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ON THE OSCILLATION OF SECOND-ORDER PERTURBED NONLINEAR DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we will use the Riccati transformation technique to establish some new oscillation criteria for the second-order perturbed nonlinear difference equation of the general form

$$\Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}) + F(n, x_n) = G(n, x_n, \Delta x_{n-1}), \quad n \geq 1.$$

Our results complement and improve some well known results in the literature. Some examples are given to illustrate our main results.

AMS (MOS) Subject Classification. 39A10.

1. INTRODUCTION

In recent years, the oscillation theory and asymptotic behavior of difference equations and their applications have been and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have been written, see for example the monographs [1, 2] and the papers [3-5, 7-31] and the references therein.

Recently, Saker [17] considered the second-order perturbed nonlinear difference equation

$$\Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}) + F(n, x_n) = G(n, x_n, \Delta x_{n-1}), \quad n \geq 1, \tag{1.1}$$

where Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$ for any sequence $\{x_n\}$ of real numbers, $\gamma > 0$ is a quotient of odd positive integers, $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $a_n > 0$ and

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n}\right)^{1/\gamma} = \infty, \tag{1.2}$$

or

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n}\right)^{1/\gamma} < \infty, \tag{1.3}$$

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for some positive integer $n_0 \geq 1$. Using the Riccati transformation technique, the author presented some new oscillation criteria for Eq.(1.1) under the condition (1.2) or (1.3) which improved many known criteria discussed in [1, 3, 5, 8, 14, 15, 26, 27, 28, 30, 31]. Like [17], throughout this paper we will assume that there exist two real sequences $\{q_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ such that $q_n - p_n \geq 0$, and

$$(1.4)\qquad \begin{cases} \liminf_{n\to\infty} \sum_{i=n_0}^{n} (q_i - p_i) > 0 \quad \text{for all large } n_0, \\ \dfrac{F(n,u)}{u^{\beta}} \geq q_n, \quad \dfrac{G(n,u,v)}{u^{\beta}} \leq p_n \quad \text{for } u \neq 0, \end{cases}$$

where β is a quotient of odd positive integers.

It is easy to see that the solution of Eq.(1.1) satisfying initial values $x_0 = a$ and $x_1 = b$ is unique and is defined for $n \geq 1$. By a solution of (1.1), we mean a nontrivial sequence $\{x_n\}_{n=1}^{\infty}$ satisfying Eq.(1) for $n \geq 1$. A solution x_n of (1.1) is said to be oscillatory if for every $n_1 \geq n_0 \geq 1$, there exists $n \geq n_1$ such that $x_n x_{n+1} \leq 0$; otherwise, it is nonoscillatory. If every solution of Eq.(1.1) is oscillatory, we say Eq.(1.1) is oscillatory.

In the recent paper of Saker [17], we note that the main results, Theorem 2.1 and Theorem 2.5, hold only for the case when $\gamma \geq \beta$ and for the case when $\gamma > 1$, respectively. It is natural to ask whether Eq.(1.1) is oscillatory for the cases when $\gamma < \beta$ and $0 < \gamma < 1$. In this paper, we will further study the oscillatory behavior of Eq.(1.1) for $\gamma < \beta$ and $0 < \gamma < 1$, which answers this question. In Section 2, some new oscillation criteria for Eq.(1.1) are established, which complements the main results established in [17]. In Section 3, two examples are given to illustrate our main results.

2. MAIN RESULTS

In this section, we will use the Riccati transformation technique to establish some sufficient conditions for oscillation of (1.1) when (1.2) holds, and when (1.3) holds we establish some sufficient conditions which insure that every solution $\{x_n\}$ of (1.1) oscillates or converges to zero. The section will be divided into four parts.

(I) The case when (1.2) holds and $\gamma < \beta$.

Theorem 2.1. *Assume that (1.2) and (1.4) hold. Furthermore, assume that there exists a positive sequence $\{\rho_n\}_{n=1}^{\infty}$ such that for every positive constant M and for some $n_0 \geq 1$;*

$$(2.1)\qquad \limsup_{n\to\infty} \sum_{l=n_0}^{n} \left[\rho_l (q_l - p_l) - \frac{M a_l (\Delta_+ \rho_l)^2}{\rho_l} \right] = \infty \quad \textit{when } \gamma \geq 1,$$

and

$$\limsup_{n\to\infty} \sum_{l=n_0}^{n} \left[\rho_l(q_l - p_l) - \frac{Ma_l(\Delta_+\rho_l)^{\gamma+1}}{\rho_l^{\gamma}}\right] = \infty \quad \text{when } \beta \geq 1,\ 0 < \gamma < 1, \tag{2.2}$$

where $\Delta_+\rho_n = \max\{0, \Delta\rho_n\}$. *Then every solution of Eq.(1.1) oscillates.*

Proof. Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1.1), say $x_n > 0$ for all $n \geq n_0$. Similar to the proof of Theorem 2.1 in [17], we have that there exists $n_1 \geq n_0$ such that

$$x_n > 0,\ \Delta x_{n-1} > 0 \text{ and } \Delta(a_{n-1}(\Delta x_{n-1})^{\gamma}) \leq 0 \text{ for } n \geq n_1. \tag{2.3}$$

Firstly, we consider the case when $\gamma \geq 1$. Define the sequence $\{w_n\}$ by

$$w_n = \rho_n \frac{a_{n-1}(\Delta x_{n-1})^{\gamma}}{x_n^{\gamma}}; \tag{2.4}$$

then, $w_n > 0$ for $n \geq n_1$ and

$$\begin{aligned} \Delta w_n = &\ a_n(\Delta x_n)^{\gamma}\Delta\left(\frac{\rho_n}{x_n^{\gamma}}\right) + \frac{\rho_n\Delta(a_{n-1}(\Delta x_{n-1})^{\gamma})}{x_n^{\gamma}} \\ \leq &\ -\rho_n(q_n - p_n)x_n^{\beta-\gamma} + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\rho_n a_n(\Delta x_n)^{\gamma}\Delta(x_n^{\gamma})}{x_n^{\gamma}x_{n+1}^{\gamma}}. \end{aligned} \tag{2.5}$$

From (2.3), we can assume, without loss of generality, that there exists a constant $c > 0$ such that $x_n^{\beta-\gamma} \geq c$ for $n \geq n_1$. Using this, (2.3), (2.5), the assumption $\beta > \gamma$, and the inequality (see [6])

$$x^{\gamma} - y^{\gamma} \geq 2^{1-\gamma}(x-y)^{\gamma} \text{ for all } x \geq y > 0 \text{ and } \gamma \geq 1,$$

we have

$$\begin{aligned} \Delta w_n \leq &\ -c\rho_n(q_n - p_n) + \frac{\Delta_+\rho_n}{\rho_{n+1}}w_{n+1} - 2^{1-\gamma}\rho_n a_n\frac{(\Delta x_n)^{2\gamma}}{x_{n+1}^{2\gamma}} \\ = &\ -c\rho_n(q_n - p_n) + \frac{\Delta_+\rho_n}{\rho_{n+1}}w_{n+1} - \frac{2^{1-\gamma}\rho_n}{a_n\rho_{n+1}^2}w_{n+1}^2 \\ = &\ -c\rho_n(q_n - p_n) + \frac{a_n(\Delta_+\rho_n)^2}{2^{3-\gamma}\rho_n} - \left[\frac{\sqrt{2^{1-\gamma}\rho_n}}{\rho_{n+1}\sqrt{a_n}}w_{n+1} - \frac{\sqrt{a_n}\Delta_+\rho_n}{2\sqrt{2^{1-\gamma}\rho_n}}\right]^2 \\ \leq &\ -\left[c\rho_n(q_n - p_n) - \frac{a_n(\Delta_+\rho_n)^2}{2^{3-\gamma}\rho_n}\right] \end{aligned} \tag{2.6}$$

for $n \geq n_1$. Summing (2.6) from n_1 to n, we obtain

$$-w_{n_1} \leq w_{n+1} - w_{n_1} \leq -\sum_{l=n_1}^{n}\left[c\rho_l(q_l - p_l) - \frac{a_l(\Delta_+\rho_l)^2}{2^{3-\gamma}\rho_l}\right],$$

which yields

$$\sum_{l=n_1}^{n}\left[\rho_l(q_l - p_l) - \frac{Ma_l(\Delta_+\rho_l)^2}{\rho_l}\right] \leq \frac{w_{n_1}}{c} < \infty \quad \text{for all } n \geq n_1,$$

where $M = 1/(c2^{3-\gamma})$, which contradicts (2.1).

Second, let us consider the case where $0 < \gamma < 1$ and $\beta \geq 1$. From (2.3), we see that there exist two constants $c_1, c_2 > 0$ such that $x_n^{\beta-1} \geq c_1$ and $x_n^{(1/\gamma)-1} \geq c_2$ for $n \geq n_1$. Define the sequence $\{w_n\}$ by

$$w_n = \rho_n \frac{a_{n-1}(\Delta x_{n-1})^\gamma}{x_n}; \tag{2.7}$$

then $w_n > 0$ for $n \geq n_1$ and

(2.8)

$$\begin{aligned} \Delta w_n \leq& \ -\rho_n(q_n - p_n)x_n^{\beta-1} + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n a_n (\Delta x_n)^{\gamma+1}}{x_{n+1}^2} \\ =& \ -\rho_n(q_n - p_n)x_n^{\beta-1} + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \left[\frac{\rho_{n+1} a_n (\Delta x_n)^\gamma}{x_{n+1}}\right]^{(\gamma+1)/\gamma} \cdot \frac{\rho_n x_{n+1}^{(1/\gamma)-1}}{\rho_{n+1}^{(\gamma+1)/\gamma} a_n^{1/\gamma}} \\ \leq& \ -c_1 \rho_n (q_n - p_n) + \frac{\Delta_+ \rho_n}{\rho_{n+1}} w_{n+1} - \frac{c_2 \rho_n}{\rho_{n+1}^\lambda a_n^{\lambda-1}} w_{n+1}^\lambda, \end{aligned}$$

where $\lambda = (\gamma + 1)/\gamma$. Setting

$$A = \left(\frac{c_2 \rho_n}{\rho_{n+1}^\lambda a_n^{\lambda-1}}\right)^{1/\lambda} w_{n+1} \quad \text{and} \quad B = \left[\frac{1}{\lambda} \frac{\Delta_+ \rho_n}{\rho_{n+1}} \left(\frac{c_2 \rho_n}{\rho_{n+1}^\lambda a_n^{\lambda-1}}\right)^{-1/\lambda}\right]^{1/(\lambda-1)},$$

and using the inequality (see [6])

$$A^\lambda - \lambda A B^{\lambda-1} + (\lambda - 1)B^\lambda \geq 0, \quad A, B \geq 0, \ \lambda > 1,$$

we have

$$\begin{aligned} \frac{\Delta_+ \rho_n}{\rho_{n+1}} w_{n+1} - \frac{c_2 \rho_n}{\rho_{n+1}^\lambda a_n^{\lambda-1}} w_{n+1}^\lambda \leq& \ (\lambda - 1)\lambda^{\frac{\lambda}{1-\lambda}} \left(\frac{\Delta_+ \rho_n}{\rho_{n+1}}\right)^{\frac{\lambda}{\lambda-1}} \left(\frac{c_2 \rho_n}{\rho_{n+1}^\lambda a_n^{\lambda-1}}\right)^{\frac{1}{1-\lambda}} \\ =& \ \frac{C a_n (\Delta_+ \rho_n)^{\lambda/(\lambda-1)}}{\rho_n^{1/(\lambda-1)}} = \frac{C a_n (\Delta_+ \rho_n)^{\gamma+1}}{\rho_n^\gamma}, \end{aligned} \tag{2.9}$$

where $C = (\lambda - 1)\lambda^{\lambda/(1-\lambda)}(c_2)^{1/(1-\lambda)} = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}(c_2)^\gamma}$. Thus, from (2.8) and (2.9) we obtain

$$\Delta w_n \leq -c_1 \rho_n (q_n - p_n) + \frac{C a_n (\Delta_+ \rho_n)^{\gamma+1}}{\rho_n^\gamma}.$$

Summing the above inequality, we have

$$\sum_{l=n_1}^{n} \left[\rho_l (q_l - p_l) - \frac{M a_l (\Delta_+ \rho_l)^{\gamma+1}}{\rho_l^\gamma}\right] \leq \frac{w_{n_1}}{c_1} \quad \text{for all } n \geq n_1,$$

where $M = C/c_1$, which contradicts the assumption (2.2). This completes the proof of Theorem 2.1. □

Theorem 2.2. *Assume that (1.2) and (1.4) hold. Let $\{\rho_n\}_{n=1}^\infty$ be a positive sequence. Furthermore, we assume that there exists a double sequence $\{H_{m,n} : m \geq n \geq 0\}$ such that (i) $H_{m,m} = 0$ for $m \geq 0$, (ii) $H_{m,n} > 0$ for $m > n \geq 0$, (iii) $\Delta_2 H_{m,n} =$*

$H_{m,n+1} - H_{m,n} \le 0$ for $m \ge n \ge 0$ and $\Delta_2 H_{m,n} = h_{m,n}\sqrt{H_{m,n}}$. If for every positive number M and for some $n_0 \ge 1$,

$$\lim_{m\to\infty}\sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} H_{m,n}\left[\rho_n(q_n - p_n) - \frac{M a_n(\Delta_+\rho_n\sqrt{H_{m,n}} + \rho_{n+1}h_{m,n})^2}{\rho_n}\right] = \infty, \tag{2.10}$$

for the case when $\gamma > 1$, and

$$\lim_{m\to\infty}\sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} H_{m,n}\left[\rho_n(q_n - p_n) - \frac{M a_n(\Delta_+\rho_n)^{\gamma+1}}{\rho_n^{\gamma}}\right] = \infty, \tag{2.11}$$

for the case when $0 < \gamma < 1$ and $\beta \ge 1$, where $\Delta_+\rho_n$ is the same as in Theorem 2.1. Then every solution of Eq.(1.1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.1, we assume that Eq.(1.1) has a nonoscillatory solution, say $x_n > 0$ for all $n \ge n_0$. Then, we have (2.3) holds. For the case $\gamma > 1$, again define w_n by (2.4); then, $w_n > 0$ for $n \ge n_1$. Similar to the proof of Theorem 2.1, we have that (2.6) holds, i.e., there exists a constant $c > 0$ such that

$$c\rho_n(q_n - p_n) \le -\Delta w_n + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{2^{1-\gamma}\rho_n}{a_n\rho_{n+1}^2}w_{n+1}^2.$$

Therefore, we have

$$\begin{aligned} c\sum_{n=n_1}^{m-1} H_{m,n}\rho_n(q_n - p_n) \le \;& -\sum_{n=n_1}^{m-1} H_{m,n}\Delta w_n + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} \\ & -\sum_{n=n_1}^{m-1} H_{m,n}\frac{2^{1-\gamma}\rho_n}{a_n\rho_{n+1}^2}w_{n+1}^2, \end{aligned}$$

which, after summing by parts, yields

$$\begin{aligned} c\sum_{n=n_1}^{m-1} H_{m,n}\rho_n(q_n - p_n) \le \;& H_{m,n_1}w_{n_1} + \sum_{n=n_1}^{m-1} h_{m,n}\sqrt{H_{m,n}}w_{n+1} \\ & + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \sum_{n=n_1}^{m-1} H_{m,n}\frac{2^{1-\gamma}\rho_n}{a_n\rho_{n+1}^2}w_{n+1}^2 \\ \le \;& H_{m,n_1}w_{n_1} + \sum_{n=n_1}^{m-1}\left(H_{m,n}\frac{\Delta_+\rho_n}{\rho_{n+1}} + h_{m,n}\sqrt{H_{m,n}}\right)w_{n+1} \\ & - \sum_{n=n_1}^{m-1} H_{m,n}\frac{2^{1-\gamma}\rho_n}{a_n\rho_{n+1}^2}w_{n+1}^2. \end{aligned} \tag{2.12}$$

Completing the square in (2.12), we obtain that

$$\frac{1}{H_{m,n_1}} \sum_{n=n_1}^{m-1} H_{m,n}\left[\rho_n(q_n - p_n) - \frac{M a_n(\Delta_+\rho_n\sqrt{H_{m,n}} + \rho_{n+1}h_{m,n})^2}{\rho_n}\right] \le \frac{w_{n_1}}{c},$$

for all $m > n_1$, where $M = 1/(c2^{3-\gamma})$, which contradicts the assumption (2.10).

For the case when $0 < \gamma < 1$ and $\beta \geq 1$, define w_n by (2.7). Similar to the proof of Theorem 2.1, we have that there exist two positive constants c_1 and c_2 such that

$$c_1\rho_n(q_n - p_n) \leq -\Delta w_n + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{c_2\rho_n}{\rho_{n+1}^{\lambda}a_n^{\lambda-1}}w_{n+1}^{\lambda}.$$

Noting that $\Delta_2 H_{m,m} \leq 0$, we have

$$c_1\sum_{n=n_1}^{m-1} H_{m,n}\rho_n(q_n - p_n) \leq H_{m,n_1}w_{n_1} + \sum_{n=n_1}^{m-1} H_{m,n}\left[\frac{\Delta_+\rho_n}{\rho_{n+1}}w_{n+1} - \frac{c_2\rho_n}{\rho_{n+1}^{\lambda}a_n^{\lambda-1}}w_{n+1}^{\lambda}\right].$$

Let

$$A = \left(\frac{c_2\rho_n}{\rho_{n+1}^{\lambda}a_n^{\lambda-1}}\right)^{1/\lambda} w_{n+1} \quad \text{and} \quad B = \left[\frac{1}{\lambda}\frac{\Delta_+\rho_n}{\rho_{n+1}}\left(\frac{c_2\rho_n}{\rho_{n+1}^{\lambda}a_n^{\lambda-1}}\right)^{-1/\lambda}\right]^{1/(\lambda-1)}.$$

Similar to the proof of (2.9), we have

$$c_1\sum_{n=n_1}^{m-1} H_{m,n}\rho_n(q_n - p_n) \leq H_{m,n_1}w_{n_1} + \sum_{n=n_1}^{m-1} H_{m,n}\frac{Ma_n(\Delta_+\rho_n)^{\gamma+1}}{\rho_n^{\gamma}},$$

where $\lambda = (\gamma+1)/\gamma$ and $C = (\lambda-1)\lambda^{\lambda/(1-\lambda)}(c_2)^{1/(1-\lambda)} = \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}(c_2)^{\gamma}}$. It follows that

$$\frac{1}{H_{m,n_1}}\sum_{n=n_1}^{m-1} H_{m,n}\left[\rho_n(q_n - p_n) - \frac{Ma_n(\Delta_+\rho_n)^{\gamma+1}}{\rho_n^{\gamma}}\right] \leq \frac{w_{n_1}}{c_1},$$

for all $m > n_1$, where $M = C/c_1$, which contradicts the assumption (2.11). This completes the proof of Theorem 2.2. □

(II) The case when (1.3) holds and $\gamma < \beta$.

Theorem 2.3. *Assume that (1.3) and (1.4) hold. Furthermore, we assume that there exist positive sequences $\{\rho_n\}$ and $\{\delta_n\}$ such that (2.1) and (2.2) hold for every positive constant M, and*

$$\text{(2.13)} \qquad \Delta\delta_n \geq 0,\ \sum_{n=n_0}^{\infty}\delta_n(q_n - p_n) = \infty,\ \sum_{n=n_0}^{\infty}\left(\frac{1}{a_n\delta_n}\sum_{i=n_0}^{n+1}\delta_i(q_i - p_i)\right)^{1/\gamma} = \infty,$$

for some $n_0 \geq 1$. Then every solution $\{x_n\}$ of (1.1) oscillates or $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose, to the contrary, that $\{x_n\}$ is an eventually positive solution of (1) such that $x_n > 0$ for all $n \geq n_0$. By (1.4) and Theorem 2.2 (a) in [26] (case 2), we see that $\{\Delta x_n\}$ does not oscillate and there exist two possible cases for the sign of Δx_n.

Case (a): Suppose that $\{\Delta x_n\}$ is eventually positive. We are then back to the case where (2.3) holds. Thus, the proof of Theorem 2.1 goes through, and we may conclude that $\{x_n\}$ cannot be eventually positive, which is not possible.

Case (b): Suppose that $\Delta x_n < 0$ for $n \geq n_1 \geq n_0$. It follows that $\lim_{t\to\infty} x_n = b \geq 0$. Now, we claim that $b = 0$. Otherwise, there exists a constant $M > 0$ such that

$x_n^\beta \geq M$. Therefore, from (1.1) and (1.4) we have $\Delta(a_{n-1}(\Delta x_{n-1})^\gamma) \leq -M(q_n - p_n)$. Define the sequence $u_n = \delta_{n-1}(a_{n-1}(\Delta x_{n-1})^\gamma)$ for $n \geq n_1$. Then, we have

$$\Delta u_n \leq -M\delta_n(q_n - p_n) + \Delta\delta_n(a_{n-1}(\Delta x_{n-1})^\gamma) \leq -M\delta_n(q_n - p_n).$$

Summing it from n_1 to n, we obtain $u_{n+1} \leq u_{n_1} - M\sum_{l=n_1}^{n+1}\delta_l(q_l - p_l)$. In view of (2.13), it is possible to choose an integer n_2 sufficiently large such that for all $n \geq n_2$,

$$u_{n+1} \leq -\frac{M}{2}\sum_{l=n_1}^{n+1}\delta_l(q_l - p_l).$$

Summing the above inequality from n_2 to n, we obtain

$$x_{n+1} \leq x_{n_2} - \left(\frac{M}{2}\right)^{1/\gamma}\sum_{s=n_2}^{n}\left(\frac{1}{a_s\delta_s}\sum_{l=n_1}^{s+1}\delta_l(q_l - p_l)\right)^{1/\gamma}.$$

This implies that x_n is eventually negative, which is a contradiction, and completes the proof of Theorem 2.3. □

Similar to the proof of Theorem 2.3, we have the following theorem.

Theorem 2.4. *Assume that (1.3) and (1.4) hold. Let $\{\delta_n\}$ be a positive sequence such that (2.13) holds. Furthermore, we assume that there exist a positive sequence $\{\rho_n\}$ and a double sequence $\{H_{m,n}\}$ as defined in Theorem 2.2 such that (2.10) and (2.11) hold for every positive constant M. Then every solution $\{x_n\}$ of (1.1) oscillates or* $\lim_{n\to\infty} x_n = 0$.

(III) The case when (1.2) holds and $0 < \beta, \gamma < 1$.

Theorem 2.5. *Assume that (1.2) and (1.4) hold. Furthermore, assume that there exist a positive sequence $\{\rho_n\}$ and a double sequence $\{H_{m,n}\}$ such that for every positive constant M and for some $n_0 \geq 1$,*

$$\limsup_{n\to\infty}\sum_{l=n_0}^{n}\left[\rho_l(q_l - p_l)A_l^{\beta-1} - \frac{Ma_l(\Delta_+\rho_l)^{\gamma+1}}{\rho_l^\gamma}\right] = \infty, \tag{2.14}$$

or

$$\limsup_{m\to\infty}\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1}H_{m,n}\left[\rho_n(q_n - p_n)A_n^{\beta-1} - \frac{Ma_n(\Delta_+\rho_n)^{\gamma+1}}{\rho_n^\gamma}\right] = \infty, \tag{2.15}$$

where $A_n = \sum_{s=0}^{n-1}\left(\frac{1}{a_s}\right)^{1/\gamma}$ *and $\Delta_+\rho_n$ is defined as in Theorem 2.1. Then every solution of (1.1) oscillates.*

Proof. Assume that Eq. (1.1) has a nonoscillatory solution, say $x_n > 0$ for all $n \geq n_0$. Thus, we have that (2.3) holds and

$$a_n(\Delta x_n)^\gamma \leq a_{n_1}(\Delta x_{n_1})^\gamma = m,$$

which means

$$\Delta x_n \le \left(\frac{m}{a_n}\right)^{1/\gamma}.$$

Summing from n_1 to $n-1$, we get

$$x_n \le x_{n_1} + m^{1/\gamma} \sum_{s=n_1}^{n-1} \left(\frac{1}{a_s}\right)^{1/\gamma}.$$

Noting that (1.2) holds, we have that there exists $b_1 > 0$ such that $x_n \le b_1 A_n$ where $A_n = \sum_{s=0}^{n-1} \left(\frac{1}{a_s}\right)^{1/\gamma}$. On the other hand, from (2.3), there exists $b_2 > 0$ such that $x_n^{(1/\gamma)-1} \ge b_2$ for $n \ge n_1$. Define again w_n by (2.7); then we obtain

$$\begin{aligned} \Delta w_n \le & -\rho_n(q_n - p_n)x_n^{\beta-1} + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \left[\frac{\rho_{n+1} a_n (\Delta x_n)^\gamma}{x_{n+1}}\right]^{(\gamma+1)/\gamma} \cdot \frac{\rho_n x_n^{(1/\gamma)-1}}{\rho_{n+1}^{(\gamma+1)/\gamma} a_n^{1/\gamma}} \\ \le & -\rho_n(q_n - p_n)(b_1 A_n)^{\beta-1} + \frac{\Delta_+\rho_n}{\rho_{n+1}} w_{n+1} - \frac{b_2\rho_n}{\rho_{n+1}^{\lambda} a_n^{\lambda-1}} w_{n+1}^{\lambda}, \end{aligned}$$

where $\lambda = (\gamma+1)/\gamma$. The remainder of the proof is similar to that of Theorems 2.1 and 2.2, and hence is omitted. This completes the proof of Theorem 2.5. □

(V) The case when (1.3) holds and $0 < \beta, \gamma < 1$.

Theorem 2.6. *Assume that (1.3) and (1.4) hold, and let $\{\delta_n\}$ be a positive sequence such that (2.13) holds. Furthermore, we assume that there exist a positive sequence $\{\rho_n\}$ and a double sequence $\{H_{m,n}\}$ as defined in Theorem 2.1 such that (2.14) or (2.15) holds for every positive constant M. Then every solution $\{x_n\}$ of (1.1) oscillates or $\lim_{n\to\infty} x_n = 0$.*

Proof. The proof of Theorem 2.6 is similar to that of Theorem 2.3, and hence is omitted. □

Remark 2.7. By choosing $\{\rho_n\}$ and $\{H_{m,n}\}$ in appropriate manners, we can derive many oscillation criteria for Eq.(1.1) from Theorems 2.1-2.6, and due to the limited space, we left the details to the reader.

3. EXAMPLES

In this section, we will present two examples to illustrate our results. However, none of the results in [17] can be applied to these two examples.

Example 3.1. Consider the following perturbed difference equation

$$\Delta(n^a(\Delta x_{n-1})^\gamma) + n^b(1 + x_n^2)x_n^\beta = n^c \frac{(\Delta x_{n-1})^2 x_n^\beta}{1 + (\Delta x_{n-1})^2}, \quad n \ge 1, \tag{3.1}$$

where a, b, c, γ, β are constants, $b > c$, $\gamma > 0$, $\beta > 0$ are quotients of odd positive integers, and $a \le \gamma$ which guarantees (1.2) holds. It is easy to see that (1.4) holds with $q_n = n^b$ and $p_n = n^c$. Now let us consider Eq.(3.1) in the following cases where we choose $\rho_n = n$:

Case 1: $\gamma < \beta$ and $\gamma \geq 1$. Then we have

$$\lim_{n\to\infty} \sup \sum_{l=1}^{n} \left[\rho_l(q_l - p_l) - \frac{Ma_l(\Delta_+\rho_l)^2}{\rho_l}\right] = \lim_{n\to\infty} \sup \sum_{l=1}^{n} (l^{b+1} - l^{c+1} - Ml^{a-1}).$$

Noting that $b > c$, we have that (2.1) holds when $b + 1 > -1$ and $b + 1 > a - 1$. By Theorem 2.1 we have Eq.(3.1) is oscillatory when $b > \max\{a - 2, -2\}$.

Case 2: $\gamma < \beta$, $\beta \geq 1$ and $0 < \gamma < 1$. Then we have

$$\lim_{n\to\infty} \sup \sum_{l=1}^{n} \left[\rho_l(q_l - p_l) - \frac{Ma_l(\Delta_+\rho_l)^{\gamma+1}}{\rho_l^{\gamma}}\right] = \lim_{n\to\infty} \sup \sum_{l=1}^{n} (l^{b+1} - l^{c+1} - Ml^{a-\gamma}).$$

Similar to the analysis of Case 1, we have that (2.2) holds when $b > \max\{a-\gamma-1, -2\}$. Therefore, Eq.(3.1) is oscillatory when $b > \max\{a - \gamma - 1, -2\}$ by Theorem 2.1.

Case 3: $0 < \beta, \gamma < 1$. It is easy to see that $A_n = \sum_{s=1}^{n-1} \left(\frac{1}{a_s}\right)^{1/\gamma} \leq n$ for $0 \leq a \leq \gamma$, and

$$\lim_{n\to\infty} \sup \sum_{l=1}^{n} \left[\rho_l(q_l - p_l)A_l^{\beta-1} - \frac{Ma_l(\Delta_+\rho_l)^{\gamma+1}}{\rho_l^{\gamma}}\right] \geq \lim_{n\to\infty} \sup \sum_{l=1}^{n} (l^{b+\beta} - l^{c+\beta} - Ml^{a-\gamma}).$$

Thus, we have that (2.14) holds for $b > \max\{-\beta - 1, a - \gamma - \beta\} = a - \gamma - \beta$ since $0 \leq a \leq \gamma$ and $0 < \gamma < 1$. By theorem 2.5, we have that Eq.(3.1) is oscillatory when $b > a - \gamma - \beta$.

Example 3.2. Consider the following perturbed difference equation

$$(3.2) \qquad \Delta(n^{\alpha}(\Delta x_{n-1})^{\gamma}) + n^{(2)}(1 + \sin^2 x_n)x_n^{\beta} = n^{\lambda}\frac{(\Delta x_{n-1})^2 x_n^{\beta}}{1 + (\Delta x_{n-1})^2}, \quad n \geq 1,$$

where $\alpha, \lambda < 2$, γ, β are constants, $\gamma > 0$ and $\beta > 0$ are quotients of odd positive integers, $a > \gamma$ which means that (1.3) holds, $n^{(2)} = n(n - 1)$. It is easy to see that (1.4) holds with $q_n = n^{(2)}$ and $p_n = n^{\lambda}$. Choose $\rho_n = n$ and $\delta_n = 1$, then we can easily obtain that (2.13) holds for $\alpha \leq 3 + \gamma$. Now, let us consider Eq.(3.2) in the following cases:

Case 1: $\gamma < \beta$ and $\gamma \geq 1$. Then we have

$$\lim_{n\to\infty} \sup \sum_{l=1}^{n} \left[\rho_l(q_l - p_l) - \frac{Ma_l(\Delta_+\rho_l)^2}{\rho_l}\right] = \lim_{n\to\infty} \sup \sum_{l=1}^{n} (l^3 - l^2 - l^{\lambda+1} - Ml^{\alpha-1}).$$

In order to guarantee (2.1) holds we need $3 > \alpha-1$, i.e., $\alpha < 4$. Thus, by Theorem 2.3, every solution $\{x_n\}$ of Eq.(3.2) is oscillatory or $\lim_{n\to\infty} x_n = 0$ when $\alpha < \min\{4, 3 + \gamma\} = 4$.

Case 2: $\gamma < \beta$, $\beta \geq 1$ and $0 < \gamma < 1$. Then we have

$$\lim_{n\to\infty} \sup \sum_{l=1}^{n} \left[\rho_l(q_l - p_l) - \frac{Ma_l(\Delta_+\rho_l)^{\gamma+1}}{\rho_l^{\gamma}}\right] = \lim_{n\to\infty} \sup \sum_{l=1}^{n} (l^3 - l^2 - l^{\lambda+1} - Ml^{\alpha-\gamma}).$$

It is easy to see that (2.2) holds when $\alpha < 3+\gamma$. Therefore, by Theorem 2.3 we have every solution $\{x_n\}$ of Eq.(3.2) is oscillatory or $\lim_{n\to\infty} x_n = 0$ when $\alpha < 3+\gamma$.

Case 3: $0 < \beta, \gamma < 1$. Since (1.3) holds, we have that $A_n \leq N$ where $N > 0$ is a constant, and

$$\lim_{n\to\infty} \sup \sum_{l=1}^{n} \left[\rho_l(q_l - p_l)A_l^{\beta-1} - \frac{M a_l(\Delta_+\rho_l)^{\gamma+1}}{\rho_l^{\gamma}}\right]$$

$$\geq \lim_{n\to\infty} \sup \sum_{l=1}^{n} [N^{\beta-1}(l^3 - l^2 - l^{\lambda+1}) - Ml^{\alpha-\gamma}].$$

It is easy to show that (2.14) holds for $\alpha < 3+\gamma$. Thus, by theorem 2.6, we have that every solution $\{x_n\}$ of Eq.(3.2) is oscillatory or $\lim_{n\to\infty} x_n = 0$ when $\alpha < 3+\gamma$.

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