Dynamic Systems and Applications 15 (2006) 501-514

DETERMINISTIC APPROXIMATION IN N-PERSON STOCHASTIC DIFFERENTIAL GAMES WITH A SMALL PARAMETER

K. M. RAMACHANDRAN

Department of Mathematics, University of South Florida,
Tampa, FL 33620-5700 USA

ABSTRACT. In this work, we consider the fluid approximation of Nash equilibria in N-player stochastic differential games with wideband noise perturbations. We present conditions under which the games perturbed by noise converges to the deterministic games. It will be shown, under appropriate conditions, that applying the near optimal policies of the limi game to the stochastic games are near optimal for the stochastic games. Weak convergence methods will be utilized.

1. INTRODUCTION

Consider an N-person noncooperative dynamic game problem where the evolution of the system is given by the following deterministic ordinary differential equation:

$$\begin{aligned} dx(t) &= [\overline{a}(x(t)) + \sum_{i=1}^{N} b_i(x(t))u_i(t)]dt \\ x(0) &= x_0 \end{aligned} \tag{1.1}$$

where $x(t)$ is controlled deterministic process, $u_i(t), i = 1, 2, \ldots, N$, are deterministic controls for each of the N-players. Let $U_i, i = 1, \ldots, N,$ be compact metric spaces (we can take U_i as compact subsets of R^d). Let $U = U_1 \times \cdots \times U_N$. An $u \in U$ is called an N-dimentional strategy vector. We denote $u_i(t) \in U_i$ as the ith component of u and u_{-i} denotes the $N-1$ dimentional vector obtained by removing the ith component $(i = 1, 2, \ldots, N)$ of vector u. Define payoff to player k by

$$J_k(u_1, \ldots, u_N) = \int_0^T [g_k(x(t)) + p_k(u_1(t), \ldots, u_N(t))]dt + r_k(x(T)) \tag{1.2}$$

where $T < \infty$ is the fixed terminal time for the game. An N-tuple of strategies $u^* = (u_1^*, \ldots, u_N^*) \in U$ is said to be equilibrium (in the sense of Nash) if for each $k = 1, \ldots, N$,

$$J_k[u^*] \geq J_k[u_{-k}^*, u_k]$$

Received August 6, 2005

for any $u_k \in U_k$. That is, it is not beneficial for player k to deviate from the equillibrium. The following concept of δ- equilibrium is important in the theory of approximations. Fix a $k \in \{1, \ldots, N\}$. An N-tuple of strategies $u^\delta = (u_1^\delta, \ldots, u_N^\delta)$ is said to be a δ-equilibrium if for any $k = 1, \ldots, N$,

$$J_k[u^\delta] \geq \sup_{u_k \in U_k} J_k[u_{-k}^\delta, u_k] - \delta.$$

Since most of the physical systems are stochastic in nature, the deterministic models are only approximations to the real systems. Now consider a more realistic physical model for an N-person game problem described by a family of stochastic equations parametrized by a small parameter ϵ ($\epsilon \downarrow 0$), with dynamics

$$dX^\epsilon(t) = [a(X^\epsilon(t), \xi^\epsilon(t)) + \sum_{i=1}^{N} b_i(X^\epsilon(t))u_i^\epsilon(t)]dt + dM^\epsilon(t) \tag{1.3}$$

and with initial condition $X^\epsilon(0)$. Here $X^\epsilon = (X^\epsilon(t))$ is the controlled state process, $\xi = (\xi^\epsilon(t))$ is the contamination process affecting the drift of X^ϵ, and $M = (M^\epsilon(t))$ is the process representing the noise in the system. Also $u_i^\epsilon = (u_i^\epsilon(t)), i = 1, \ldots, N$, are controls for each of the players. Given a finite horizon $T > 0$, with each strategy vector $u^\epsilon = (u_1^\epsilon, u_2^\epsilon, \ldots, u_N^\varepsilon)$, we associate the payoff to player k by

$$J_k^\epsilon(u_1^\epsilon, u_2^\epsilon, \ldots, u_N^\varepsilon) = E\{\int_0^T [g_k(X^\epsilon(t)) + p_k(u_1^\epsilon(t), \ldots, u_N^\varepsilon(t))]dt + r_k(X^\epsilon(T))\} \tag{1.4}$$

where $g_i(x), p_i(u_1, \ldots, u_N)$, and $r_i(x),\ i = 1, \ldots, N$, are nonnegative functions on the real line referred to as holding cost, control costs, and terminal cost functions, respectively. Nash equilibrium and δ-equilibrium are defined analogously. Our objective is to find an N-tuple of strategies $u^{\varepsilon *} = (u_1^{\varepsilon *}, \ldots, u_N^{\varepsilon *})$ that is in an equilibrium for each $k = 1, \ldots, N$, and the corresponding value function V_k^ϵ, that is,

$$V_k^\epsilon = \max_{u_k^\epsilon \in A_k} J_k^\epsilon(u_k^\epsilon, u_{-k}^{\epsilon *}). \tag{1.5}$$

The value of the game can be defined as the vector $V^\varepsilon = (V_1^\epsilon, \ldots, V_N^\epsilon)$. The sets A_1, $A_2, \ldots, A_N$ will be defined in the next section.

The process $\xi^\epsilon(\cdot)$ is said to be *exogenous* or state independent if for each t and for the set B in the $\sigma-$field $\sigma(\xi^\epsilon(s), s > t)$,

$$P\{B \mid \xi^\epsilon(s), s \leq t\} = P\{B \mid \xi^\epsilon(s), X^\epsilon(s), s \leq t\}.$$

In order for desired convergence to occur, the 'rate of fluctuations' of $\xi^\epsilon(\cdot)$ must increase as $\epsilon \to 0$. We consider the case in which the 'intensity' of the random noise disturbance M^ε becomes very small with ε, while the 'contaminating' process ξ^ε fluctuates with increasing speed. In this work, we assume that the controlled state process X is completely observed.

It is very hard to obtain optimal strategies and values satisfying (1.3) and (1.5), in such generality. It is well known that only few stochastic differential game or stochastic control problems can be solved in closed form. For practical purposes one may just as well be interested in finding a near optimal or an asymptotically optimal strategy vector. Considerable effort has been put into developing approximation techniques for such problems. One such approach used in the stochastic control literature is, in lieu of the original model, a model where the underlying processes are replaced by simpler ones, ([2], [4], [6], [7], [9]). In stochastic game problems such an effort was made in [11] using diffusion approximation techniques.

In the present work, deterministic approximation techniques (i.e., the simpler model is deterministic) to an N-person non-zero sum differential game model will be developed. To this end, we will now introduce a deterministic model, which we will show to be the limiting model corresponding to (1.3) to (1.5) under appropriate conditions introduced in the next section.

These type of results has two major benefits. From the theoretical point of view, one obtains a stability result for the optimal strategy pair of a deterministic system in the sense that this strategy vector is asymptotically optimal for a large class of complicated problems of stochastic differential games. From a practical point of view, when a direct approach would be impossible, these results allow one to compute an asymptotically optimal strategy vector for a variety of stochastic differential game problems under quite general conditions.

In the next section various definitions, assumptions and some standrad preliminaries on the weak convergence are described. In section 3, we will state and prove the main convergence result. Also in this section, we will present results on the convergence of payoffs and near optimality of strategies. In section 4, we will give a L^2-convergence result. Some concluding remarks will be given in section 5.

2. PRELIMINARIES

Let $F^\epsilon = \{\Im_t^\epsilon\}_{t\geq 0}$ denote the minimal σ-algebra over which $\{X^\epsilon(s), \xi^\epsilon(s), M^\epsilon(s), s \leq t\}$ is measurable. For each ε, let $(\Omega, \Im, F^\varepsilon, P)$ be a fixed stochastic basis, where $(\Omega, \Im, P)$ is a complete probability space. Let E_t^ϵ denote the expectation conditioned on $\Im_t^\epsilon$. Let $U_1, U_2, \ldots, U_N$ are compact metric spaces with metric $d_i(\cdot)$. The control process $u_i^\epsilon(t)$ with values in U_i is said to be *admissible strategy* for the i^{th} player if it is $\Im_t^\epsilon$ adapted and $\int_0^T |u_i^\epsilon(s)|\,ds < \infty$ *a.s.* Let $A_i, i = 1, 2, \ldots, N$ denote the set of admissible strategies. Let $A = A_1 \times A_2 \times \cdots \times A_N$. Similarly define the admissible control space in the deterministic game by $\widetilde{A}_i = \{u_i : u_i$ is measurable and $\int_0^T |u_i(t)|\,dt < \infty\}$ and $\widetilde{A} = \widetilde{A}_1 \times \widetilde{A}_2 \times \cdots \times \widetilde{A}_N$. Note that $\widetilde{A} \subset A$.

Now we will give some weak convergence preliminaries. Let $D[0,\infty)$ denote the space of real valued functions which are right continuous and have left-hand limits endowed with the Skorokhod topology. Following [3] and [4], we define the notion of '$p-\lim$' and an operator $\hat{\mathcal{A}}^{\epsilon}$ as follows. Let $\widetilde{M}$ denote the set of real valued functions of (ω,t) that are nonzero only on a bounded $t-$ interval. Let

$$\overline{M}^{\epsilon}=\{f\in\widetilde{M};\sup_t E\,|f(t)|<\infty \text{ and } f(t) \text{ is } \Im_t^{\epsilon} \text{ measurable}\}.$$

For each $\Delta>0$, let $f(\cdot),f^{\Delta}(\cdot)\in\overline{M}^{\epsilon}$. Then we say that $f=p-\lim_{\Delta}f^{\Delta}$ if and only if

$$\sup_{t,\Delta}E\left|f^{\Delta}(t)\right|<\infty,$$

and $\lim_{\Delta\to0}E\left|f(t)-f^{\Delta}(t)\right|=0$, for each t. A function $f(\cdot)$ is said to be in the domain of $\hat{A}^{\epsilon}$, i.e., $f(\cdot)\in D(\hat{A}^{\epsilon})$, and $\hat{A}^{\epsilon}f=g$, if

$$p-\lim_{\Delta\to0}\left(\frac{E_t^{\epsilon}f(t+\Delta)-f(t)}{\Delta}-g(t)\right)=0.$$

If $f(\cdot)\in D(\hat{A}^{\epsilon})$, then

$$f(t)-\int_0^t\hat{A}^{\epsilon}f(u)du \quad \text{is a martingale,}$$

and

$$E_t^{\epsilon}f(t+s)-f(t)=\int_t^{t+s}E_t^{\epsilon}\hat{A}^{\epsilon}f(u)du, \text{ w.p.1.}$$

Let $\Im_s^t$ denote the minimal $\sigma-$algebra over which $\{\xi(\tau);s\le\tau\le t\}$ is measurable. The following result from [4] tells us the type of process that can be used for $\xi(\cdot)$ in (1.3).

Lemma 2.1. *Let $\xi(\cdot)$ be $\phi-$mixing process with mixing rate $\phi(\cdot)$, and let $h(\cdot)$ be a function of ξ which is bounded and measurable on $\Im_t^{\infty}$. Then, there exist $K_i, i=1,2,3$ such that*

$$\left|E(h(t+s)\mid\Im_0^t)-Eh(t+s)\right|\le K_1\phi(s).$$

If $t<u<v$, and $Eh(s)=0$ for all s, then,

$$\left|E(h(u)h(v)\mid\Im_{\tau}^t)-Eh(u)h(v)\right|\le\begin{cases}K_2\phi(v-u), & u<\tau<v\\ K_3\phi(u-t), & t<\tau<u\end{cases}, \tag{2.1}$$

where $\Im_{\tau}^t=\sigma\{\xi(s);\tau\le s\le t\}$. (2.1) is bounded by $\max(K_2,K_3)\phi^{1/2}(v-u)\phi^{1/2}(u-t)$.

The truncation procedure

The following K$-$truncation procedure will simplify the steps necessary to varify the condition

$$\lim_{n\to\infty}\limsup_{\epsilon\to0}P(\sup_{t\le T}|X^{\epsilon}(t)|\ge n)=0, \qquad \text{for each } T<\infty,$$

that is necessary in proving convergence results in this work.

For each $K > 0$, let

$$S_K = \{x : |x| \leq K\} \text{ be the } K- \text{ ball.}$$

Let $X^{\epsilon,K}(0) = X^{\epsilon}(0), X^{\epsilon,K}(t) = X^{\epsilon}(t)$ up until first exit from S_K, and

$$\lim_{n\to\infty} \limsup_{\epsilon\to 0} P(\sup_{t\leq T} \left|X^{\epsilon,K}(t)\right| \geq n) = 0 \quad \text{ for each } T < \infty.$$

$X^{\epsilon,K}(t)$ is said to be the $K-$ truncation of $X^{\epsilon}(\cdot)$Let

$$q^K(x) = \begin{cases} 1 & \text{for } x \in S_K \\ 0 & \text{for } x \in R^d - S_{K+1} \\ \text{smooth} & \text{otherwise.} \end{cases}$$

For any funtion $a(x, \alpha)$, define $a_K(x, \alpha) = a(x, \alpha)q^K(x)$. Let $X^{\epsilon,K}(\cdot)$ denote the solution of (1.3) corresponding to the use of truncated coefficients. Then $X^{\epsilon,K}(\cdot)$is bounded uniformly in t and $\epsilon > 0$.

Let

$$b(x, u) = \sum_{i=1}^{N} b_i(x(t))u_i(t)$$

Define the operator $\widehat{\mathcal{A}}^u$ as follows

$$\widehat{\mathcal{A}}^u f(x) = f_x(x)[\overline{a}(x) + b(x, u)]$$

We will use following result from [4] for the proof of Theorem 3.1.

Theorem 2.2. *Let the martingale problem corresponding to the operator $\widehat{\mathcal{A}}$ have a unique solution $x(\cdot)$ in $D[0, \infty)$ for each initial condition. Suppose that for each $T < \infty$ and $f(\cdot) \in C_1$, a dense set (in the sup norm sense) in $\widehat{C}_0$, there are $f^{\epsilon}(\cdot) \in \mathcal{D}(\widehat{\mathcal{A}}^{\epsilon})$ such that*

$$p - \lim_{\epsilon} [f^{\epsilon}(\cdot) - f^{\epsilon}(X^{\epsilon}(\cdot))] = 0 \tag{2.2}$$

and

$$p - \lim_{\epsilon} [\widehat{\mathcal{A}}^{\epsilon} f^{\epsilon}(\cdot) - \widehat{\mathcal{A}} f(X^{\epsilon}(\cdot))] = 0 \text{ for each } t \leq T, \tag{2.3}$$

Then $X^{\epsilon}(\cdot) \Rightarrow x(\cdot)$, the solution of the martingale problem for the operator $\widehat{\mathcal{A}}$.

3. CONVERGENCE RESULTS

We will now present the main results of this work. We will use following general assumptions.

(B1) $\xi^{\epsilon}(t) = \xi(t/\epsilon)$, where $\xi(\cdot)$ is a stationary process which is strong mixing, right continuous and bounded with mixing rate function $\phi(\cdot)$ satisfying $\int_0^{\infty} \phi(s)ds < \infty$.

(B2) $b_i(\cdot)$, $1 = 1, 2, \ldots, N$ are bounded and Lipshitz continuous. $a(\cdot,\cdot)$ and its gradient $a_x(\cdot,\cdot)$ are continuous in (x, ξ) and satisfy uniform Lipschitz condition with same constant.

(B3) There is a continuously differentiable function $\overline{a}(\cdot)$ such that for each $t < T$ and x

$$\int_t^T [E_t^\epsilon a(x, \xi^\epsilon(s)) - \overline{a}(x)]ds \to 0$$

in probability as $\epsilon \to 0$.

(B4) The cost functions $k(\cdot)$ and $r(\cdot)$ are continuous nonnegative satisfying

$$f_i(x), r_i(x) \leq c_0(1 + |x|^\gamma), c_0, \gamma > 0,\ i = 1, \ldots, N.$$

Also, $p_i(u_1(t), \ldots, u_i(t)) \geq c_2(\sum_{i=1}^N |u_i|^{1+\gamma_2})$, $c_2, \gamma_2 > 0$, and $p_i(u_1(t), \ldots, u_i(t))$ are non-negative convex.

(B5) The process $M^\epsilon = (M^\epsilon(t))_{t\geq 0}$ is a square integrable martingale with paths in the Skorokhod space $D[0, \infty)$ whose predictable quadratic variations $\langle M^\epsilon \rangle (t)$ satisfies

(i) $\langle M^\epsilon \rangle (t) = \epsilon \int_0^t m^\epsilon(s)ds$

with bounded density $m^\epsilon(s)$. That is, there exists a constat c_1 such that

(ii) $m^\epsilon(t) \leq c_1$; $t \leq T$, $P - a.s.$

The jumps $\Delta M^\epsilon(s) \doteq M^\epsilon(s) - \lim_{v\uparrow s} M^\epsilon(s)$ are bounded, i.e., there exists a constant $K > 0$ such that

(iii) $|\Delta M^\epsilon(t)| \leq K$; $t \leq T$, $\epsilon \in (0, 1]$.

(B6) $p - \lim_{\epsilon\to 0} X^\epsilon(0) = x_0$, $x_0 \in R$.

Note: These assumptions are general enough, but need not be most general. For instance, assumption (B2) could be relaxed to say that the system 1.1 has a unique solution.

Theorem 3.1. *Suppose that (B1)–(B6) hold. Let* $X_0^\epsilon \Rightarrow x_0$ *and* $u^\epsilon(\cdot) \equiv (u_1^\epsilon(\cdot), u_2^\epsilon(\cdot), \ldots, u_N^\varepsilon) \Rightarrow u(\cdot) \equiv (u_1(\cdot), u_2(\cdot), \ldots, u_N(\cdot))$, *where* $u(\cdot)$ *is an admissible strategy vector for (1.1). Then* $(X^\epsilon(\cdot), u^\epsilon(\cdot))$ *of (1.3) converges weakly to* $(x(\cdot), u(\cdot))$ *where* $u(\cdot)$ *is measurable (admissible) and satisfies*

$$dx(t) = [\overline{a}(x(t)) + \sum_{i=1}^N b_i(x(t))u_i(t)]dt. \tag{3.1}$$

Also

$$J_k^{\epsilon_n}(u_1^{\epsilon_n}, u_2^{\epsilon_n}, \ldots, u_N^{\varepsilon_n}) \to J_k(u_1, u_2, \ldots, u_N). \tag{3.2}$$

Proof. Define a process $\widetilde{X}^\epsilon(\cdot)$ by

$$\widetilde{X}^\epsilon(t) = \widetilde{X}^\epsilon(0) + \int_0^t [a(\widetilde{X}^\epsilon(s), \xi^\epsilon(s)) + \sum_{i=1}^N b_i(\widetilde{X}^\epsilon(s)) u_i^\epsilon(s)] ds \tag{3.3}$$

Let $Y^\epsilon(s) = \sup_{s \le t} \left| X^\epsilon(s) - \widetilde{X}^\epsilon(s) \right|$. Then by (B2),

$$Y^\epsilon(t) \le K[\int_0^t Y^\epsilon(s) ds + Y^\epsilon(s) \sum_{i=1}^N \int_0^s |u_i^\epsilon(w)|\, dw] + \sup_{s \le T} |M^\epsilon(s)|, \quad t \le T,$$

where K is the Lipschitz constant. By the Gronwall-Bellman inequality we get

$$Y^\epsilon(t) \le K \sup_{s \le T} |M^\epsilon(s)| \exp\{K[T + \sum_{i=1}^N \int_0^T |u_i^\epsilon(w)|\, dw]\}.$$

By (B5) (see [9]), $\sup_{s \le T} |M^\epsilon(s)| \to 0, \epsilon \to 0$, in probability and by (B2) and (B4)

$$\lim_{\delta \to 0} \limsup_{\epsilon \to 0} P(\sup_{t,s \le T: |t-s| \le \delta} \sum_{i=1}^N \int_s^t |u_i(w)|\, dw > \eta) = 0.$$

Consequently $Y^\epsilon(T) \to 0, \epsilon \to 0$, in probability and the theorem remains true if its statements are proved only for $(\widetilde{X}^\epsilon(\cdot), u^\epsilon(\cdot))$. We will prove the weak convergence for the process $(\widetilde{X}^\epsilon(\cdot), u^\epsilon(\cdot))$ using the so called perturbed test function method. For notational convenience we will use $(X^\epsilon(\cdot), u^\epsilon(\cdot))$ for $(\widetilde{X}^\epsilon(\cdot), u^\epsilon(\cdot))$.

Define the perturbation $f_1^\epsilon(t) = f_1^\epsilon(X^\epsilon(t), t)$, where

$$f_1^\epsilon(x, t) = \int_t^T f_x(x)[E_t^\epsilon a(x, \xi^\epsilon(s)) - \overline{a}(x)] ds \tag{3.4}$$

It is important to note that (3.4) averages only the noise, not the state $X^\epsilon(\cdot)$. The state $x = X^\epsilon(t)$ is considered as parameter in (3.4). Now

$$\begin{aligned} f_1^\epsilon(x, t) &= \int_t^T f_x(x)[E_t^\epsilon a(x, \xi^\epsilon(s)) - \overline{a}(x)] ds \\ &= \epsilon \int_{t/\epsilon}^{T/\epsilon} f_x(x)[E_t^\epsilon a(x, \xi(s)) - \overline{a}(x)] ds \end{aligned}$$

In view of Lemma 2.1, (B1) and (B2), for some $L > 0$

$$
\begin{aligned}
\sup_{t\le T}|f_1^\epsilon(t)| &= \epsilon \sup_{t\le T}\left|\int_{t/\epsilon}^{T/\epsilon} f_x(x)[E_t^\epsilon a(x,\xi(s)) - \overline{a}(x)] - [Ea(x,\xi(s)) - \overline{a}(x)]ds\right| \\
&\le L\epsilon \sup_{t\le T}\left(\int_{t/\epsilon}^{T/\epsilon} \phi\left(s - \frac{t}{\epsilon}\right)ds\right) \\
&= O(\epsilon)
\end{aligned}
$$

Hence

$$
\lim_{\epsilon} \sup_{t<T} E\,|f_1^\epsilon(t)| = 0 \tag{3.5}
$$

Write $\widetilde{a}(x,\xi) = f_x(x)(a(x,\xi) - \overline{a}(x))$. We have

$$
\widehat{\mathcal{A}}^\epsilon f_1^\epsilon(t) = -\widetilde{a}(X^\epsilon(t),\xi^\epsilon(t)) + \int_t^T (E_t^\varepsilon \widetilde{a}(X^\epsilon(t),\xi^\epsilon(s)))_x \dot{X}^\epsilon(t)ds + \circ(1)
$$

where $p - \lim_\epsilon \circ(1) = 0$ uniformly in t. Define the perturbed test function $f^\epsilon(t) = f(X^\epsilon(t)) + f_1^\epsilon(t)$. For simplicity write x for $X^\epsilon(t)$. Then

$$
\widehat{\mathcal{A}}^\epsilon f^\epsilon(t) = f_x(x)[a(x,\xi^\epsilon(t)) + \sum_{i=1}^N b_i(x)u_i^\epsilon(t)] \tag{3.6}
$$

$$
\begin{aligned}
&- f_x(x)(a(x,\xi) - \overline{a}(x)) \qquad (3.7)\\
&+ \int_t^T (E_t^\varepsilon \widetilde{a}(x,\xi^\epsilon(s)))_x[a(x,\xi^\epsilon(t)) + \sum_{i=1}^N b_i(x)u_i^\epsilon(t)]ds + \circ(1) \\
&= f_x(x)[\overline{a}(x) + \sum_{i=1}^N b_i(x)u_i^\epsilon(t)] \\
&+ \epsilon \int_{t/\epsilon}^{T/\epsilon} (E_t^\varepsilon \widetilde{a}(x,\xi(s)))_x[a(x,\xi^\epsilon(t)) + \sum_{i=1}^N b_i(x)u_i^\epsilon(t)]ds + \circ(1)
\end{aligned}
$$

Under (B2), the second term in (3.6) is o(1) where o(1) terms goes to zero in p-limit as $\epsilon \to 0$. Then (3.5) and (3.6) imply that

$$
p - \lim_\epsilon [f^\epsilon(\cdot) - f(X^\epsilon(\cdot))] = 0
$$

and

$$
p - \lim_\epsilon [\widehat{\mathcal{A}}^\epsilon f^\epsilon(\cdot) - \widehat{\mathcal{A}}^u f(X^\epsilon(\cdot))] = 0 \text{ for } t \le T.
$$

Hence (3.1) is proved.

By the above methods,

$$\int_0^T [g_k(X^\epsilon(t)) + p_k(u_1^\epsilon(t),\dots,u_N^\epsilon(t))]dt \tag{3.8}$$
$$\Rightarrow \int_0^T [g_k(x(t)) + p_k(u_1(t),\dots,u_N(t))]dt.$$

Also,

$$r_k(X^\epsilon(T)) \Rightarrow r_k(x(T)), \quad k = 1,\dots,N.$$

By (B2), each moment of $X^\epsilon(t)$ is bounded uniformly in ϵ and $t \le T$. By (B2) and (B4), the left hand terms in (3.8) are uniformly (in ϵ) integrable and the convergence in (3.2) follows. □

Remark 1: The condition in the theorem stating that $u^\epsilon(\cdot) \Rightarrow u(\cdot)$ is a reasonable one. This follows, if $p_i(u_1(t),\dots,u_1(t)) \ge c_2(\sum_{i=1}^N |u_i|^{1+\gamma})$, $c_2, \gamma > 0$, and $p_i(u_1(t),\dots,u_1(t))$, $i = 1,\dots,N$ are nonnegative convex, then mimicking the proof of Theorem 5.1 of [9], we can obtain the weak convergence of theorem 3.1. The analytic method used in [9], under their conditions, could also be adapted to prove Theorem 3.1, however, our conditions are general.

Remark 2: In order to obtain (3.5), in lieu of Lemma 2.1, we could use the following differential inequality assumption:

(**B3′**) Set $E_t^\varepsilon[a(x,\xi^\varepsilon(s)) - \overline{a}(x)] = V^\varepsilon(t,x)$, $s \ge t > 0$. Let

$$\frac{\partial V^\varepsilon}{\partial t} + \frac{\partial V^\varepsilon}{\partial x}(a(x,\xi^\varepsilon(t)) + \sum_{i=1}^N b_i u_i^\varepsilon(t) \le g(t,V^\varepsilon).$$

Let $u^\varepsilon(t)$ be the solution of

$$u^{\varepsilon'} = g(t,u^\varepsilon),\ u^\varepsilon(0) = g(0,V^\varepsilon(0,X^\varepsilon(0))$$

The funtion $u^\varepsilon(t) \in L_1(0,\infty)$. Under this assumption, we have $\mid E_t^\varepsilon[a(x,\xi^\varepsilon(s)) - \overline{a}^\varepsilon(t)] \mid \le u^\varepsilon(t)$. This in conjunction with (B1) and (B2) implies (3.5).

Remark 3: Combining the procedure used in the Theorem 3.1 with the relaxed controls set up in [12], we could easily generalize the convergence proof to more general nonlinear control policies in system (1.3).

Following result states that the optimal strategy vector for the limit deterministic system is near optimal and asymptotically optimal for the stochastic system.

Theorem 3.2. *Assume (B1)–(B6). Let $(u_1^*, u_2^*, \dots, u_N^*)$ be the unique optimal strategy vector for (1.1)-(1.2). Then $\{X^\epsilon(\cdot), u_1^*, u_2^*, \dots, u_N^*\} \Rightarrow (x(\cdot), u_1^*, u_2^*, \dots, u_N^*)$ and the limit satisfies (3.1). Also*

$$J_k^\epsilon(u_1^*, u_2^*, \dots, u_N^*) \to J_k(u_1^*, u_2^*, \dots, u_N^*), \quad k = 1,2,\dots,N \tag{3.9}$$

In addition, let $\widehat{u}_k^\epsilon, k = 1, 2, \ldots, N$ be a $\delta-$optimal strategy vector for each player k with $X^\epsilon(\cdot)$ of (1.3). Then

$$\liminf_{\epsilon} \left| \sup_{u_i^\epsilon \in U_i} J_k^\epsilon(u_1^\epsilon, \widehat{u}_2^\epsilon, \ldots, \widehat{u}_N^\epsilon) - J_k^\epsilon(u_1^*, u_2^*, \ldots, u_N^*) \right| \leq \delta, \quad k = 1, 2, \ldots, N \tag{3.10}$$

Proof. By Theorem 3.1, the weak convergence is straight forward. By the assumed uniqueness, the limit satisfies (3.1). Also, by this weak convergence and the fact that $T < \infty$, by the bounded convergence,

$$\lim_{\epsilon} J_k^\epsilon(u_1^*, u_2^*, \ldots, u_N^*) = J_k(u_1^*, u_2^*, \ldots, u_N^*).$$

Now to show (3.10), repeat the procedure with admissible strategies $u_i^\epsilon, i = 1, 2, \ldots, N$. The limit $(u_1, u_2, \ldots, u_N)$ might depend on the chosen subsequence. For any convergent subsequence, we obtain,

$$\lim_{\epsilon=\epsilon_n \to 0} J_k^\epsilon(u_1^\epsilon, u_2^\epsilon, \ldots, u_N^\varepsilon) = J_k(u_1, u_2, \ldots, u_N).$$

Now by the definition of $\delta-$optimality, (3.10) follows. □

Note: If $(u_1^*(t), u_2^*(t), \ldots, u_n^*)$ is the optimal strategies for (1.1) and (1.2), then $\{X^\epsilon(t), u_1^*(t), u_2^*(t), \ldots, u_N^*\}_{0 \leq t \leq T}$ is the process associated with policies $(u_1^\epsilon(t), u_2^\epsilon(t), \ldots, u_N^\varepsilon(t)) \equiv (u_1^*(t), u_2^*(t), \ldots, u_N^*(t))$. Since $(u_1^*(t), u_2^*(t), \ldots, u_N^*(t))$ is deterministic, corresponding $(u_1^\epsilon(t), u_2^\epsilon(t), \ldots, u_N^\varepsilon(t))$ is admissible control for the systems (1.3)-(1.5).

If for each ϵ, and for each player, there is a value for the stochastic game, then the following result shows that they converge to the value of the deterministic game. To prove this we will now introduce a new game through which we will connect the values of stochastic and deterministic games.

Define a continuous map ϕ by

$$\phi : A \to \widetilde{A} \text{ such that if } u = (u_1, u_2, \ldots, u_N) \in \widetilde{A}, \text{ then } \phi(u) = u.$$

Note that example of one such map is $\phi(u) = Eu$ for $u \in A$. Clearly, if $u \in \widetilde{A}$, $\phi(u) = Eu = u$. Define the map $L_k : \widetilde{A} \to \widetilde{A}$ by letting $L_k u_{-k}$ $(u_{-k} \in \widetilde{A}_{-k})$ to be:

$$J_k(L_k u_{-k}, u_{-k}) = \sup_{u_k \in \widetilde{A}_k} J_k(u_k, u_{-k}), u_{-k} \in \widetilde{A}_{-k}$$

We will now make following simplifying assumption.

(B7) $L_k, k = 1, 2, \ldots, N$ are continuous, that is, for any sequence $\{u_{-k,i}\}$ of admissible controls in $\widetilde{A}_{-k}$ such that $u_{-k,i} \to u_{-k} \in \widetilde{A}_{-k}$, $L_k(u_{-k,i}) \to L_k(u_{-k})$, in $L_2(0,T)$ norm.

Remark: The continuity of L_k may be justified by the fact that the controls u_k are state dependent feedback controls.

Following result states that if the value exists for the stochastic game for each ε, then asymptotically (as $\epsilon \to 0$) they coincide with the value of the limit deterministic model.

Theorem 3.3. *Assume (B1)–(B7) and that for each ϵ, value exists for the stochastic game (1.3) to (1.5). Also value for the deterministic game (1.1) to (1.2) exists. Then*$\lim_{\epsilon\to 0} V_k^\epsilon = v_k$, *where*

$$v_k = \max_{u_k \in \widetilde{A}_k} J_k(u_k, u_{-k}^*).$$

Proof. For the proof, we introduce following game which is played as follows. All players except k choose their controls first, say u_{-k}^ϵ, which is known to player k. Then player k chooses the control $L_k(\phi(u_{-k}^\epsilon))$. Define $\widetilde{u}_{-k}^\epsilon$ be optimal strategies for all players except player k with corresponding payoff for player k being $J^\epsilon(L_1\phi(\widetilde{u}_{-k}^\epsilon), \widetilde{u}_{-k}^\epsilon)$. By relative compactness, $\widetilde{u}_{-k}^\epsilon \to \widetilde{u}_{-k}^* \in \widetilde{A}_{-k}$. By continuity and by the definition of ϕ and L_k, we have $L_k(\phi(\widetilde{u}_{-k}^\epsilon)) \to L_k(\phi(\widetilde{u}_{-k}^*)) = L_k(u_{-k}^*)$. Now using (3.9),

$$\lim_{\epsilon\to 0} V_k^\epsilon = v_k.$$

□

Following result is direct from Theorem 3.2 and Theorem 3.3.

Theorem 3.4. *Assume (B1)–(B7). Let $(u_1^*(t), u_2^*(t), \ldots, u_N^*), 0 \le t \le T$ be an optimal deterministic strategy vector for (1.1), (1.2), then $(u_1^*(t), u_2^*(t), \ldots, u_N^*)$ is asymptotically optimal for (1.3), (1.5) in the sense that*

$$\lim_{\epsilon\to 0} |J_k^\epsilon(u_1^*, u_2^*, \ldots, u_N^*) - V_k^\epsilon| = 0. \tag{3.11}$$

4. $\mathbf{L^2}$-CONVERGENCE

Consider a simpler physical system of the form (4.1).Now we will show how to obtain an L^2-convergence in lieu of the weak convergence of X^ε to x. Rewrite the systems (3.3) and (1.1) respectively in the integral form:

$$X^\varepsilon(t) = X^\varepsilon(0) + \int_0^t [a(X^\varepsilon(s), \xi^\varepsilon(s)) + \sum_{i=1}^N b_i(X^\varepsilon(s))u_i^\varepsilon(s)]ds \tag{4.1}$$

and

$$x(t) = x_0 + \int_0^t [\overline{a}(x(s)) + \sum_{i=1}^N b_i(x(s))u_i(s)]ds \tag{4.2}$$

Note that using Theorem 3, it is enough to consider the system (3.3) instead of the system (1.3). The conditions on $a, b_i, i = 1, 2, \ldots, N$ are same as (B3).

Define $\|X^\varepsilon(t)\| = \sup_{0\le t\le T} \{E(X^\varepsilon(t))^2\}^{1/2}$. Assume the following.

(B8) **(i)** $\|X^\varepsilon(0) - x_0\| \to 0$ as $\varepsilon \to 0$.

(ii) $\|u_i^\varepsilon - u_i\| \to 0$ as $\varepsilon \to 0$ $(i = 1, 2)$.

(iii) $E(a(x, \xi^\varepsilon) - \overline{a}(x))^2 \le \phi(\varepsilon, x)$, where $\phi(\varepsilon, x) \to 0$ as $\varepsilon \to 0$.

Theorem 4.1. *Assume (B1), (B2), (B5), and (B8). Then*

$$\|X^\varepsilon(t) - x(t)\| \to 0 \text{ as } \varepsilon \to 0, \tag{4.3}$$

where $X^\varepsilon(\cdot)$ is the solution of (4.1) and $x(\cdot)$ is the solution of (4.2).

Proof:

$$E\,|X^\varepsilon - x|^2 \le N\{E(X^\varepsilon(0) - x_0)^2 + \int_0^t E\,|a(X^\varepsilon, \xi^\varepsilon) - \overline{a}(x)|^2\, ds$$

$$+ \sum_{i=1}^N \int_0^t E\,|b_i(X^\varepsilon)u_i^\varepsilon - b_i(x)u_i|^2\, ds \tag{4.4}$$

$$\le N\{E(X^\varepsilon(0) - x_0)^2 + \int_0^t E\,|a(X^\varepsilon, \xi^\varepsilon) - \overline{a}(x, \xi^\varepsilon)|^2\, ds \tag{4.5}$$

$$+ \int_0^t E\,|a(x, \xi^\varepsilon) - \overline{a}(x)|^2\, ds \tag{4.6}$$

$$+ \sum_{i=1}^N \int_0^t E\,|b_i(X^\varepsilon) - b_i(x)|^2\, |u_i|^2\, ds \tag{4.7}$$

$$+ \sum_{i=1}^N \int_0^t E\,|b_i(X^\varepsilon)|^2\, |u_i^\varepsilon - u_i|^2\, ds\} \tag{4.8}$$

Note that b_i's are bounded and Lipschtzian, and so is a. Hence

$$E\,|X^\varepsilon - x_0|^2 \le NK\{E\,|X^\varepsilon(0) - x_0|^2 + \int_0^t E\,|X^\varepsilon - x|^2\, ds \tag{4.9}$$

$$+ \int_0^T E(a(x, \xi^\varepsilon) - \overline{a}(x))^2 ds + \int_0^t E\,|X^\varepsilon - x|^2\, ds \tag{4.10}$$

$$+ \sum_{i=1}^N \int_0^t E(u_i^\varepsilon - u_i)^2 ds\} \tag{4.11}$$

Using the assumptions (B8) in (4.9) we get (for some K)

$$E\left|X^{\varepsilon}-x\right|^{2} \leq KE[\left|X^{\varepsilon}(0)-x_{0}\right|^{2}+\int_{0}^{T}\phi(\varepsilon,x)ds$$

$$+\sum_{i=1}^{N}\int_{0}^{T}E(u_{1}^{\varepsilon}-u_{1})^{2}ds \tag{4.12}$$

$$+\int_{0}^{t}E(X(s)^{\varepsilon}-x(s))^{2}ds] \tag{4.13}$$

Using Grownwall-Bellman inequality

$$E\left|X^{\varepsilon}(t)-x(t)\right|^{2} \leq KE[\left|X^{\varepsilon}(0)-x_{0}\right|^{2}+\int_{0}^{T}\phi(\varepsilon,x)ds \tag{4.14}$$

$$+\sum_{i=1}^{N}\int_{0}^{T}E(u_{i}^{\varepsilon}-u_{i})^{2}ds]e^{KT}. \tag{4.15}$$

Equation (4.14) implies that

$$\sup_{0\leq t\leq T}E\left|X^{\varepsilon}(t)-x(t)\right|^{2}\to 0 \text{ as } \varepsilon\to 0. \tag{4.16}$$

Hence $\|X^{\varepsilon}-x\|\to 0$. □

Once we have L^2-convergence, we can obtain pathwise convergence using following arguments. Suppose there is no pathwise convergence of $X^{\varepsilon}(t,\omega)$ for $\omega\in A$, with $P(A)=\lambda>0$. Then there is a sequence $\{\varepsilon_n\}\to 0$, such that for each ε_n, there is a $t_n\in(0,T]$ such that

$$\left|X^{\varepsilon_n}(t_n,\omega)-x(t_n,\omega)\right|>\delta>0,\quad \omega\in A.$$

Hence

$$0 < \varepsilon^{2}\delta<\int_{A}\left|X^{\varepsilon_n}-x\right|^{2}dP\leq E\left|X^{\varepsilon_n}-x\right|^{2}$$

$$\leq \sup_{0\leq t\leq T}E\left|X^{\varepsilon_n}-x\right|^{2} \tag{4.17}$$

Since $\sup_{0\leq t\leq T}E\left|X^{\varepsilon_n}-x\right|^{2}\to 0$ as $\varepsilon_n\to 0$, this leads to a contradiction. The convergence of payoffs and near optimality for this setup follows as in earlier sections.

5. CONCLUSION

From a practical point of view, when a direct approach would be impossible, the results of this work allow us to compute an asymptotically optimal strategies for a variety of problems under quite general conditions. These results could easily be extended, by combining the ideas of [11] and the methods used in this paper, to other commonly used payoff criteria such as total discounted payoff, ergodic payoff and payoff up to first exit time. In [5], such approximation techniques are utilized in developing numerical methods for stochastic control problems. Due to the ode limit here, we will be able to develop computational methods for the randomly perturbed differential game problems in much simpler way. The details of numerical methods for the system (1.1) will be discussed elsewhere.

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