterates. We hope to develop linear iterates and yet have the order of convergence as m. At present this problem is still open.

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