Dynamic Systems and Applications 15 (2006) 429-450

OSCILLATION AND NONOSCILLATION CRITERIA FOR HALF-LINEAR SECOND ORDER DIFFERENCE EQUATIONS

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ABSTRACT. New oscillation and nonoscillation criteria are established for the second order half-linear difference equation

$$\Delta(r_n\Phi(\Delta x_n)) + q_n\Phi(x_{n+1}) = 0, \quad \Phi(x) = |x|^{p-2}x, \quad p > 1,$$

via the Riccati technique. Some known results are also improved including the discrete version of the Hille-Wintner comparison theorem.

AMS (MOS) Subject Classification. 39A10.

1. INTRODUCTION

The aim of this paper is to study the oscillatory character of the half-linear difference equation

$$\Delta(r_n\Phi(\Delta x_n)) + q_n\Phi(x_{n+1}) = 0, \tag{1.1}$$

in which $\Phi(x) = |x|^{p-2}x$, $p > 1$, and $\{r_n\}$, $\{q_n\}$ are real sequences.

An interval $(m, m+1]$ is said to contain a generalized zero of a solution $\{x_n\}$ if $x_m \neq 0$ and $r_m x_m x_{m+1} \leq 0$. A solution is called oscillatory if it has infinitely many generalized zeros in the set of positive integers. Otherwise, the solution is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory. Equation (1.1) is said to be nonoscillatory if it has at least one nonoscillatory solution. If $r_n > 0$ for all $n \geq n_0$ and some positive integer n_0, then the nonoscillation of $\{x_n\}$ is equivalent to saying that x_n is either eventually positive or eventually negative.

Recently, there have been an extensive investigation on the various qualitative properties of equation (1.1) (e.g., oscillation, nonoscillation, conjugacy). Among the papers dealing with the oscillation and/or the nonoscillation of (1.1) and some related equations we refer to [3, 7, 8, 14, 18], [20]-[25] and to [1] for further results on the oscillation as well as the general theory of the difference equations. The study of (1.1)

Received September 11, 2005

draws its importance from the fact that (1.1) can be viewed as the discrete version of the second order half-linear differential equation

$$(r(t)\Phi(x'(t)))' + q(t)\Phi(x(t)) = 0, \quad t > 0, \tag{1.2}$$

which has been studied by many authors, (see, e.g. [4]-[6], [13] and [15]-[17]). Also, when $p = 2$ equation (1.1) is reduced to the famous second order linear difference equation

$$\Delta(r_n \Delta x_n) + q_n x_{n+1} = 0. \tag{1.3}$$

It is known (see [7, 20]) that many fundamental results of the oscillation theory of (1.3) can be extended to the half-linear generalization (1.1). In particular, the Sturm comparison and separation results have been proved for (1.1) (see [7]). Accordingly, equation (1.1) can not have both oscillatory and nonoscillatory solutions. Also, many known results for (1.2) have their discrete versions holding true for (1.1) with sometimes extra conditions due to the discrete nature of the utilized techniques. However, these similarities do not mean necessarily that all oscillation (nonoscillation) criteria for (1.2) must have discrete analogies for (1.1). For instance, when $r_n \equiv 1$, [9]-[11] have proved that (1.3) is oscillatory provided that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=n_0}^{n} \sum_{i=n_0}^{k} q_i = \infty \tag{1.4}$$

although the continuous version of this condition, i.e.,

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(r)\,dr\,ds = \infty \tag{1.5}$$

does not ensure the oscillation of (1.2) when $p = 2$ and $r(t) \equiv 1$ according to Hartman [12].

Our approach to the oscillation/nonoscillation of equation (1.1) will depend on the so called Riccati technique by which (1.1) is related to the generalized Riccati difference inequality

$$R_p(w_n) \le 0, \quad n \ge n_0 \tag{1.6}$$

by the substitution

$$w_n = \frac{r_n \Phi(\Delta x_n)}{\Phi(x_n)}, \quad n \ge n_0$$

for some positive integer n_0, where

$$R_p(w_n) = \Delta w_n + q_n + w_n \left(1 - \frac{r_n}{\Phi(\Phi^{-1}(r_n) + \Phi^{-1}(w_n))}\right),$$

and Φ^{-1} is the inverse function of Φ, that is $\Phi^{-1}(x) = |x|^{\rho-2}x$, $\rho = \frac{p}{p-1}$. This relation is established by the following fundamental result which generalizes Chen and Erbe [2, Lemma 1.2].

Lemma 1.1. [7, Lemma 2] *Equation (1.1) is nonoscillatory if and only if there exists a sequence w_n with $r_n + w_n > 0$ for large n such that (1.6) holds.*

A brief outline of the paper is as follows. The next section studies the effect of changing the value of p on the nonoscillation of (1.1). The third section investigates the nonoscillation of (1.1) when $\sum^{\infty} q_i$ exists. A comparison technique is developed and used to extract an improved discrete version of the well known Hille-Wintner comparison criterion. We also improve the condition

$$\lim_{n\to\infty} \frac{r_n^{1-\rho}}{\sum^{n-1} r_i^{1-\rho}} = 0 \tag{1.7}$$

and hence give a partial answer to a related open problem raised by [7]. The last section contains some new oscillation criteria. Most of these results do not require the positivity restriction of r_n.

2. COMPARISON RESULTS WITH RESPECT TO p

Throughout this section we assume that $r_n > 0$, $n \geq n_0$ for some positive integer n_0.

Lemma 2.1. *Let f be a function defined by*

$$f(x) = \frac{x}{\Phi(\Phi^{-1}(a) + \Phi^{-1}(x)) - a}, \quad -a < x \neq 0, a > 0.$$

Then

$$f(x) \begin{cases} \in (0,\,1], & \text{if } p \geq 2 \\ > 1, & \text{if } p \in (1,\,2) \end{cases}$$

for all $x \in (-a,\,\infty)$, $x \neq 0$.

Proof. Note that f is well defined since $\Phi(\Phi^{-1}(a) + \Phi^{-1}(x)) - a \neq 0$ for any $x \neq 0$ in $(-a,\,\infty)$. If $p = 2$, then $f(x) = 1$ for all $x > -a$. If $p > 2$, we have two separate cases; when $x > 0$ and when $x < 0$. Consider the first case; then

$$f(x) = \frac{x}{(a^{\rho-1} + x^{\rho-1})^{p-1} - a} > 0,$$

and

$$f'(x) = \frac{(a^{\rho-1} + x^{\rho-1})^{p-2} a^{\rho-1} - a}{((a^{\rho-1} + x^{\rho-1})^{p-1} - a)^2} > 0, \quad ('= d/dx).$$

In view of the fact that

$$\begin{aligned} (a^{\rho-1} + x^{\rho-1})^{p-2} a^{\rho-1} - a &> a^{(\rho-1)(p-2)} a^{\rho-1} - a \\ &= a - a = 0, \end{aligned}$$

we conclude that $f'(x) > 0$ for all $x > 0$. Thus,

$$0 < f(x) < \lim_{x\to\infty} f(x) = 1, \quad x > 0$$

which is our desired conclusion for this case.

When $x \in (-a, 0)$, we have

$$f(x) = \frac{x}{(a^{\rho-1} - (-x)^{\rho-1})^{p-1} - a} > 0.$$

On the other hand, the first derivative of f is found to be

$$f'(x) = \frac{(a^{\rho-1} - (-x)^{\rho-1})^{p-2} a^{\rho-1} - a}{((a^{\rho-1} - (-x)^{\rho-1})^{p-1} - a)^2} > 0,$$

now

$$\begin{aligned} (a^{\rho-1} - (-x)^{\rho-1})^{p-2} a^{\rho-1} - a &< a^{(\rho-1)(p-2)} a^{\rho-1} - a \\ &= a - a = 0, \quad x \in (-a, 0) \end{aligned}$$

so $f'(x) < 0$ for all $x \in (-a, 0)$, and hence,

$$0 < f(x) < \lim_{x \to -a} f(x) = 1, \quad x \in (-a, 0)$$

as required. When $p < 2$, similar arguments imply the proof. We omit the details to avoid repetition. The proof is complete. $\square$

Now assume that $\{w_n\}$ and $\{r_n\}$ are such that $w_n + r_n > 0$ for all $n \geq n_0$. One can show that $\Phi(\Phi^{-1}(r_n) + \Phi^{-1}(w_n)) > 0$ for $n \geq n_0$. Moreover,

$$\Phi(\Phi^{-1}(r_n) + \Phi^{-1}(w_n)) = r_n$$

if and only if $\Phi^{-1}(w_n) = 0$ that is; if and only if $w_n = 0$. Therefore, the sequence $\{\rho_n\}$ given by

$$\rho_n = \begin{cases} r_n, & \text{if } w_n = 0, \\ \frac{w_n r_n}{\Phi(\Phi^{-1}(r_n) + \Phi^{-1}(w_n)) - r_n}, & \text{if } w_n \neq 0, \end{cases} \tag{2.1}$$

is well defined for $n \geq n_0$. On the other hand, if for each n the values a and x in Lemma 2.1 are replaced by r_n and w_n respectively, it follows, when $w_n \neq 0$, that

$$\frac{w_n}{\Phi(\Phi^{-1}(r_n) + \Phi^{-1}(w_n)) - r_n} \begin{cases} \in (0, 1], & \text{if } p \geq 2 \\ > 1, & \text{if } p < 2. \end{cases}$$

This arguments proves the following result.

Lemma 2.2. *Let r_n and w_n be such that $w_n + r_n > 0$ for all $n \geq n_0$. Then the sequence $\{\rho_n\}$ defined by (2.1) satisfies*

$$\begin{aligned} &0 < \rho_n \leq r_n, \quad \textit{if } p \geq 2, \\ &\rho_n \geq r_n, \quad \textit{if } p < 2, \end{aligned} \tag{2.2}$$

for all $n \geq n_0$.

Lemma 2.3. *If there exists a sequence $\{w_n\}$ with $w_n + r_n > 0$ for all $n \geq n_0$ then w_n satisfies (1.6) if and only if it satisfies*

$$\Delta w_n + q_n + \frac{w_n^2}{w_n + \rho_n} \leq 0, \quad n \geq n_0, \tag{2.3}$$

where ρ_n is defined by (2.1) and $w_n + \rho_n > 0$ for all $n \geq n_0$.

Proof. When $w_n = 0$, the inequalities (1.6) and (2.3) coincide. For $w_n \neq 0$, (2.1) implies that

$$\rho_n \left(\Phi(\Phi^{-1}(r_n)) + \Phi^{-1}(w_n)) - r_n\right) = w_n r_n, \quad n \geq n_0,$$

which yields

$$\rho_n \Phi(\Phi^{-1}(r_n) + \Phi^{-1}(w_n)) = r_n(w_n + \rho_n), \quad n \geq n_0,$$

and hence,

$$\begin{aligned} F(w_n, r_n) &= w_n \left(1 - \frac{r_n}{\Phi(\Phi^{-1}(r_n) + \Phi^{-1}(w_n))}\right) \\ &= w_n \left(1 - \frac{\rho_n}{w_n + \rho_n}\right) \\ &= \frac{w_n^2}{w_n + \rho_n}, \quad n \geq n_0. \end{aligned}$$

Therefore, (1.6) and (2.3) coincide also in this case. The proof is complete. □

Theorem 2.4. *Assume that (1.1) is nonoscillatory for some $p > 2$. Then (1.3) is nonoscillatory.*

Proof. Since (1.1) is nonoscillatory then, according to Lemma 1.1, there exists a sequence $\{w_n\}$ with $w_n + r_n > 0$ for large n (say $n \geq n_0$) such that $R_p(w_n) \leq 0$, for all $n \geq n_0$. Let ρ_n be defined by (2.1). Then (2.2) holds for $p > 2$ which implies that

$$\frac{w_n^2}{w_n + \rho_n} \geq \frac{w_n^2}{w_n + r_n}, \quad n \geq n_0.$$

Then (2.3) yields

$$\Delta w_n + q_n + \frac{w_n^2}{w_n + r_n} \leq 0, \quad n \geq n_0,$$

which, in view of Lemma 1.1, leads to the nonoscillation of (1.3). This completes the proof. □

Theorem 2.5. *Assume that $p < 2$. If (1.3) is nonoscillatory, then (1.1) is nonoscillatory.*

Proof. Since (1.3) is nonoscillatory, there exists w_n such that $w_n + r_n > 0$ for $n \geq n_0$ and

$$\Delta w_n + q_n + \frac{w_n^2}{w_n + r_n} \leq 0, \quad n \geq n_0.$$

Using (2.2) when $p < 2$, we get

$$w_n + \rho_n \geq w_n + r_n > 0, \quad n \geq n_0.$$

Then

$$\frac{w_n^2}{w_n + r_n} \geq \frac{w_n^2}{w_n + \rho_n}, \quad n \geq n_0.$$

Therefore, w_n satisfies the Riccati inequality (2.3). This means that (1.1) is nonoscillatory according to Lemma 1.1. The proof is complete. □

The contrapositives of Theorem 2.4 and Theorem 2.5 are the following two results respectively.

Corollary 2.6. *Assume that $p > 2$. If the linear equation (1.3) is oscillatory, then the half-linear equation (1.1) is oscillatory.*

Corollary 2.7. *Assume that $p < 2$. If the half-linear equation (1.1) is oscillatory, then the linear equation (1.3) is also oscillatory.*

Combining Theorem 2.4 and Theorem 2.5, we obtain the following interesting result.

Corollary 2.8. *Assume that (1.1) is nonoscillatory for some $p > 2$. Then it is also nonoscillatory for all $p \leq 2$.*

The following result is a partial improvement of Corollary 2.8.

Theorem 2.9. *Assume that (1.1), with $p = p_0$, has a nonoscillatory solution $\{x_n\}$ such that $w_n = r_n \frac{\Phi(\Delta x_n)}{\Phi(x_n)} > 0$ for all $n \geq n_0$. Then (1.1) is nonoscillatory for all $1 < p < p_0$.*

Proof. Let

$$F(x, y, p) = x(1 - \frac{x}{\Phi(\Phi^{-1}(x) + \Phi^{-1}(y))}), \quad x > 0, \ y > 0.$$

Then

$$F(x, y, p) = x(1 - (1 + (x/y)^{\rho-1})^{1-p}$$

Differentiating with respect to p, we get

$$\begin{aligned} F_p(x, y, p) &= \frac{x((1 + (x/y)^{\rho-1}) \ln(1 + (x/y)^{\rho-1}) - (x/y)^{\rho-1} \ln(x/y)^{\rho-1})}{1 + (x/y)^{\rho-1}} \\ &> \frac{x \ln(1 + (x/y)^{\rho-1})}{1 + (x/y)^{\rho-1}} > 0 \quad \text{for all} \quad x > 0, y > 0. \end{aligned}$$

Thus $F(w_n, r_n, p_0) > F(w_n, r_n, p)$ for all $1 < p < p_0$ and $n \geq n_0$. Since $R_{p_0}(w_n) = 0$, then $R_p(w_n) \leq 0$, for all $1 < p < p_0$, $n \geq n_0$. Therefore, (1.1) is nonoscillatory, when $1 < p < p_0$, according to Lemma 1.1. □

Note that to apply Theorem 2.9, one needs further assumptions on the coefficients r_n, q_n to guarantee the positivity of w_n. Such conditions are not difficult to obtain. For example, the reader is referred to Lemma 2 and Lemma 4 in [23] as well as Lemma 4.2 in this work. However, it would be better if Theorem 2.9 holds without those extra restrictions. Indeed, this author believes that the following conjecture is true.

Conjucture 2.10. Assume that (1.1) is nonoscillatory for some $p = p_0$. Then (1.1) is also nonoscillatory for all other values of p satisfying that $1 < p < p_0$.

Using Theorem 2.4 and Theorem 2.5, one can apply many known linear oscillation (nonoscillation) criteria to the half-linear equation (1.1). Some of these criteria seem to be difficult to drive directly from (1.1) due to the presence of the half-linear parameter p. For example, an application to [19, Lemma 3] leads, using Theorem 2.4, to the following result.

Corollary 2.11. *If there exists a subsequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$, such that $r_{n_k-1} + r_{n_k} - q_{n_k} \leq 0$ for large k, then (1.1) oscillates for all $p \geq 2$.*

If the above corollary fails, one may use the following one which is an immediate application of [10, Corollary 2.5] and Theorem 2.4.

Corollary 2.12. *Let $\{\lambda_n\}$ be a positive real sequence satisfying*

$$(\frac{r_n + r_{n-1} - q_n}{2r_n})\frac{\lambda_n}{\lambda_{n-1}} \leq d < 1 \quad \textit{eventually.}$$

If there exists a subsequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$ and a nonnegative real number M such that

$$r_{n_k-1} + r_{n_k} - q_{n_k} - M\lambda_{n_k-1}r^2_{n_k-1} \leq 0 \quad \textit{for large } k,$$

then (1.1) oscillates for all $p \geq 2$.

3. SOME RESULTS WHEN $\sum^\infty q_i$ CONVERGES

Lemma 3.1. *Assume that $r_n > 0$ for $n \geq n_0$,*

$$\sum_{i=n_0}^{\infty} \frac{1}{r_i^{\rho-1}} = \infty, \tag{3.1}$$

and

$$Q_n = \sum_{i=n}^{\infty} q_i \quad \textit{exists}, \; n \geq n_0, \tag{3.2}$$

such that $Q_n > 0$ for infinitely many values of n. If (1.1) is nonoscillatory, then there exists an eventually nonnegative solution z_n of the inequality

$$\sum_{i=n}^{\infty} F(u_i, r_i) + Q_n \leq u_n, \quad n \geq n_0. \tag{3.3}$$

Proof. Let x_n be a nonoscillatory solution of (1.1). Then x_n is either eventually positive or eventually negative. It is easy to see that x_n is a solution of (1.1) if and only if $-x_n$ is a solution of (1.1), so without loss of generality, we assume that x_n is eventually positive. That is, there exists $n_1 \geq n_0$ such that $x_n > 0$ for all $n \geq n_1$. Using the Riccati transformation

$$w_n = \frac{r_n \Phi(\Delta x_n)}{\Phi(x_n)}, \quad n \geq n_1,$$

equation (1.1) implies that

$$\Delta w_n + q_n + F(w_n,\, r_n) = 0, \quad n \geq n_1 \tag{3.4}$$

where $w_n + r_n > 0$ for all $n \geq n_1$ (see Lemma 1.1). Summing from n_1 to $n-1$, we obtain

$$w_n - w_{n_1} + \sum_{i=n_1}^{n-1} q_i + \sum_{i=n_1}^{n-1} F(w_i,\, r_i) = 0. \tag{3.5}$$

Since $F(w_i,\, r_i) \geq 0$ for all $i \geq n_1$, we have either

$$\sum_{i=n_1}^{\infty} F(w_i,\, r_i) = \infty, \tag{3.6}$$

or

$$\sum_{i=n_1}^{\infty} F(w_i,\, r_i) < \infty \quad \text{exists.} \tag{3.7}$$

If (3.6) holds, then (3.5) yields

$$\lim_{n\to\infty} w_n = -\infty$$

which, in view of the definition of w_n, implies that $\Phi(\Delta x_n)$ is eventually negative. Consequently, Δx_n is also eventually negative. Thus there exists $n_2 \geq n_1$ such that $\Delta x_n < 0$ for all $n \geq n_2$. But Φ is increasing on the real line, so $\Delta\Phi(y_n) < 0$ for all $n \geq n_2$. Choosing the integer n_2 so large that $Q_{n_2} > 0$, we can find another integer $N > n_2$ such that

$$\sum_{i=n_2}^{N-1} q_i \leq 0 \quad \text{and} \quad \sum_{i=n_2}^{n} q_i > 0, \quad n \geq N. \tag{3.8}$$

Note that

$$\Delta(\Phi(x_{n+1}) \sum_{i=n_2}^{n-1} q_i) = (\Delta\Phi(x_{n+1})) \sum_{i=n_2}^{n} q_i + q_n \Phi(x_{n+1}).$$

Summing both sides of this identity from N to n and using (3.8), we see that

$$\begin{aligned}\sum_{i=N}^{n} q_i \Phi(x_{i+1}) &= \Phi(x_{n+1}) \sum_{i=n_2}^{n} q_i - \Phi(x_{N+1}) \sum_{i=n_2}^{N-1} q_i \\ &\quad - \sum_{i=N}^{n} (\sum_{k=n_2}^{i} q_k) \Delta\Phi(x_{i+1}) > 0 \quad \text{for all} \quad n \geq N.\end{aligned}$$

Therefore, (1.1) yields

$$\begin{aligned}r_{n+1}\Phi(\Delta x_{n+1}) - r_N \Delta\Phi(\Delta x_N) &= -\sum_{i=N}^{n} q_i \Phi(x_{i+1}) \\ &\leq 0, \quad \text{for all} \quad n \geq N.\end{aligned}$$

Hence,

$$\Phi(\Delta x_{n+1}) \leq r_N \Delta\Phi(\Delta x_N) \frac{1}{r_{n+1}},$$

or equivalently,

$$\Delta x_{n+1} \leq \Phi^{-1}(r_N \Delta\Phi(\Delta x_N)) \frac{1}{r_{n+1}^{\rho-1}}, \quad n \geq N.$$

Summing, we have

$$x_{n+1} - x_N \leq \Phi^{-1}(r_N \Delta\Phi(\Delta x_N)) \sum_{i=N}^{n-1} \frac{1}{r_{i+1}^{\rho-1}}, \quad n \geq N.$$

But

$$\Phi^{-1}(r_N \Delta\Phi(\Delta x_N)) < 0 \quad \text{and} \quad \sum_{i=N}^{\infty} \frac{1}{r_{i+1}^{\rho-1}} = \infty,$$

so $x_n \to -\infty$ as $n \to \infty$, which is a contradiction.

Suppose now that (3.7) holds. Let $n \to \infty$ in (3.5). It follows that $\lim_{n\to\infty} w_n$ exists. In fact $\lim_{n\to\infty} w_n \geq 0$, since otherwise we have $\lim_{n\to\infty} w_n < 0$ which yields $\lim_{n\to\infty} x_n = -\infty$ as in the above case.

Now summing both sides of (3.4) from $n \geq n_1$ to ∞, we obtain

$$\sum_{i=n}^{\infty} q_i + \sum_{i=n}^{\infty} F(w_i, r_i) \leq w_n, \quad n \geq n_1.$$

Let $W_n = \max\{0, w_n\}$, $n \geq n_1$. Then, $W_n \geq 0$ ($\not\equiv 0$ eventually), $W_n \geq w_n$ for all $n \geq n_1$, and

$$\sum_{i=n}^{\infty} F(w_i, r_i) \geq \sum_{i=n}^{\infty} F(W_i, r_i), \quad n \geq n_1.$$

Therefore, W_n satisfies the inequality

$$\sum_{i=n}^{\infty} q_i + \sum_{i=n}^{\infty} F(W_i, r_i) \leq W_n, \quad n \geq n_1,$$

which is our desired conclusion. $\square$

Lemma 3.2. *Assume that (3.2) holds,* $r_n > 0$ *for* $n \geq n_0$*, and*

$$\sum_{i=n}^{\infty} F(Q_i^+, r_i) + Q_n \geq 0, \quad \text{for } n \geq n_0 \tag{3.9}$$

where $Q_n^+ = \max\{0, Q_n\}$*. If (3.3) has a nonnegative solution* z_n*, then (1.1) has a nondecreasing positive solution.*

Proof. Let us define a mapping $T : l^\infty \to l^\infty$ by

$$(Tv)_n = \sum_{i=n}^{\infty} F(v_i, r_i) + Q_n, \quad n \geq n_0.$$

Consider the successive approximation sequences $\{y_n^0\}$, $\{y_n^1\}$,... defined by

$$\begin{aligned} y_n^0 &= Q_n^+, \quad n \geq n_0 \\ y_n^{(m+1)} &= (Ty^{(m)})_n, \quad \text{for} \quad m = 0, 1, 2, \dots \quad n \geq n_0. \end{aligned} \tag{3.10}$$

Since $(Tz)_n \leq z_n$, $n \geq n_0$, then $z_n \geq Q_n^+$ for $n \geq n_0$ and this in turn leads to

$$z_n \geq (TQ^+)_n = (Ty^{(0)})_n = y_n^{(1)}, \quad n \geq n_0.$$

But

$$y_n^{(1)} = \sum_{i=n}^{\infty} F(Q_i^+, r_i) + Q_n \geq 0, \quad n \geq n_0,$$

so

$$y_n^{(1)} > \max\{0, Q_n\} = Q_n^+ = y_n^{(0)}, \quad n \geq n_0.$$

Consequently, a simple induction yields

$$0 \leq y_n^{(0)} \leq y_n^{(1)} \leq \dots \leq z_n, \quad n \geq n_0.$$

Thus, there exists a sequence $\{y_n\}_{n \geq n_0} \in l^\infty$ such that

$$\lim_{m \to \infty} y_n^{(m)} = y_n, \quad y_n \geq 0 \quad \text{for} \quad n \geq n_0.$$

Allowing $m \to \infty$ in (3.10), Lebesgue's dominated convergence theorem implies that $y_n = (Ty)_n$ for all $n \geq n_0$, that is;

$$y_n = \sum_{i=n}^{\infty} F(y_i, r_i) + \sum_{i=n}^{\infty} q_i, \quad n \geq n_0,$$

and hence

$$\Delta y_n + F(y_n, r_n) + q_n = 0, \quad n \geq n_0.$$

Let $\{x_n\}$ be defined by

$$x_n = \sqcap_{i=n_0}^{n-1} \left(1 + (\frac{y_i}{r_i})^{\rho - 1}\right), n > n_0.$$

Then x_n is a solution of (1.1) for $n > n_0$ (see [7]). Moreover, the nonnegativity of y_n implies that

$$\begin{aligned} x_{n+1} &= \sqcap_{i=n_0}^{n}\left(1+(\frac{y_i}{r_i})^{\rho-1}\right) \\ &= x_n\left(1+(\frac{y_n}{r_n})^{\rho-1}\right) \geq x_n \quad \text{for all} \quad n \geq n_0. \end{aligned}$$

Thus, x_n is a positive nondecreasing solution of (1.1). The proof is complete. □

Combining Lemma 3.1 and Lemma 3.2, we obtain the following fundamental result that improves [14, Theorem 1] and [23, Lemma 3].

Theorem 3.3. *Assume that $r_n > 0$ for $n \geq n_0$, (3.1), (3.2) and (3.9) are satisfied. Then (1.1) is nonoscillatory if and only if the inequality (3.3) has a nonnegative solution.*

The following Hille-Wintner type comparison result is an improvement to [14, Corollary 1]. Its proof is a direct application of Theorem 3.3 and hence is omitted.

Corollary 3.4. *Assume that $\{R_n\}$ and $\{c_n\}$ are real sequences such that*

$$\sum_{i=n}^{\infty} c_i \quad \textit{exists} \quad \textit{for all} \quad n \geq n_0 > 0,$$

$$\sum_{i=n}^{\infty} F(C_i^+,\, R_i) + \sum_{i=n}^{\infty} c_i \geq 0, \quad n \geq n_0,$$

where $C_n^+ = \max\{0,\, \sum_{i=n}^{\infty} c_i\}$ ($\not\equiv 0$ eventually) for $n \geq n_0$,

$$\sum_{i=n_0}^{\infty} \frac{1}{R_i^{\rho-1}} = \infty,$$

$$0 < r_n \leq R_n, \quad n \geq n_0,$$

$$\sum_{i=n}^{\infty} q_i \quad \textit{exists},$$

and

$$\sum_{i=n}^{\infty} q_i \geq \sum_{i=n}^{\infty} c_i, \quad n \geq n_0.$$

Then the half-linear equation

$$\Delta(R_n\Phi(\Delta y_n)) + c_n\Phi(y_{n+1}) = 0$$

is nonoscillatory provided that (1.1) is nonoscillatory.

The argument used in the proofs of Lemma 3.1 and Lemma 3.2 leads to the following result which will be used in the next section.

Corollary 3.5. *Assume that $r_n > 0$ for $n \geq n_0$, (3.1), (3.2) and (3.9) are satisfied. If (1.1) is nonoscillatory, then the sequence $\{y_n^{(m)}\}$ in (3.10) is well defined and there exists a nonnegative nontrivial sequence $\{y_n\}$ such that $\lim_{m\to\infty} y_n^{(m)} = y_n$ for $n \geq n_0$.*

Throughout the rest of this section, we consider the real numbers β, λ and γ such that $0 < \beta < \frac{\lambda}{p}$, $\lambda = (\frac{p-1}{p})^{p-1}$, and $\gamma = \frac{\lambda-\beta}{\beta(p-1)}$.

Lemma 3.6. *Assume that $r_n > 0$ for $n \geq n_0$,*

$$q_n = -\beta\Delta(\sum_{i=n_0}^{n-1} \frac{1}{r_i^{\rho-1}})^{1-p}, \quad n > n_0,$$

and

$$r_n^{\rho-1}\sum_{i=n_0}^{n-1}\frac{1}{r_i^{\rho-1}} \geq \begin{cases} \frac{1}{\gamma^{\rho-1}-1}, & \text{if } p \geq 2 \\ \frac{1}{\gamma-1}, & \text{if } p < 2. \end{cases} \tag{3.11}$$

Then $w_n = \lambda(\sum_{i=n_0}^{n-1}\frac{1}{r_i^{\rho-1}})^{1-p}$ is a positive solution of the Riccati inequality (1.6) for $n > n_0$.

Proof. Note that $q_n = -\frac{\beta}{\lambda}\Delta w_n$ for all $n > n_0$. Then,

$$\begin{aligned} R_p(w_n) &= (1-\frac{\beta}{\lambda})\Delta w_n + F(w_n, r_n) \\ &= (1-\frac{\beta}{\lambda})w_{n+1} + \frac{\beta}{\lambda}w_n - \frac{w_n r_n}{(r_n^{\rho-1}+w_n^{\rho-1})^{p-1}}, \quad n > n_0. \end{aligned} \tag{3.12}$$

Let $v_n = \sum_{n_0}^{n-1} r_i^{1-\rho}$ for all $n > n_0$. Then

$$v_{n+1} = r_n^{1-\rho} + v_n, \quad w_n = \frac{\lambda}{v_n^{p-1}},$$

and

$$w_{n+1} = \frac{\lambda}{(r_n^{1-\rho}+v_n)^{p-1}} = \frac{\lambda r_n}{(1+r_n^{\rho-1}v_n)^{p-1}}, \quad \text{for all} \quad n > n_0.$$

Substituting into (3.12), we obtain

$$\begin{aligned} R_p(w_n) &= (1-\frac{\beta}{\lambda})\frac{\lambda r_n}{(1+r_n^{\rho-1}v_n)^{p-1}} + \frac{\beta}{v_n^{p-1}} - \frac{\lambda r_n}{v_n^{p-1}(r_n^{\rho-1}+\frac{\lambda^{\rho-1}}{v_n})^{p-1}} \\ &= (\lambda-\beta)\frac{r_n}{(1+r_n^{\rho-1}v_n)^{p-1}} + \frac{\beta}{v_n^{p-1}} - \frac{\lambda r_n}{(v_n r_n^{\rho-1}+\lambda^{\rho-1})^{p-1}} \\ &= \frac{D_n}{v_n^{p-1}(1+r_n^{\rho-1}v_n)^{p-1}(v_n r_n^{\rho-1}+\lambda^{\rho-1})^{p-1}}, \quad n > n_0, \end{aligned}$$

where

$$\begin{aligned} D_n = \quad & (\lambda-\beta)r_n v_n^{p-1}(v_n r_n^{\rho-1}+\lambda^{\rho-1})^{p-1} + \beta(1+r_n^{\rho-1}v_n)^{p-1}(v_n r_n^{\rho-1}+\lambda^{\rho-1})^{p-1} \\ & -\lambda r_n v_n^{p-1}(1+r_n^{\rho-1}v_n)^{p-1}, \quad n > n_0. \end{aligned}$$

Our proof will be complete if $D_n < 0$ for all $n > n_0$. Let $\lambda - \beta = \sigma$ and $z_n = r_n^{\rho-1} v_n$; then we see that

$$D_n = \sigma z_n^{p-1}\left((z_n + \lambda^{\rho-1})^{p-1} - (1+z_n)^{p-1}\right) + \beta(1+z_n)^{p-1}\left((z_n+\lambda^{\rho-1})^{p-1} - z_n^{p-1}\right).$$

Using the Mean Value Theorem, we have

$$D_n = \sigma z_n^{p-1}(p-1)(\lambda^{\rho-1}-1)\eta_n^{p-2} + \beta(1+z_n)^{p-1}\lambda^{\rho-1}(p-1)\xi_n^{p-2}, \tag{3.13}$$

where $\eta \in (z_n + \lambda^{\rho-1},\, 1+z_n)$ and $\xi \in (z_n,\, z_n + \lambda^{\rho-1})$ for all $n > n_0$. Assume that $p \geq 2$. Then,

$$\begin{aligned} D_n &\leq \sigma(p-1)(\lambda^{\rho-1}-1)z_n^{p-1}(z_n+\lambda^{\rho-1})^{p-2} + \beta(p-1)\lambda^{\rho-1}(1+z_n)^{p-1}(z_n+\lambda^{\rho-1})^{p-2} \\ &= (p-1)(z_n+\lambda^{\rho-1})^{p-2}\left(\sigma(-\frac{1}{p})z_n^{p-1} + \beta\frac{p-1}{p}(1+z_n)^{p-1}\right), \quad n > n_0. \end{aligned}$$

Suppose that for a certain $n_1 > n_0$, we have $D_{n_1} > 0$. Then

$$\sigma(-\frac{1}{p})z_{n_1}^{p-1} + \beta\frac{p-1}{p}(1+z_{n_1})^{p-1} > 0,$$

which implies that

$$\left(\frac{1+z_{n_1}}{z_{n_1}}\right)^{p-1} > \frac{\rho}{\beta(p-1)} = \gamma.$$

Note that $\gamma > 1$ since otherwise we get $\lambda - \beta = \sigma \leq \beta(p-1)$, i.e., $\lambda \leq p\beta$ which is impossible according to our assumption that $\beta < \frac{\lambda}{p}$. Now solving the above inequality with respect to z_{n_1}, we obtain

$$z_{n_1} < \frac{1}{\gamma^{\rho-1}-1}, \text{ i.e., } r_{n_1}^{\rho-1}\sum_{i=n_0}^{n_1} r_i^{1-\rho} < \frac{1}{\gamma^{\rho-1}-1},$$

which contradicts (3.11). Thus, $D_n \leq 0$ for all $n > n_0$ as required and the proof is complete for this case.

When $p < 2$, combining the inequalities

$$\eta_n < 1 + z_n \quad \text{and} \quad \xi_n > z_n$$

with (3.13), it follows that

$$D_n \leq (p-1)(1+z_n)^{p-2}z_n^{p-2}\left(\sigma z_n(-\frac{1}{p}) + \beta(\frac{p-1}{p})(1+z_n)\right), \quad n > n_0.$$

Using the same argument as in the above case, we can prove that

$$\frac{-\sigma}{p}z_n + \beta(\frac{p-1}{p})(1+z_n) \leq 0, \quad n > n_0,$$

that is $D_n \leq 0$ for all $n > n_0$. Hence, $R_p(w_n) \leq 0$ for all $n > n_0$, which is our desired conclusion. The proof is complete. □

Note that the above result implies that $w_n = \lambda(\sum_{i=n_0}^{n-1} r_i^{1-\rho})^{1-p}$ is a positive solution of the inequality

$$\sum_{i=n}^{\infty} F(w_i,\, r_i) + \sum_{i=n}^{\infty} q_i \leq w_n, \quad n \geq n_0,$$

where condition (3.1) ($\sum^{\infty} r_i^{1-\rho} = \infty$) is not needed here due to the fact that $w_n > 0$ for $n \geq n_0$. We claim also that this w_n satisfies the inequality

$$\sum_{i=n}^{\infty} F(w_i,\, r_i) + \beta(\sum_{i=n_0}^{n-1} r_i^{1-\rho})^{1-p} < w_n, \quad n \geq n_0.$$

For this purpose, let

$$I(w_n) = \sum_{i=n}^{\infty} F(w_i,\, r_i) + \beta(\sum_{i=n_0}^{n-1} r_i^{1-\rho})^{1-p} - w_n, \quad n \geq n_0.$$

Then, using (3.12), we get

$$\begin{aligned} I(w_n) &= \sum_{i=n}^{\infty} F(w_i,\, r_i) + (\frac{\beta}{\lambda} - 1)w_n \\ &= \sum_{i=n}^{\infty} (F(w_i,\, r_i) + (1 - \frac{\beta}{\lambda})\Delta w_i) + (\frac{\beta}{\lambda} - 1)w_\infty \\ &= \sum_{i=n}^{\infty} R_p(w_i) + (\frac{\beta}{\lambda} - 1)w_\infty, \quad n \geq n_0, \end{aligned}$$

where $w_\infty = \lambda(\sum_{i=n_0}^{\infty} r_i^{1-\rho})^{1-p} \geq 0$, $\beta < \lambda$ and $R_p(w_n) \leq 0$ for all $n \geq n_0$. Therefore, $I(w_n) \leq 0$ for all $n \geq n_0$, which proves our claim. So, if $\{q_n\}$ is any real sequence such that $\sum^{\infty} q_i$ exists and

$$\sum_{i=n_0}^{n} q_i \leq \beta(\sum_{i=n_0}^{n-1} r_i^{1-\rho})^{1-p}, \quad n > n_0, \tag{3.14}$$

then (3.3) has a positive solution. This conclusion, together with Lemma 3.2, imply the following important result.

Theorem 3.7. *Assume that $r_n > 0$ for all $n \geq n_0$ and that (3.2), (3.9), (3.11) and (3.14) are satisfied. Then (1.1) has a nondecreasing positive solution.*

Corollary 3.8. *Assume that $r_n > 0$ for all $n \geq n_0$ and that (3.2), (3.9) and (3.11) are satisfied. If*

$$\limsup_{n\to\infty}\, (\sum_{i=n_0}^{n-1} r_i^{1-\rho})^{p-1} \sum_{i=n}^{\infty} q_i < \beta,$$

then (1.1) has a nondecreasing positive solution.

Corollary 3.9. *Assume that $r_n > 0$ for all $n \geq n_0$ and (3.11) holds. If $\sum_{i=n}^{\infty} q_i^+$ exists for all $n \geq n_0$ and*

$$\limsup_{n\to\infty} \Big(\sum_{i=n_0}^{n-1} r_i^{1-\rho}\Big)^{p-1} \sum_{i=n}^{\infty} q_i^+ < \beta,$$

then (1.1) is nonoscillatory.

Proof. In view of the assumptions and Corollary 3.8, we see that the equation

$$\Delta(r_n\Phi(\Delta x_n)) + q_n^+\Phi(x_{n+1}) = 0$$

is nonoscillatory. But $q_n^+ \geq q_n$, so (1.1) is also nonoscillatory according to the Sturm comparison theorem (see [7]). ☐

Remark 3.10. Došlý and Řehák [7] questioned the need of condition (1.7) in some of their results. The reader can see that condition (3.11) has weakened this restriction, so that Corollary 3.8 improves [7, Theorem 3] particularly when $Q_n \geq 0$ eventually.

The following example is illustrative.

Example 3.11. Consider the difference equation (1.3) in which $r_n = 2^{-n}$ and

$$q_n = -\Delta\left(\frac{1+3(-1)^n}{3^{n+1}} + \left(\left(\frac{2+2(-1)^n}{3^{n+1}}\right)^2 + \frac{4+4(-1)^n}{2^n3^{n+1}}\right)^{1/2}\right), \quad n > n_0 > 1.$$

Then the reader can see that

$$\sum_{i=n_0}^{n-1} \frac{1}{r_i} = 2^n - 2^{n_0},$$

and

$$Q_n = \sum_{i=n}^{\infty} q_i = \frac{1+3(-1)^n}{3^{n+1}} + \left(\left(\frac{2+2(-1)^n}{3^{n+1}}\right)^2 + \frac{4+4(-1)^n}{2^n3^{n+1}}\right)^{1/2}, \quad n > n_0.$$

Moreover,

$$Q_n^+ = \frac{2+2(-1)^n}{3^{n+1}} + \left(\left(\frac{2+2(-1)^n}{3^{n+1}}\right)^2 + \frac{4+4(-1)^n}{2^n3^{n+1}}\right)^{1/2}, \quad n > n_0,$$

$$F(Q_n^+, r_n) = \frac{4+4(-1)^n}{3^{n+1}},$$

$$\sum_{i=n}^{\infty} F(Q_i^+, r_i) = \frac{2+(-1)^n}{3^n},$$

and

$$r_n \sum_{i=n_0}^{n-1} \frac{1}{r_i} = 1 - 2^{n_0-n} \geq \frac{1}{2}, \quad n > n_0.$$

Therefore, condition (3.13) is satisfied with $\beta = 1/8$ (since $\gamma = 3$),

$$\sum_{i=n}^{\infty} F(Q_i^+,\, r_i) + Q_n \geq \frac{1}{3^{n+1}}, \quad n \geq n_0,$$

and

$$\limsup_{n\to\infty} \sum_{i=n_0}^{n-1} \frac{1}{r_i} Q_n = 0 < \frac{1}{8}.$$

Thus, all requirements of Corollary 3.8 are satisfied. Accordingly, the given equation has a nondecreasing positive solution.

4. SOME OSCILLATION CRITERIA

The first result in this section is the contrapositive of Corollary 3.5. It improves [23, Theorem 1].

Theorem 4.1. *Assume that (3.1), (3.2) and (3.9) are satisfied. Let y_n^m be defined by (3.10). Then equation (1.1) is oscillatory if either one of the following conditions is satisfied:*
(i)

$$\lim_{m\to\infty} y_n^m = \infty \quad \textit{for all} \quad n \geq n_0.$$

or
(ii) There exists an integer $m_0 \geq 1$ such that

$$y_n^{m_0} = \infty \quad \textit{for all} \quad n \geq n_0.$$

Lemma 4.2. *Let $\{w_n\}$ satisfy $R_p(w_n) = 0$ with $w_n + r_n > 0$ for all $n \geq n_0$ (i.e., (1.1) is nonoscillatory). If*

$$\limsup_{n\to\infty} \sum_{i=n_0}^{n-1} q_i - r_n > -\infty, \tag{4.1}$$

then

$$\sum_{i=n_0}^{\infty} F(w_i,\, r_i) < \infty.$$

Moreover, if

$$\limsup_{n\to\infty} \sum_{i=n_0}^{n-1} q_i - r_n \geq 0, \tag{4.2}$$

then $w_n > 0$ for all $n \geq n_0$.

Proof. Summing both sides of the Riccati equation $R_p(w_n) = 0$ from $k \geq n_0$ to $n-1$, we obtain

$$w_n + r_n - w_k + \sum_{i=k}^{n-1} F(w_i,\, r_i) + \sum_{i=k}^{n-1} q_i - r_n = 0, \quad n > k \geq n_0. \tag{4.3}$$

Since $w_n + r_n > 0$ for all $n \geq n_0$, it follows from (4.1) and (4.3) that $\sum_{i=n_0}^{\infty} F(w_i, r_i)$ is convergent as required.

If condition (4.2) is satisfied, then (4.3) yields

$$
\begin{aligned}
w_k &= w_n + r_n + \sum_{i=k}^{n-1} F(w_i, r_i) + \sum_{i=k}^{n-1} q_i - r_n \\
&> \liminf_{n\to\infty} \Big(w_n + r_n + \sum_{i=k}^{n-1} F(w_i, r_i)\Big) + \limsup_{n\to\infty} \Big(\sum_{i=k}^{n-1} q_i - r_n\Big) \\
&> 0, \quad n > k \geq n_0.
\end{aligned} \tag{4.4}
$$

The proof is complete. □

Theorem 4.3. *Assume that (4.1) holds. If there exists a subsequence of positive integers $\{n_k\}$ with $n_k \to \infty$ as $k \to \infty$ such that $\limsup_{k\to\infty} r_{n_k} < 0$, then (1.1) is oscillatory.*

Proof. Suppose, for the sake of obtaining a contradiction, that (1.1) is nonoscillatory. Then (1.1) has a solution $\{x_n\}$ such that $r_n x_n x_{n+1} > 0$ for all $n \geq n_0$. Therefore, $w_n = \frac{r_n \Phi(\Delta x_n)}{x_n}$ satisfies the Riccati equation $R_p(w_n) = 0$, $n \geq n_0$ with $w_n + r_n > 0$ for all $n \geq n_0$. Applying Lemma 4.2, we obtain

$$
\sum_{i=n_0}^{\infty} F(w_i, r_i) < \infty,
$$

so, it is necessary that

$$
\lim_{n\to\infty} F(w_n, r_n) = 0. \tag{4.5}
$$

In view of the assumption that $\limsup_{k\to\infty} r_{n_k} < 0$, we obtain

$$
\lim_{k\to\infty} \frac{1}{r_{n_k}} F(w_{n_k}, r_{n_k}) = 0,
$$

that is,

$$
\lim_{k\to\infty} \left(1 - \frac{1}{\Phi(1 + \Phi^{-1}(\frac{w_{n_k}}{r_{n_k}}))}\right) \frac{w_{n_k}}{r_{n_k}} = 0,
$$

which implies that

$$
\lim_{k\to\infty} \frac{w_{n_k}}{r_{n_k}} = 0. \tag{4.6}
$$

Then

$$
\lim_{k\to\infty} \frac{w_{n_k} + r_{n_k}}{r_{n_k}} = \lim_{k\to\infty} \frac{w_{n_k}}{r_{n_k}} + 1 = 1,
$$

which is impossible as $w_{n_k} + r_{n_k}$ is eventually positive while r_{n_k} is eventually negative. This completes the proof. □

Theorem 4.4. *Assume that (4.1) holds. If there exist a positive real number M and a subsequence of positive integers $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$ such that $|r_{n_k}| < M$ for all $n_k > n_0$, and*

$$\lim_{k\to\infty} \sum_{i=n_0}^{n_k-1} q_i \text{ does not exist,} \tag{4.7}$$

then (1.1) is oscillatory.

Proof. Proceed as in the proof of Theorem 4.3. Then (4.5) yields

$$\lim_{k\to\infty} \left(1 - \frac{1}{\Phi(1+\Phi^{-1}(\frac{w_{n_k}}{r_{n_k}}))}\right) w_{n_k} = 0,$$

which holds only if $\lim_{k\to\infty} w_{n_k} = 0$ (since $|r_{n_k}|$ is bounded). From (4.3), we obtain

$$w_{n_k} - w_{n_0} + \sum_{i=n_0}^{n_k-1} F(w_i, r_i) + \sum_{i=n_0}^{n_k-1} q_i = 0, \quad n_k > n_0. \tag{4.8}$$

Then,

$$\begin{aligned} \lim_{k\to\infty} \sum_{i=n_0}^{n_k-1} q_i &= -\lim_{k\to\infty} (w_{n_k} - w_{n_0} + \sum_{i=n_0}^{n_k} F(w_i, r_i)) \\ &= w_{n_0} - \sum_{i=n_0}^{\infty} F(w_i, r_i) \quad \text{exists,} \end{aligned}$$

which contradicts (4.7). The proof is complete. □

Combining the above two theorems, we obtain the following result:

Corollary 4.5. *Assume that $\{n_k\}$ is a subsequence of positive integers such that $n_k \to \infty$ as $k \to \infty$ and $r_{n_k} < M$ for some $M > 0$ and all n_k. If (4.1) and (4.7) are satisfied, then (1.1) is oscillatory.*

Remark 4.6.

1. As $\{r_n\}$ is not required to be bounded above, Corollary 4.5 improves [21, Corollary 1].
2. Since

$$\liminf_{n\to\infty} \sum^{n} q_i \le \liminf_{n\to\infty} \frac{1}{n}\sum^{n}\sum^{k} q_i \le \limsup_{n\to\infty} \frac{1}{n}\sum^{n}\sum^{k} q_i \le \limsup_{n\to\infty} \sum^{n} q_i, \tag{4.9}$$

then (4.7) is better than the averaging criterion

$$\liminf_{n\to\infty} \frac{1}{n}\sum^{n}\sum^{k} q_i < \limsup_{n\to\infty} \frac{1}{n}\sum^{n}\sum^{k} q_i.$$

Thus, Theorem 4.4 improves Corollary 2.7 and Corollary 2.8 of [2] for the linear case (1.3).

The following result completes partially Corollary 4.5.

Theorem 4.7. *Assume that $\{n_k\}$ is a subsequence of positive integers such that $n_k \to \infty$ as $k \to \infty$. If (4.1) holds,*

$$\lim_{k\to\infty} r_{n_k} = \infty,$$

and

$$\lim_{k\to\infty} \frac{1}{r_{n_k}} \sum^{n_k-1} q_i \neq 0 \quad \text{or} \quad \text{does not exist,} \tag{4.10}$$

the (1.1) is oscillatory.

Proof. Using the same reasoning of the proofs of Theorem 4.3 and Theorem 4.4, we obtain (4.6) and (4.8). Dividing both sides of (4.8) by r_{n_k}, $r_{n_k} > n_0$, it follows that

$$\frac{w_{n_k}}{r_{n_k}} - \frac{w_{n_0}}{r_{n_k}} + \frac{1}{r_{n_k}} \sum_{i=n_0}^{n_k-1} F(w_i,\, r_i) + \frac{1}{r_{n_k}} \sum_{i=n_0}^{n_k-1} q_i = 0, \quad n_k > n_0.$$

Taking into account that (4.6) holds and $\sum_{i=n_0}^{\infty} F(w_i,\, r_i) < \infty$, then

$$\begin{aligned} \lim_{k\to\infty} \frac{1}{r_{n_k}} \sum_{i=n_0}^{n_k-1} q_i &= \lim_{k\to\infty} \left(\frac{w_{n_k}}{r_{n_k}} - \frac{w_{n_0}}{r_{n_k}} + \frac{1}{r_{n_k}} \sum_{i=n_0}^{n_k-1} F(w_i,\, r_i)\right) \\ &= 0, \end{aligned}$$

which contradicts (4.6). The proof is complete. □

If (4.1) holds and $\lim_{n\to\infty} r_n = \infty$, then one can find a number $A > 0$ and a positive integer sequence $\{n_k\}$ with $\lim_{k\to\infty} n_k = \infty$ such that

$$\sum_{i=n_0}^{n_k-1} q_i - r_{n_k} > -A, \quad \text{for all} \quad n_k.$$

Dividing by r_{n_k}, it follows that

$$\frac{1}{r_{n_k}} \sum_{i=n_0}^{n_k-1} q_i - 1 > -\frac{A}{r_{n_k}}.$$

As $k \to \infty$, we obtain

$$\liminf_{k\to\infty} \frac{1}{r_{n_k}} \sum_{i=n_0}^{n_k-1} q_i - 1 \geq 0$$

which, clearly, implies that condition (4.10) is satisfied. Therefore, we conclude the following corollary of Theorem 4.7.

Corollary 4.8. *Assume that (4.1) holds and $\lim_{n\to\infty} r_n = \infty$. Then (1.1) is oscillatory.*

Theorem 4.9. *Assume that*

$$\limsup_{n\to\infty} \sum_{i=n_0}^{n-1} q_i - r_n = \infty.$$

Then (1.1) is oscillatory.

Proof. Assume that (1.1) is nonoscillatory. Proceeding as in the proof of Lemma 4.2, we obtain (4.4). Taking the upper limit of both sides of (4.4) as $n \to \infty$, we obtain $w_k = \infty$ which is impossible. This completes the proof. □

If the upper limit in the above result is a finite number, one may use the following result.

Theorem 4.10. *Assume that there exists a sequence $\{\varphi_n\}$ such that*

$$\limsup_{n\to\infty} \Big(\sum_{i=k}^{n-1} q_i - r_n\Big) \geq \varphi_k \quad \textit{for all} \quad k \geq n_0.$$

If $\liminf_{n\to\infty} \varphi_n > -\infty$, and

$$\limsup_{n\to\infty} \sum^{n} q_i = \infty, \tag{4.11}$$

then (1.1) is oscillatory.

Proof. Assume that (1.1) is nonoscillatory. Proceeding as in the proof of Lemma 4.2, we obtain (4.3). Remembering that $w_n + r_n > 0$ for $n \geq n_1$, (4.3) yields

$$\sum_{i=k}^{n-1} F(w_i, r_i) + \Big(\sum_{i=k}^{n-1} q_i - r_n\Big) \leq w_k, \quad n > k \geq n_1. \tag{4.12}$$

Note that (4.1) is satisfied due to the given assumptions and hence $\sum_{i=k}^{\infty} F(w_i, r_i) < \infty$. Accordingly, taking the upper limit of both sides of inequality (4.12), we obtain

$$\sum_{i=k}^{\infty} F(w_i, r_i) + \varphi_k \leq w_k, \quad k \geq n_1.$$

Thus, $\liminf_{n\to\infty} w_n > -\infty$. On the other hand, letting $n \to \infty$ in (3.5) and using (4.11), we get

$$\liminf_{n\to\infty} w_n = -\infty,$$

which is a contradiction. The proof is complete. □

The following result is an immediate consequence of the above theorem.

Corollary 4.11. *Assume that $r_n \equiv 1$ and (4.11) is satisfied. Then (1.1) is oscillatory.*

Taking into account the relation (4.9), we conclude from Corollary 4.11 the following result.

Corollary 4.12. *Assume that $r_n \equiv 1$ and (1.4) is satisfied. Then (1.1) is oscillatory.*

Remark 4.13. Corollary 4.12 generalizes [11, Corolary 2.2](see [11, Remark 2.1(2)]) to the half-linear case. As we have mentioned in the introduction of this work, Corollary 4.12 does not have a continuous analogue for $p = 2$. So it is very interesting to see if it does not have a continuous analogue for all $p > 1$.

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