Dynamic Systems and Applications 15 (2006) 549-568

OSCILLATION AND ASYMPTOTIC BEHAVIOR OF THIRD-ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

S. H. SAKER

Department of Mathematics, Faculty of Science, Mansoura University,
Mansoura 35516, Egypt
E-mail: shsaker@mans.edu.eg

ABSTRACT. By means of the Riccati transformation techniques, we will establish some new oscillation criteria for certain class of third order nonlinear neutral delay difference equations. Our results extend as well as improve the well known oscillation results in the literature. Comparison between our theorems and those previously known results are indicated throughout the paper. Some examples are given to illustrate the main results.

AMS (MOS) Subject Classification. 34K11, 39A10.

1. INTRODUCTION

In recent years, the oscillation theory and asymptotic behavior of difference equations and their applications have been and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have been written, see for example the monographs [1, 2].

Determining oscillation criteria for third order nonlinear difference equations has not received a great deal of attention in the literature even though such equations arise in the study of economics, mathematical Biology, and other areas of mathematics which discrete models are used (see for example [3]). Some recent results on third order difference equations can be found in [4, 7-13]. In this paper, we consider the third-order nonlinear neutral delay difference equation

$$\Delta(c_n\Delta\left(d_n\Delta(x_n+p_nx_{n-\tau})\right)^{\gamma})+q_nf(x_{n-\sigma})=0,\quad n\geq n_0, \tag{1.1}$$

where Δ denotes the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$ for any sequence $\{x_n\}$ of real numbers, $\gamma\geqslant 1$ is quotient of odd positive integers, τ and σ are nonnegative integers such that $\tau\leq\sigma$ and the real sequences $\{c_n\}_{n=n_0}^{\infty}$, $\{d_n\}_{n=n_0}^{\infty}$, $\{p_n\}_{n=n_0}^{\infty}$, $\{q_n\}_{n=n_0}^{\infty}$ and the function f satisfies the following conditions:

Received January 16, 2006

$(h1)$ $\{c_n\}_{n=n_0}^{\infty}$, $\{d_n\}_{n=n_0}^{\infty}$ are positive sequences of real numbers such

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{c_n}\right)=\sum_{n=n_0}^{\infty}\left(\frac{1}{d_n}\right)=\infty;$$

$(h2)$ $0 \leq p_n < 1$, $q_n \geqslant 0$ and $\{q_n\}_{n=n_0}^{\infty}$ has a positive subsequence;

$(h3)$ f: $\mathbf{R} \to \mathbf{R}$ is a continuous function such that $uf(u) > 0$ for $u \neq 0$ and $f(u)/u^{\gamma} \geqslant K > 0$.

By a solution of (1.1) we mean a nontrivial real sequence $\{x_n\}$ that is defined for $n \geqslant n_0 - \sigma$ and satisfies equation (1.1) for $n \geq n_0$. Clearly if $x_n = A_n$ for $n = n_0 - \sigma$, $n_0 - \sigma + 1, \dots, n_0 - 1$ are given, then Eq. (1.1) has a unique solution satisfying the above initial conditions.

A solution $\{x_n\}$ of (1.1) is said to be oscillatory if for every $n_1 \geq n_0$ there exists $n \geq n_1$ such that $x_n x_{n+1} \leq 0$, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

A number of dynamical behavior of solutions of third-order difference equations are possible; here we will only be concerned with conditions which are sufficient for all solutions of (1.1) to be oscillatory or tends to zero as $n \to \infty$.

Our concern is motivated by recent results by Graef and Thandapani [4] and Thandapani and Mahalingam [13].

In [4], the authors considered the equation

$$\Delta(c_n\Delta(d_n\Delta(x_n))) + q_n f(x_{n-\sigma+1}) = 0, \quad n \geq n_0, \tag{1.1}$$

and supposed that:

(i) $\{c_n\}_{n=n_0}^{\infty}$, $\{d_n\}_{n=n_0}^{\infty}$ are positive sequences of real numbers such

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{c_n}\right)=\sum_{n=n_0}^{\infty}\left(\frac{1}{d_n}\right)=\infty, \quad \text{and} \quad \Delta c_n \geqslant 0;$$

(ii) $q_n \geqslant 0$ and $\{q_n\}_{n=n_0}^{\infty}$ has a positive subsequence;

(iii) f: $\mathbf{R} \to \mathbf{R}$ is a continuous function such that $uf(u) > 0$ for $u \neq 0$ and

$$f(u) - f(v) = g(u,v)(u-v), \text{ for } u, v \neq 0 \text{ and } g(u,v) \geqslant \mu > 0. \tag{1.2}$$

In the linear case when $f(u) = u$, the authors assumed that (i) and (ii) hold and used the Riccati transformation techniques and established oscillation criterion for Eq. (1.1) which is the discrete analogy of Philos's type oscillation criterion for second-order differential equations (see [6]). In the nonlinear case, the authors assumed that $(i)-(iii)$ hold and established some new oscillation criteria for Eq. (1.2) by reducing it to Riccati difference inequality.

In [13], the authors considered Eq. (1.1) when $\gamma = 1$ and established some sufficient conditions for oscillation. They, used the generalized Riccati technique and given oscillation criterion of the linear case which improves the result of [4] when $p_n = 0$. Also, they assumed that $(i) - (iii)$ hold and established another oscillation criteria.

Our aim in this paper, by using the Riccati transformation techniques we present some new oscillation criteria for Eq. (1.1) bypass the condition (1.5) and do not require the condition $\Delta c_n \geqslant 0$. Our results improve the results in [4] as well as extend and improve the results in [13]. Some comparison between our theorems and those previously known results are indicated throughout the paper. The paper is organized as follows: In section 2, we will state and prove some lemmas which are useful in the proof of our main results. In Section 3, we will state and prove our main oscillation results. In Section 4, we present some examples to illustrate our main results.

2. SOME PRELIMINARY LEMMAS

In this section, we state and prove some basic lemmas, which we will use in the proof of our main results. We begin with the following lemma.

Lemma 2.1. *Suppose that $\{x_n\}$ is an eventually positive solution of (1.1). Set*

$$z_n := x_n + p_n x_{n-\tau}. \tag{2.1}$$

Then there are only the following two cases for $n \geqslant n_0$ sufficiently large:

(I) $z_n > 0$, $\Delta z_n > 0$, $\Delta(d_n \Delta z_n)^\gamma > 0$.

(II) $z_n > 0$, $\Delta z_n < 0$, $\Delta(d_n \Delta z_n)^\gamma > 0$.

Proof. Let $\{x_n\}$be an eventually positive solution of (1.1) and there exists a $n_1 \geqslant n_0$ such that $x_{n-\tau} > 0$ and $x_{n-\sigma} > 0$ for $n \geqslant n_1$. From (2.1) and (h2), it is clear that $z_n > 0$ for all $n \geqslant n_1$ and from (1.1) $\Delta(c_n \Delta(d_n \Delta z_n)^\gamma) \leq 0$ for $n \geqslant n_1$. Then $\{z_n\}$, $\{\Delta z_n\}$ and $\{\Delta(d_n \Delta z_n)^\gamma\}$ are monotone and eventually of one sign. We claim that there is $n_2 \geqslant n_1$ such that for $n \geqslant n_2$, $\Delta(d_n \Delta z_n)^\gamma > 0$. Suppose to the contrary that $\Delta(d_n \Delta z_n)^\gamma \leq 0$ for $n \geqslant n_2$. Since $c_n > 0$ and $c_n \Delta(d_n \Delta z_n)^\gamma$ is nonincreasing there exists a negative constant C and $n_3 \geqslant n_2$ such that $c_n \Delta(d_n \Delta z_n)^\gamma \leq C$ for $n \geqslant n_3$. Dividing by c_n and summing from n_3 to $n-1$, we obtain

$$(d_n \Delta z_n)^\gamma \leq (d_{n_3} \Delta z_{n_3})^\gamma + C \sum_{s=n_3}^{n-1} \left(\frac{1}{c_s}\right).$$

Letting $n \to \infty$, then $d_n \Delta z_n \to -\infty$ by $(h1)$. Thus, there is an integer $n_4 \geqslant n_3$ such that for $n \geqslant n_4$, $d_n \Delta z_n \leq d_{n_4} \Delta z_{n4} < 0$. Dividing by d_n and summing from n_4 to $n-1$ we obtain

$$z_n - z_{n_4} \leq d_{n_4} \Delta z_{n_4} \sum_{s=n_3}^{n-1} \left(\frac{1}{d_s}\right),$$

which implies that $z_n \to -\infty$ as $n \to \infty$ by $(h1)$, a contradiction with the fact that $z_n > 0$. Then $\Delta(d_n\Delta z_n)^\gamma > 0$. The proof is complete.

Lemma 2.2. *Let $\{x_n\}$ be an eventually positive solution of (1.1) and suppose Case (I) of Lemma 2.1 holds and set z_n be as defined by (2.1). Then $\{z_n\}$ is a positive solution of the inequality*

$$\Delta(c_n\Delta(d_n\Delta z_n)^\gamma) + Kq_n(1-p_{n-\sigma})^\gamma z_{n-\sigma}^\gamma \leq 0, \qquad n \geq n_1, \tag{2.2}$$

for some n_1 sufficiently large.

Proof. From (2.1) and $(h2)$ it is clear that z_n is positive and from Lemma (2.2), the case (I) implies that $y_n \geqslant (1-p_n)z_n$ for $n \geqslant n_1$. Then there exists $n_2 \geqslant n_1 + \sigma$ such that $y_{n-\sigma} \geqslant (1-p_{n-\sigma})z_{n-\sigma}$. This and $(h3)$ imply that (2.2) holds. The proof is complete.

Lemma 2.3 [13]. *Let $\{x_n\}$ be an eventually positive solution of (1.1) and suppose Case (II) of Lemma 2.1 holds. Then there exists $n_1 \geqslant n_0$ such that*

$$x_{n-\tau} \geqslant \frac{z_n}{1+p_n} \quad \text{for} \quad n \geqslant n_1. \tag{2.2}$$

Lemma 2.4. *Assume that $(h1)-(h3)$ hold and suppose that Case (II) of Lemma 2.1 holds and the following conditions are satisfied:*

$(h4)$ $\sum_{n=n_0}^{\infty} \frac{q_n}{(1+p_{n-\sigma+\tau})^\gamma} = \infty$;

$(h5)$ $\sum_{n_0}^{\infty} \frac{1}{d_n}\left[\sum_{n_0}^{n-1}\frac{1}{c_t}\sum_{n_0}^{t-1}\frac{q_s}{(1+p_{s-\sigma+\tau})^\gamma}\right]^{\frac{1}{\gamma}} = \infty$.

Then every nonoscillatory solution $\{x_n\}$ of (1.1) satisfies

$$\lim_{n\to\infty}(x_n + p_n x_{n-\tau}) = 0, \tag{2.3}$$

and if $\lim_{n\to\infty} p_n = p^* \in [0,1)$ *then* $\lim_{n\to\infty} x_n = 0$.

Proof: Let $\{x_n\}$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x_{n-\sigma} > 0$ for $n \geqslant n_1$ where n_1 is chosen so large that Lemma 2.1 holds. (The proof when $\{x_n\}$ is eventually negative is similar, since the substitution $y_n = -x_n$ transforms Eq. (1.1) into an equation of the same form.) From Lemma 2.3, (2.3) implies that there exists an $n_2 \geqslant n_1$ such that

$$x_{n-\sigma} \geqslant \frac{z_{n-(\sigma-\tau)}}{1+p_{n-\sigma+\tau}} \quad \text{for } n \geqslant n_2. \tag{2.4}$$

From Case (II) since $\sigma \geqslant \tau$ and z_n is decreasing, (2.5) implies that

$$x_{n-\sigma} \geqslant \frac{z_n}{1+p_{n-\sigma+\tau}} \quad \text{for} \quad n \geqslant n_2. \tag{2.5}$$

From (h3), (1.1) and (2.6) we obtain

$$\Delta(c_n\Delta(d_n\Delta z_n)^\gamma) + \frac{Kq_n}{(1+p_{n-\sigma+\tau})^\gamma} z_n^\gamma \leq 0, \quad n \geq n_2, \tag{2.6}$$

Since $\{z_n\}$ is positive and decreasing it follows that $\lim_{n\to\infty} z_n = b \geqslant 0$. Now we claim that $b = 0$. If not then $z_n^\gamma \to b^\gamma > 0$ as $n \to \infty$, and hence there exists $n_2 \geq n_1$ such that $z_n^\gamma \geq b^\gamma$. Therefore from (2.7) we have

$$\Delta(c_n\Delta(d_n\Delta z_n)^\gamma) + \frac{Kq_n}{(1+p_{n-\sigma+\tau})^\gamma} b^\gamma \leq 0, \quad n \geq n_2, \tag{2.7}$$

Define the sequence $u_n = c_n\Delta(d_n\Delta z_n)^\gamma$ for $n \geq n_2$. Then we have

$$\Delta u_n \leq -\frac{Kq_n}{(1+p_{n-\sigma+\tau})^\gamma} b^\gamma$$

Summing the last inequality from n_2 to $n-1$, we have

$$u_n \leq u_{n_2} - b^\gamma K \sum_{s=n_2}^{n-1} \frac{q_s}{(1+p_{s-\sigma+\tau})^\gamma},$$

In view of (h_4), since $\sum\limits_{n=n_0}^{\infty} \frac{q_n}{(1+p_{n-\sigma+\tau})^\gamma} = \infty$, it is possible to choose an integer n_3 sufficiently large such that for all $n \geq n_3$

$$u_n \leq -\frac{b^\gamma K}{2} \sum_{s=n_2}^{n-1} \frac{q_s}{(1+p_{s-\sigma+\tau})^\gamma},$$

and hence

$$\Delta(d_n\Delta z_n)^\gamma \leq -\frac{b^\gamma K}{2}\frac{1}{c_n} \sum_{s=n_2}^{n-1} \frac{q_s}{(1+p_{s-\sigma+\tau})^\gamma},$$

Summing the last inequality from n_3 to $n-1$ we obtain

$$(d_n\Delta z_n)^\gamma \leq (d_{n_3}\Delta z_{n_3})^\gamma - \frac{b^\gamma K}{2} \sum_{t=n_3}^{n-1} \left(\frac{1}{c_t} \sum_{s=n_2}^{t-1} \frac{q_s}{(1+p_{s-\sigma+\tau})^\gamma} \right).$$

Since $\Delta z_n < 0$ for $n \geqslant n_0$, the last inequality implies that

$$(d_n\Delta z_n)^\gamma \leq -\frac{b^\gamma K}{2} \sum_{t=n_3}^{n-1} \frac{1}{c_t} \sum_{s=n_2}^{t-1} \frac{q_s}{(1+p_{s-\sigma+\tau})^\gamma},$$

or

$$\Delta z_n \leq -\frac{b^\gamma K}{2}\frac{1}{d_n} \left[\sum_{t=n_3}^{n-1} \frac{1}{c_t} \sum_{s=n_2}^{t-1} \frac{q_s}{(1+p_{s-\sigma+\tau})^\gamma} \right]^{\frac{1}{\gamma}}.$$

Summing from n_3 to $n-1$ we have

$$z_n \leq z_{n_3} - \frac{b^\gamma K}{2} \sum_{l=n_3}^{n-1} \frac{1}{d_l} \left[\sum_{t=n_3}^{l-1} \frac{1}{c_t} \sum_{s=n_2}^{t-1} \frac{q_s}{(1+p_{s-\sigma+\tau})^\gamma} \right]^{\frac{1}{\gamma}}.$$

Condition $(h5)$ implies that $z_n \to -\infty$ as $n \to \infty$ which is a contradiction with the fact that z_n is positive. Then $b = 0$ and this completes the proof.

Lemma 2.5. *Let $\{x_n\}$ be an eventually positive solution of (1.1) and suppose Case (I) of Lemma 2.1 holds. Then there exists $n_1 \geqslant n_0$ such that*

$$(\Delta z_{n-\sigma})^{\gamma} \geqslant \frac{\delta_{n-\sigma} c_n}{(d_{n-\sigma})^{\gamma}} \Delta(d_n \Delta z_n)^{\gamma} \quad \text{for} \quad n \geqslant n_1,, \tag{2.9}$$

where $\delta_n := \sum_{s=n_0}^{n-1} \frac{1}{c_s}$.

Proof: From Case (I) of Lemma 2.1 and Eq.(1.1) we have $(d_n \Delta z_n)^{\gamma} > 0$, $c_n \Delta(d_n \Delta z_n)^{\gamma} > 0$ and $\Delta(c_n \Delta(d_n \Delta z_n)^{\gamma}) \leq 0$ for $n \geqslant n_1$. Hence

$$(d_n \Delta z_n)^{\gamma} = (d_{n_1} \Delta z_{n_1})^{\gamma} + \sum_{s=n_1}^{n-1} \frac{c_s \Delta(d_s \Delta z_s)^{\gamma}}{c_s} \geqslant c_n \delta_n \Delta(d_n \Delta z_n)^{\gamma}, \quad n \geqslant n_1. \tag{2.10}$$

Since $\Delta(c_n \Delta(d_n \Delta z_n)^{\gamma}) \leq 0$, we have $c_{n-\sigma} \Delta(d_{n-\sigma} \Delta z_{n-\sigma})^{\gamma} \geqslant c_n \Delta(d_n \Delta z_n)^{\gamma}$. This and (2.10) imply that

$$\begin{aligned} (d_{n-\sigma} \Delta z_{n-\sigma})^{\gamma} &\geqslant c_{n-\sigma} \delta_{n-\sigma} \Delta(d_{n-\sigma} \Delta z_{n-\sigma})^{\gamma} \\ &\geqslant c_n \delta_{n-\sigma} \Delta(d_n \Delta z_n)^{\gamma}, \quad n \geqslant n_2 = n_1 + \sigma, \end{aligned}$$

and then we have

$$(d_{n-\sigma} \Delta z_{n-\sigma})^{\gamma} \geqslant c_n \delta_{n-\sigma} \Delta(d_n \Delta z_n)^{\gamma}, \quad n \geqslant n_2 = n_1 + \sigma.$$

The proof is complete.

3. **OSCILLATION CRITERIA**

In this section we establish some sufficient conditions which guarantee that every solution $\{x_n\}$ of (1.1) oscillates or satisfies (2.4).

First, we use the Riccati transformation technique.

Theorem 3.1: *Assume that $(h1) - (h5)$ hold. Furthermore, assume that there exists a positive sequence $\{\rho_n\}_{n=n_0}^{\infty}$ such that,*

$$\lim_{n \to \infty} \sup \sum_{l=n_0}^{n} \left[K \rho_l q_l (1 - p_{l-\sigma})^{\gamma} - \frac{(d_{l-\sigma})^{\gamma} (\Delta \rho_l)^2}{2^{3-\gamma} \delta_{l-\sigma} \rho_l} \right] = \infty. \tag{3.1}$$

Then every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

Proof: Let $\{x_n\}$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x_{n-\sigma} > 0$ for $n \geqslant n_1$ where n_1 is chosen so large that Lemma 2.1 to Lemma 2.3 and Lemma 2.5 hold. We shall consider only this case, because the proof when $x_n < 0$ is similar. Define z_n be as in (2.1), then $z_n > 0$ and from Lemma 2.1 there are two possible cases. First we consider the Case (I): From Lemma 2.2, we see that z_n be a positive solution of the delay difference inequality (2.2). Define the sequence $\{w_n\}$ by the Riccati substitution

$$w_n := \rho_n \frac{c_n \Delta \left(d_n \Delta z_n\right)^{\gamma}}{z_{n-\sigma}^{\gamma}}, \quad n \geqslant n_1. \tag{3.2}$$

Then $w_n > 0$, and

$$\Delta w_n = c_{n+1}\Delta\left(d_{n+1}\Delta z_{n+1}\right)^{\gamma}\Delta\left[\frac{\rho_n}{z_{n-\sigma}^{\gamma}}\right] + \frac{\rho_n\Delta(c_n\Delta\left(d_n\Delta z_n\right)^{\gamma})}{z_{n-\sigma}^{\gamma}}.$$

This and (2.2), imply that

$$\Delta w_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\rho_n c_{n+1}\Delta\left(d_{n+1}\Delta z_{n+1}\right)^{\gamma}\Delta(z_{n-\sigma}^{\gamma})}{z_{n-\sigma}^{\gamma}z_{n-\sigma+1}^{\gamma}}, \tag{3.3}$$

where $Q_n := Kq_n(1-p_{n-\sigma})^{\gamma}$. From Lemma 2.1 Case (I), we have $z_{n-\sigma+1} \geq z_{n-\sigma}$. Then from (3.3), we obtain

$$\Delta w_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - \frac{\rho_n c_{n+1}\Delta\left(d_{n+1}\Delta z_{n+1}\right)^{\gamma}\Delta(z_{n-\sigma}^{\gamma})}{z_{n-\sigma+1}^{2\gamma}}. \tag{3.4}$$

Using the inequality

$$x^{\beta} - y^{\beta} \geqslant 2^{1-\beta}(x-y)^{\beta} \text{ for all } x \geq y > 0 \text{ and } \beta \geqslant 1,$$

we have

$$\begin{aligned} \Delta(z_{n-\sigma}^{\gamma}) &= z_{n-\sigma+1}^{\gamma} - z_{n-\sigma}^{\gamma} \geqslant 2^{1-\beta}(z_{n-\sigma+1} - z_{n-\sigma})^{\gamma} \\ &= 2^{1-\beta}\left(\Delta z_{n-\sigma}\right)^{\gamma}, \qquad \gamma \geqslant 1. \end{aligned} \tag{3.5}$$

Substituting from (3.5) in (3.4), we obtain

$$\Delta w_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - 2^{1-\gamma}\frac{\rho_n c_{n+1}\Delta\left(d_{n+1}\Delta z_{n+1}\right)^{\gamma}\left(\Delta z_{n-\sigma}\right)^{\gamma}}{z_{n-\sigma+1}^{2\gamma}}. \tag{3.6}$$

From Lemma 2.5, we have

$$\Delta w_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - 2^{1-\gamma}\frac{\rho_n c_{n+1}\Delta\left(d_{n+1}\Delta z_{n+1}\right)^{\gamma}\delta_{n-\sigma}c_n\Delta(d_n\Delta z_n)^{\gamma}}{\left(d_{n-\sigma}\right)^{\gamma}z_{n-\sigma+1}^{2\gamma}}. \tag{3.7}$$

From (2.2) since $\Delta(c_n\Delta(d_n\Delta z_n)^{\gamma}) \leq 0$, then we have

$$c_{n+1}\Delta\left(d_{n+1}\Delta z_{n+1}\right)^{\gamma} \geqslant c_n\Delta(d_n\Delta z_n)^{\gamma}. \tag{3.8}$$

Then (3.7) and (3.8) imply that

$$\Delta w_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - 2^{1-\gamma}\frac{\rho_n\left(c_{n+1}\Delta\left(d_{n+1}\Delta z_{n+1}\right)^{\gamma}\right)^2\delta_{n-\sigma}}{\left(d_{n-\sigma}\right)^{\gamma}z_{n-\sigma+1}^{2\gamma}}. \tag{3.9}$$

From (3.2) and (3.9) we obtain

$$\Delta w_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1} - 2^{1-\gamma}\frac{\rho_n\delta_{n-\sigma}}{\left(d_{n-\sigma}\right)^{\gamma}\rho_{n+1}^2}w_{n+1}^2. \tag{3.10}$$

By completing the square, we get

$$\begin{aligned}\Delta w_n &\leq -\rho_n Q_n + \frac{(d_{n-\sigma})^{\gamma}(\Delta\rho_n)^2}{2^{3-\gamma}\delta_{n-\sigma}\rho_n} \\ &\quad - \left[\frac{\sqrt{2^{1-\gamma}\delta_{n-\sigma}\rho_n}}{\rho_{n+1}\sqrt{(d_{n-\sigma})^{\gamma}}} w_{n+1} - \frac{\sqrt{(d_{n-\sigma})^{\gamma}}\Delta\rho_n}{2\sqrt{2^{1-\gamma}\delta_{n-\sigma}\rho_n}}\right]^2 \\ &< -\left[\rho_n Q_n - \frac{(d_{n-\sigma})^{\gamma}(\Delta\rho_n)^2}{2^{3-\gamma}\delta_{n-\sigma}\rho_n}\right].\end{aligned}$$

Then, we have

$$\Delta w_n < -\left[\rho_n Q_n - \frac{(d_{n-\sigma})^{\gamma}(\Delta\rho_n)^2}{2^{3-\gamma}\delta_{n-\sigma}\rho_n}\right]. \tag{3.11}$$

Summing (3.11) from n_1 to n, we obtain

$$-w_{n_1} < w_{n+1} - w_{n_1} < -\sum_{l=n_1}^{n}\left[\rho_l Q_l - \frac{(d_{l-\sigma})^{\gamma}(\Delta\rho_l)^2}{2^{3-\gamma}\delta_{l-\sigma}\rho_l}\right].$$

which yields

$$\sum_{l=n_1}^{n}\left[\rho_l Q_l - \frac{(d_{l-\sigma})^{\gamma}(\Delta\rho_l)^2}{2^{3-\gamma}\delta_{l-\sigma}\rho_l}\right] < c_1, \tag{3.12}$$

for all large n, and this is contrary to (3.1). If the Case (II) holds, we are then back to the proof of Lemma 2.4 to prove that (2.4) holds. The proof is complete.

Remark 3.1: From Theorem 3.1, we can obtain different conditions for oscillation of all solutions of (1.1) by different choices of $\{\rho_n\}$. Let $\rho_n = n^{\lambda}$, $n \geq n_0$ and $\lambda > 1$ is a constant. Hence we have the following result.

Corollary 3.1: *Assume that all the assumptions of Theorem 3.1 hold, except that the condition (3.1) is replaced by*

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n}\left[Ks^{\lambda}q_s(1-p_{s-\sigma})^{\gamma} - \frac{(d_{s-\sigma})^{\gamma}((s+1)^{\lambda}-s^{\lambda})^2}{2^{3-\gamma}s^{\lambda}\delta_{s-\sigma}}\right] = \infty.$$

Then, every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

Remark 3.2: If $p_n = 0$, then Eq.(1.1) reduces to the nonlinear difference equation

$$\Delta(c_n\Delta(d_n\Delta(x_n))) + q_n f(x_{n-\sigma}) = 0, \qquad n \geq n_0,$$

and (3.1) reduces to

$$\lim_{n\to\infty}\sup\sum_{l=n_0}^{n}\left[K\rho_l q_l - \frac{(d_{l-\sigma})^{\gamma}(\Delta\rho_l)^2}{2^{3-\gamma}\delta_{l-\sigma}\rho_l}\right] = \infty.$$

Then, Theorem 3.1 improves Theorem 2 of Graef and Thandapani [4] in the sense that we do not need the condition (1.5) and also our results do not require that $\Delta c_n \geqslant 0$ for $n \geqslant n_0$.

Theorem 3.2: *Assume that* $(h1)-(h5)$ *hold. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=n_0}^{\infty}$ *such that for every positive number* $\lambda \geq 1$,

$$\lim_{m\to\infty} \sup \frac{1}{m^{\lambda}} \sum_{n=1}^{m-1} (m-n)^{\lambda} \left[K\rho_n q_n (1-p_{n-\sigma})^{\gamma} - \frac{(\rho_{n+1})^2}{4\bar{\rho}_n} A_{m,n} \right] = \infty, \tag{3.13}$$

where

$$\bar{\rho}_n := 2^{1-\gamma} \frac{\rho_n \delta_{n-\sigma}}{(d_{n-\sigma})^{\gamma}}, \quad A_{m,n} := \left(\frac{\Delta\rho_n}{\rho_{n+1}} - \frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}} \right)^2$$

Then every solution $\{x_n\}$ *of Eq (1.1) oscillates or satisfies (2.4).*

Proof. Proceeding as in Theorem 3.1, we assume that Eq.(1.1) has a nonoscillatory solution, say $x_{n-\sigma} > 0$ for all $n \geq n_0$. Let z_n be as defined by (2.1). Then z_n is positive and by Lemma 2.1 there are two possible cases. First, we consider the case when Case (I) holds. From Lemma 2.2, we see that z_n be a positive solution of the delay difference inequality (2.2). Defining again $\{w_n\}$ by (3.1), then from Theorem 3.1, we have $w_n > 0$ and (3.10) holds. From (3.10), we have for $n \geq n_1$

$$\rho_n Q_n \leq -\Delta w_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\bar{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2. \tag{3.14}$$

Therefore,

$$\begin{aligned} \sum_{n=n_1}^{m-1} (m-n)^{\lambda} \rho_n Q_n \quad &\leq \quad -\sum_{n=n_1}^{m-1} (m-n)^{\lambda} \Delta w_n + \sum_{n=n_1}^{m-1} (m-n)^{\lambda} \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\ &\quad -\sum_{n=n_1}^{m-1} (m-n)^{\lambda} \frac{\bar{\rho}_n}{\rho_{n+1}^2} w_{n+1}^2. \end{aligned} \tag{3.15}$$

Now, after summing by parts, we have

$$\sum_{n=n_1}^{m-1} (m-n)^{\lambda} \Delta w_n = -(m-n_1)^{\lambda} w_{n_1} - \sum_{n=n_1}^{m-1} w_{n+1} \Delta_2 (m-n)^{\lambda},$$

where, $\Delta_2(m-n)^{\lambda} = (m-n-1)^{\lambda} - (m-n)^{\lambda}$. Then

$$\sum_{n=n_1}^{m-1} (m-n)^{\lambda} \Delta w_n = -(m-n_1)^{\lambda} w_{n_1} + \sum_{n=n_1}^{m-1} w_{n+1}((m-n)^{\lambda} - (m-n-1)^{\lambda}).$$

Using the inequality, $x^{\beta} - y^{\beta} \geq \beta y^{\beta-1}(x-y)$ for all $x \geq y > 0$ and $\beta \geq 1$, we obtain

$$\sum_{n=n_1}^{m-1} (m-n)^{\lambda} \Delta w_n \geq -(m-n_1)^{\lambda} w_{n_1} + \sum_{n=n_1}^{m-1} \lambda w_{n+1} (m-n-1)^{\lambda-1}.$$

Substituting in (3.15), we have

$$\begin{aligned}
&\sum_{n=n_1}^{m-1}(m-n)^{\lambda}\rho_n Q_n \\
\leq\ & (m-n_1)^{\lambda}w_{n_1}-\sum_{n=n_1}^{m-1}\lambda w_{n+1}(m-n-1)^{\lambda-1} \\
&+\sum_{n=n_1}^{m-1}(m-n)^{\lambda}\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1}-\sum_{n=n_1}^{m-1}(m-n)^{\lambda}\frac{\bar{\rho}_n}{\rho_{n+1}^2}w_{n+1}^2.
\end{aligned}$$

Then,

$$\begin{aligned}
&\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}(m-n)^{\lambda}\rho_n Q_n\leq(\frac{m-n_1}{m})^{\lambda}w_{n_1} \\
&-\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}(m-n)^{\lambda}\left[\frac{\rho_n}{\rho_{n+1}^2}w_{n+1}^2-\left(\frac{\Delta\rho_n}{\rho_{n+1}}-\frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}}\right)w_{n+1}\right] \\
=\ & (\frac{m-n_1}{m})^{\lambda}w_{n_1}-\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}(m-n)^{\lambda} \\
&\times\left[\frac{\sqrt{\bar{\rho}_n}}{\rho_{n+1}}w_{n+1}-\frac{\rho_{n+1}}{2\sqrt{\bar{\rho}_n}}\left(\frac{\Delta\rho_n}{\rho_{n+1}}-\frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}}\right)\right]^2 \\
&+\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}(m-n)^{\lambda}\frac{(\rho_{n+1})^2}{4\bar{\rho}_n}\left(\frac{\Delta\rho_n}{\rho_{n+1}}-\frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}}\right)^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}\kappa(m-n)^{\lambda}\rho_n Q_n<(\frac{m-n_1}{m})^{\lambda}w_{n_1} \\
&+\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}(m-n)^{\lambda}\frac{(\rho_{n+1})^2}{4\bar{\rho}_n}\left(\frac{\psi_n}{\rho_{n+1}}+\frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}}\right)^2,
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}(m-n)^{\lambda}\left[\rho_n Q_n-\frac{(\rho_{n+1})^2}{4\bar{\rho}_n}\left(\frac{\Delta\rho_n}{\rho_{n+1}}-\frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}}\right)^2\right] \\
<\ & (\frac{m-n_1}{m})^{\lambda}w_{n_1},
\end{aligned}$$

which yields

$$\lim_{m\to\infty}\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}(m-n)^{\lambda}\left[K\rho_n q_n(1-p_{n-\sigma})^{\gamma}-\frac{(\rho_{n+1})^2}{4\bar{\rho}_n}A_{m,n}\right]<\infty,$$

which is contrary to (3.13). If the Case (II) holds, we are then back to the proof of Lemma 2.4 to prove that (2.4) holds. The proof is complete.

As a variant of the Riccati transformation technique used above, we will derive new oscillation criteria which can be considered as a discrete analogy of Philos's condition for oscillation of second order differential equations [6].

Theorem 3.3: *Assume that (h1)−(h5) hold. Let $\{\rho_n\}_{n=n_0}^{\infty}$ be a positive sequence. Furthermore, we assume that there exists a double sequence $\{H_{m,n} : m \geq n \geq 0\}$ such that (i) $H_{m,m} = 0$ for $m \geq 0$, (ii) $H_{m,n} > 0$ for $m > n \geq 0$, (iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ for $m \geq n \geq 0$. If*

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} \left[K H_{m,n} \rho_n q_n (1 - p_{n-\sigma})^{\gamma} - \frac{(\rho_{n+1})^2}{4\bar{\rho}_n} B_{m,n} \right] = \infty, \tag{3.16}$$

where

$$B_{m,n} = \left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2, \ \bar{\rho}_n = 2^{1-\gamma} \frac{\rho_n \delta_{n-\sigma}}{(d_{n-\sigma})^{\gamma}}, \ h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}}.$$

Then, every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

Proof: Proceeding as in Theorem 3.1, we assume that Eq.(1.1) has a nonoscillatory solution, say $x_{n-\sigma} > 0$ for all $n \geq n_0$. Let z_n be as defined by (2.1). Then z_n is positive and by Lemma 2.1 there are two possible cases. First, we consider the case when Case (I) holds. From Lemma 2.2, we see that z_n be a positive solution of the delay difference inequality (2.2). Defining again $\{w_n\}$ by (3.1), then from Theorem 3.1, we have $w_n > 0$ and (3.10) holds. From (3.10), we have for $n \geq n_1$

$$\begin{aligned} \sum_{n=n_1}^{m-1} H_{m,n} \rho_n Q_n \leq & - \sum_{n=n_1}^{m-1} H_{m,n} \Delta w_n + \sum_{n=n_1}^{m-1} H_{m,n} \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\ & - \sum_{n=n_1}^{m-1} H_{m,n} \frac{\bar{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2. \end{aligned} \tag{3.17}$$

which yields after summing by parts

$$\begin{aligned} & \sum_{n=n_1}^{m-1} H_{m,n} \rho_n Q_n \leq H_{m,n_1} w_{n_1} + \sum_{n=n_1}^{m-1} w_{n+1} \Delta_2 H_{m,n} + \sum_{n=n_1}^{m-1} H_{m,n} \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\ & - \sum_{n=n_1}^{m-1} H_{m,n} \frac{\bar{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2, \end{aligned}$$

hence

$$\begin{aligned} & \sum_{n=n_1}^{m-1} H_{m,n} \rho_n Q_n \leq H_{m,n_1} w_{n_1} - \sum_{n=n_1}^{m-1} h_{m,n} \sqrt{H_{m,n}} w_{n+1} \\ & + \sum_{n=n_1}^{m-1} H_{m,n} \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_1}^{m-1} H_{m,n} \frac{\bar{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2 \end{aligned}$$

$$= H_{m,n_1} w_{n_1} - \sum_{n=n_1}^{m-1} \left[\frac{\sqrt{H_{m,n}\bar{\rho}_n}}{\rho_{n+1}} w_{n+1} + \frac{\rho_{n+1}}{2\sqrt{H_{m,n}\bar{\rho}_n}} \sqrt{B_{m,n}} \right]^2$$

$$+ \frac{1}{4} \sum_{n=n_1}^{m-1} \frac{(\rho_{n+1})^2}{\bar{\rho}_n} \left(h_{m,n} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 .$$

Then,

$$\sum_{n=n_1}^{m-1} \left[H_{m,n} \rho_n \rho_n Q_n - \frac{(\rho_{n+1})^2}{4\rho_n} \left(h_{m,n} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right] < H_{m,n_2} w_{n_2} \leq H_{m,0} w_{n_2}$$

which implies that

$$\sum_{n=n_0}^{m-1} \left[H_{m,n} \rho_n Q_n - \frac{(\rho_{n+1})^2}{4\bar{\rho}_n} \left(h_{m,n} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right]$$

$$< H_{m,n_0} \left(w_{n_1} + \sum_{n=n_0}^{n_1 - 1} \rho_n \rho_n Q_n \right).$$

Hence

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} \left[H_{m,n} \rho_n Q_n - \frac{(\rho_{n+1})^2}{4\bar{\rho}_n} B_{m,n} \right]$$

$$< \left(w_{n_1} + \sum_{n=n_0}^{n_1 - 1} \rho_n \rho_n Q_n \right) < \infty.$$

and this contradicts (3.13). If the Case (II) holds, we are then back to the proof of Lemma 2.4 to prove that (2.4) holds. The proof is complete.

As an immediate consequence of Theorem 3.3, we get the following:

Corollary 3.2: *Assume that all the assumptions of Theorem 3.3 hold, except that the condition (3.16) is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} H_{m,n} \rho_n q_n (1 - p_{n-\sigma})^{\gamma} = \infty,$$

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} \frac{\rho_{n+1}^2 (d_{n-\sigma})^{\frac{1}{\gamma}}}{\rho_n \delta_{n-\sigma}} \left(h_{m,n} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 < \infty.$$

Then, every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

Remark 3.3: By choosing the sequence $\{H_{m,n}\}$ in appropriate manners, we can derive several oscillation criteria for (1.1). For instance, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$H_{m,n} := (m-n)^{\lambda}, \quad \lambda \geq 1, m \geq n \geq 0,$$

or

$$H_{m,n} := \left(\log \frac{m+1}{n+1}\right)^{\lambda}, \quad \lambda \geq 1, \quad m \geq n \geq 0,$$

or

$$H_{m,n} := (m-n)^{(\lambda)} \quad \lambda > 2, \quad m \geq n \geq 0,$$

where $(m-n)^{(\lambda)} = (m-n)(m-n+1)\cdots(m-n+\lambda-1)$ and

$$\Delta_2 (m-n)^{(\lambda)} = (m-n-1)^{(\lambda)} - (m-n)^{(\lambda)} = -\lambda(m-n)^{(\lambda-1)}.$$

Then $H_{m,m} = 0$ for $m \geq 0$ and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n \geq 0$. Hence we have the following results.

Corollary 3.3: *Assume that all the assumptions of Theorem 3.3 hold, except that the condition (3.16) is replaced by*

$$\limsup_{m\to\infty} \frac{1}{m^{\lambda}} \sum_{n=0}^{m-1} \left[K(m-n)^{\lambda} \rho_n q_n (1-p_{n-\sigma})^{\gamma} - \frac{\rho_{n+1}^2}{4\bar{\rho}_n} C_{m.n} \right] = \infty,$$

where

$$C_{m,n} = \left(\lambda (m-n)^{\frac{\lambda-2}{2}} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{(m-n)^{\lambda}} \right)^2.$$

Then, every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

Corollary 3.4: *Assume that all the assumptions of Theorem 3.3 hold, except that the condition (3.16) is replaced by*

$$\limsup_{m\to\infty} \frac{1}{(\log(m+1))^{\lambda}} \sum_{n=0}^{m-1} \left[\left(\log \frac{m+1}{n+1}\right)^{\lambda} K \rho_n q_n (1-p_{n-\sigma})^{\gamma} \right.$$
$$\left. - \frac{\rho_{n+1}^2}{4\bar{\rho}_n} \left(\frac{\lambda}{n+1} \left(\log \frac{m+1}{n+1}\right)^{\frac{\lambda-2}{2}} - \frac{\Delta\rho_n}{\rho_{n+1}} \sqrt{\left(\log \frac{m+1}{n+1}\right)^{\lambda}} \right)^2 \right] = \infty.$$

Then, every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

Corollary 3.5: *Assume that all the assumptions of Theorem 3.3 hold, except that the condition (3.16) is replaced by*

$$\limsup_{m\to\infty} \frac{1}{m^{(\lambda)}} \sum_{n=0}^{m-1} (m-n)^{(\lambda)} \left[K \rho_n q_n (1-p_{n-\sigma})^{\gamma} - \frac{\rho_{n+1}^2}{4\bar{\rho}_n} F_{m,n} \right] = \infty,$$

where

$$F_{m,n} := \left(\frac{\lambda}{m-n+\lambda-1} - \frac{\Delta\rho_n}{\rho_{n+1}} \right)^2.$$

Then, every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

In the following we will use the generalized Riccati transformation techniques.

Theorem 3.4: *Assume that (h1)−(h5) hold. Let $\{\rho_n\}_{n=n_0}^{\infty}$ be a positive sequence Furthermore, we assume that there exists a double sequence $\{H_{m,n} : m \geq n \geq 0\}$ such* *that (i)* $H_{m,m}=0$ *for* $m\geq 0$, *(ii)* $H_{m,n}>0$ *for* $m>n\geq 0$, *(iii)* $\Delta_2 H_{m,n}=H_{m,n+1}-H_{m,n}\leq 0$ *for* $m\geq n\geq 0$. *If*

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1}\left[H_{m,n}\Psi_n-\frac{(d_{n-\sigma})^{\gamma}(\rho_{n+1})^2}{2^{3-\gamma}\rho_n\delta_{n-\sigma}}h_{m,n}^2\right]=\infty, \tag{3.18}$$

where

$$\Psi_n=\rho_n\left[Kq_n(1-p_{n-\sigma})^{\gamma}+\frac{2^{1-\gamma}\delta_{n-\sigma}c_{n+1-\sigma}^2\alpha_n^2}{(d_{n-\sigma})^{\gamma}}-\Delta(c_{n-\sigma}\alpha_{n-1})\right],$$

$$\alpha_n=-\frac{\Delta\rho_n\,(d_{n-\sigma})^{\gamma}}{2^{2-\gamma}\rho_n\delta_{n-\sigma}c_{n+1-\sigma}},\qquad h_{m,n}=-\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}},\ m>n\geq 0.$$

Then, every solution $\{x_n\}$ *of Eq.(1.1) oscillates or satisfies (2.4).*

Proof: Proceeding as in Theorem 3.1, we assume that Eq.(1.1) has a nonoscillatory solution, say $x_{n-\sigma}>0$ for all $n\geq n_0$. Let z_n be as in (2.1), then z_n is positive and from Lemma 2.2 we see that (2.2) holds. From Lemma 2.1 there are two possible cases. First, we consider the case when the Case (I) holds. Define the sequence $\{W_n\}$ by

$$W_n:=\rho_n\left[\frac{c_n\Delta\left(d_n\Delta z_n\right)^{\gamma}}{z_{n-\sigma}^{\gamma}}+c_{n-\sigma}\alpha_{n-1}\right].$$

Then follows the proof of Theorem 3.1, we obtain

$$\begin{aligned}&\Delta W_n\leq-\rho_n Q_n\\ &+\frac{\Delta\rho_n}{\rho_{n+1}}W_{n+1}-\frac{2^{1-\gamma}\rho_n\delta_{n-\sigma}}{(d_{n-\sigma})^{\gamma}}\left(\frac{W_{n+1}}{\rho_{n+1}}-c_{n+1-\sigma}\alpha_n\right)^2\\ &+\rho_n\Delta(c_{n-\sigma}\alpha_{n-1})\\ =\ &-\Psi_n-\frac{2^{1-\gamma}\rho_n\delta_{n-\sigma}}{(d_{n-\sigma})^{\gamma}\rho_{n+1}^2}W_{n+1}^2.\end{aligned}$$

Therefore, we have

$$\sum_{n=n_1}^{m-1}H_{m,n}\Psi_n\leq-\sum_{n=n_1}^{m-1}H_{m,n}\Delta W_n+\sum_{n=n_1}^{m-1}H_{m,n}\frac{2^{1-\gamma}\rho_n\delta_{n-\sigma}}{(d_{n-\sigma})^{\gamma}\rho_{n+1}^2}W_{n+1}^2. \tag{3.19}$$

The remainder of the proof is similar to that of the proof of Theorem 3.2 and hence is omitted. If Case (II) holds, we are back to the proof of Lemma 2.4 to prove that (2.4) holds. The proof is complete.

Remark 3.4: Let $f(u)=u$, then $K=1$ and $\gamma=1$. Then From Theorem 3.2 the condition (3.16) reduces to the condition (7) of Theorem 3.1 in [13]. Also it is clear that Theorem 3.2 is satisfied when $f(u)\geqslant Ku$ and this implies that Theorem 3.2 extend and improve the results in [13] in the sense our results do no require the condition (1.5) and also correct the misprint of definition of α_n in [13].

As an immediate consequence of Theorem 3.4, we get the following:

Corollary 3.6: *Assume that all the assumptions of Theorem 3.4 hold, except that the condition (3.18) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1}H_{m,n}\rho_n \times\left[Kq_n(1-p_{n-\sigma})^\gamma+\frac{2^{1-\gamma}\delta_{n-\sigma}c_{n+1-\sigma}^2\alpha_n^2}{(d_{n-\sigma})^\gamma}-\Delta(c_{n-\sigma}\alpha_{n-1})\right]=\infty,$$

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1}\frac{\rho_{n+1}^2\,(d_{n-\sigma})^\gamma}{\rho_n\delta_{n-\sigma}}h_{m,n}^2<\infty.$$

Then, every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

Remark 3.5: By choosing the sequence $\{H_{m,n}\}$ in appropriate manners as before, we can derive other several oscillation criteria for Eq.(1.1). Then from Theorem 3.3 we have the following results.

Corollary 3.7: *Assume that all the assumptions of Theorem 3.4 hold, except that the condition (3.18) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^\lambda}\sum_{n=0}^{m-1}\left[(m-n)^\lambda\Psi_n-\frac{\lambda^2\rho_{n+1}^2\,(d_{n-\sigma})^\gamma}{2^{3-\gamma}\rho_n\delta_{n-\sigma}}(m-n)^{\lambda-2}\right]=\infty.$$

Then, every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

Corollary 3.8: *Assume that all the assumptions of Theorem 3.4 hold, except that the condition (3.18) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{(\log(m+1))^\lambda} \times\sum_{n=0}^{m-1}\left[\left(\log\frac{m+1}{n+1}\right)^\lambda\Psi_n-\frac{\lambda^2\rho_{n+1}^2\,(d_{n-\sigma})^\gamma}{2^{3-\gamma}\rho_n\delta_{n-\sigma}}\left(\log\frac{m+1}{n+1}\right)^{\lambda-2}\right]=\infty.$$

Then, every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

Corollary 3.9: *Assume that all the assumptions of Theorem 3.4 hold, except that the condition (3.18) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^{(\lambda)}}\sum_{n=0}^{m-1}(m-n)^{(\lambda)}\left[\Psi_n-\frac{\rho_{n+1}^2\,(d_{n-\sigma})^\gamma}{2^{3-\gamma}\rho_n\delta_{n-\sigma}}\left(\frac{\lambda}{m-n+\lambda-1}\right)^2\right]=\infty.$$

Then, every solution $\{x_n\}$ of Eq.(1.1) oscillates or satisfies (2.4).

Remark 3.6: If $\gamma=1$ then Corollaries 3.7 and 3.8 reduce directly to Corollaries 3.2 and 3.3 in [13].

4. APPLICATIONS

In this section we present some examples to illustrate our main results.

Example 4.1. Consider the following third order nonlinear neutral delay difference equation

$$\Delta^2\left(n\Delta\left(x_n+\frac{n+1}{n+2}x_{n-1}\right)\right)^\gamma+n^2x_{n-2}(1+x_{n-2}^2)=0,\quad n\geq 2, \tag{4.1}$$

where $\tau=1$, $\sigma=2$, $c_n=1$, $d_n=n$, $q_n=n^2$, $p_n=\frac{n+1}{n+2}$ and $f(u)=u(1+u^2)\geq u$ with $K=1$. From this we have $\delta_n=\sum_{n=0}^{n}c_n=\frac{n(n+1)}{2}$. It is clear the conditions $(h_1)-(h_3)$ are satisfied. It remains to satisfy (h_4), (h_5) and the condition (3.1). From, the definitions of c_n, d_n, p_n and q_n, we see that

$$\begin{aligned}\sum_{n=n_0}^{\infty}\frac{q_n}{(1+p_{n-\sigma+\tau})^\gamma} &= \sum_{n=2}^{\infty}\frac{n^2}{(1+\frac{n}{n+1})}=\sum_{n=2}^{\infty}\frac{n^2\,(n+1)}{(2n+1)}\\ &= \sum_{n=2}^{\infty}\left[\frac{1}{2}n^2+\frac{1}{4}n-\frac{1}{8}+\frac{1}{8\,(2n+1)}\right]=\infty.\end{aligned}$$

Then (h_4) holds. Also

$$\begin{aligned}&\sum_{n_0}^{\infty}\frac{1}{d_n}\left[\sum_{n_0}^{n-1}\frac{1}{c_t}\sum_{n_0}^{t-1}\frac{q_s}{(1+p_{s-\sigma+\tau})^\gamma}\right]^{\frac{1}{\gamma}}\\ &=\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n}\left[\sum_{t=0}^{n-1}\sum_{s=0}^{t-1}\frac{s^2(s+1)}{s+\frac{1}{2}}\right]\geq\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n}\left[\sum_{t=0}^{n-1}\sum_{s=0}^{t-1}s^2\right]\\ &\geq\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n}\left[\sum_{t=0}^{n-1}\sum_{s=0}^{t-1}s\right]=\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{n}\left[\sum_{t=0}^{n-1}t(t-1)\right]\\ &\geq\frac{1}{12}\sum_{n=0}^{\infty}\frac{1}{n}n^2(n-3)=\infty.\end{aligned}$$

Then (h_5) holds. Now, by choosing $\rho_n=n$, we have

$$\begin{aligned}&\limsup_{n\to\infty}\sum_{s=n_0}^{n}\left[K\rho_sq_s(1-p_{s-\sigma})^\gamma-\frac{(d_{s-\sigma})^\gamma\,(\Delta\rho_s)^2}{2^{3-\gamma}\delta_{s-\sigma}\rho_s}\right]\\ =&\limsup_{n\to\infty}\sum_{s=0}^{n}\left[s^2s(1-\frac{s-1}{s})-\frac{(s-2)}{2(s-2)(s-1)s}\right]\\ =&\limsup_{n\to\infty}\sum_{s=0}^{n}\left[s^2-\frac{1}{2(s-1)s}\right]=\infty.\end{aligned}$$

Consequently condition (3.1) is satisfied. Hence, by Theorem 3.1, every solution of Eq.(4.1) oscillates or satisfies $\lim_{n\to\infty}(x_n+\frac{n+1}{n+2}x_{n-1})=0$.

Example 4.2. Consider the following third order nonlinear neutral delay difference equation

$$\Delta\left(n\Delta\left(\sqrt[3]{n}\Delta\left(x_n+\frac{1}{n+2}x_{n-1}\right)\right)^3\right) + \frac{(n+2)^3}{(n+1)}x_{n-2}^3(1+x_{n-2}^2)=0, \qquad n\geq 0, \tag{4.2}$$

where $\gamma=2$, $c_n=n$, $d_n=\sqrt[3]{n}$, $q_n=\frac{(n+2)^3}{(n+1)^2}$, $p_n=\frac{1}{n+2}$ and $f(u)=u^3(1+u^2)\geq u^3$ with $K=1$. From this we have $\delta_n=\sum_{n=0}^{n}c_n=\frac{n(n+1)}{2}$ and $p_{n-\sigma+\tau}=\frac{1}{n+1}$. It is clear the conditions $(h_1)-(h_3)$ are satisfied. It remains to satisfy (h_4), (h_5) and the condition (3.13). From, the definitions of c_n, d_n, p_n and q_n, we see that

$$\sum_{n=n_0}^{\infty}\frac{q_n}{(1+p_{n-\sigma+\tau})^{\gamma}}=\sum_{n=0}^{\infty}\frac{(n+2)^3}{(n+1)^2}\frac{1}{(1+\frac{1}{n+1})^3}=\sum_{n=0}^{\infty}(n+1)=\infty.$$

Then (h_4) holds. Also

$$\begin{aligned}
&\sum_{n_0}^{\infty}\frac{1}{d_n}\left[\sum_{n_0}^{n-1}\frac{1}{c_t}\sum_{n_0}^{t-1}\frac{q_s}{(1+p_{s-\sigma+\tau})^{\gamma}}\right]^{\frac{1}{\gamma}}\\
=&\sum_{n=0}^{\infty}\frac{1}{\sqrt[3]{n}}\left[\sum_{t=0}^{n-1}\frac{1}{t}\sum_{s=0}^{t-1}(s+1)\right]^{\frac{1}{3}}=\sum_{n=0}^{\infty}\frac{1}{\sqrt[3]{n}}\left[\frac{1}{2}\sum_{t=0}^{n-1}(t+1)\right]^{\frac{1}{3}}\\
=&\sum_{n=0}^{\infty}\frac{1}{\sqrt[3]{n}}\left[\frac{1}{4}n(n+1)\right]^{\frac{1}{3}}=\sqrt[3]{\frac{1}{4}}\sum_{n=0}^{\infty}\sqrt[3]{n+1}=\infty.
\end{aligned}$$

and this proves (h_5). Now, by choosing $\rho_n=n$ and $\lambda=2$, and $Q_n=Kq_n(1-p_{n-\sigma})^{\gamma}$, we have

$$\begin{aligned}
&\lim_{m\to\infty}\sup\frac{1}{m^{\lambda}}\sum_{n=0}^{m-1}(m-n)^{\lambda}\left(\rho_nQ_n-\frac{(\rho_{n+1})^2\left(\frac{\Delta\rho_n}{\rho_{n+1}}-\frac{\lambda(m-n-1)^{\lambda-1}}{(m-n)^{\lambda}}\right)^2}{4\bar{\rho}_n}\right)\\
=&\lim_{m\to\infty}\sup\frac{1}{m^2}\sum_{n=1}^{m-1}(m-n)^2\times\left(n\frac{(n+2)^3}{(n+1)^2}\left(1-\frac{1}{n}\right)^2-\frac{\left(\frac{1}{n+1}-\frac{2(m-n-1)}{(m-n)^2}\right)^2}{n(n-1)}\right)\\
\geq&\lim_{m\to\infty}\sup\frac{1}{m^2}\sum_{n=0}^{m-1}(m-n)^2\\
&\times\left[\frac{1}{n}(n+2)^3\frac{(n-1)^2}{(n+1)^2}-\frac{(m^2-4mn+3n^2-2m+4n+2)^2}{(n+1)^2(-m+n)^4n(n-1)}\right]\\
\geq&\lim_{m\to\infty}\sup\frac{1}{m^2}\sum_{n=0}^{m-1}(m-n)^2\left[(n-1)^2-\frac{(m^2-4mn+3n^2-2m+4n+2)^2}{(n+1)^2(-m+n)^4n(n-1)}\right]
\end{aligned}$$

$$
\begin{aligned}
&\geq \limsup_{m\to\infty} \frac{1}{m^2}\sum_{n=0}^{m-1}(m-n)^2\left[(n-1)^2-\frac{\left(m^2-4mn+3n^2-2m+4n+2\right)^2}{\left(n+1\right)^2\left(-m+n\right)^4 n\left(n-1\right)}\right]\\
&\geq \limsup_{m\to\infty} \frac{1}{m^2}\sum_{n=0}^{m-1}(m-n)^2 n -\\
&\quad -\limsup_{m\to\infty} \frac{1}{m^2}\sum_{n=0}^{m-1}\frac{\left(m^2-4mn+3n^2-2m+4n+2\right)^2}{\left(n+1\right)^2\left(m-n\right)^2 n\left(n-1\right)}\\
&= \limsup_{m\to\infty} \frac{1}{m^2}(\frac{1}{12}m^4-\frac{1}{12}m^2)\\
&\quad -\limsup_{m\to\infty} \frac{1}{m^2}\sum_{n=0}^{m-1}\frac{\left(m^2-4mn+3n^2-2m+4n+2\right)^2}{\left(n+1\right)^2\left(m-n\right)^2 n\left(n-1\right)}\\
&= \infty.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\limsup_{m\to\infty} \frac{1}{m^2}\sum_{n=0}^{m-1}\frac{\left(m^2-4mn+3n^2-2m+4n+2\right)^2}{\left(n+1\right)^2\left(m-n\right)^2 n\left(n-1\right)}\\
&\leq \limsup_{m\to\infty} \frac{1}{m^2}\sum_{n=0}^{m-1}\frac{(m-n)^4+4(m-n)^2+4(n+1)^4}{\left(n+1\right)^2\left(m-n\right)^2 n\left(n-1\right)}\\
&= \limsup_{m\to\infty} \frac{1}{m^2}\sum_{n=0}^{m-1}\left[\frac{(m-n)^2}{\left(n+1\right)^2 n\left(n-1\right)}+\frac{4}{\left(n+1\right)^2 n\left(n-1\right)}\right]\\
&+\limsup_{m\to\infty} \frac{1}{m^2}\sum_{n=0}^{m-1}\left[\frac{4(n+1)^4}{\left(n+1\right)^2\left(m-n\right)^2 n\left(n-1\right)}\right]<\infty.
\end{aligned}
$$

Then (3.13) holds. It follows from Theorem 3.2 that every solution of Eq.(4.2) is oscillatory or satisfies $\lim_{n\to\infty}(x_n+\frac{1}{n+2}x_{n-1})=0$.

Open Problem.1: It would be interesting to study the oscillation behavior of Eq.(1.1) when

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{c_n}\right)<\infty \quad \textit{and} \quad \sum_{n=n_0}^{\infty}\left(\frac{1}{d_n}\right)<\infty.$$

Open Problem 2: It would be interesting to study the oscillation behavior of Eq.(1.1) when $0<\gamma\leq 1$ and

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{c_n}\right)=\sum_{n=n_0}^{\infty}\left(\frac{1}{d_n}\right)=\infty,$$

or

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{c_n}\right)<\infty \quad \textit{and} \quad \sum_{n=n_0}^{\infty}\left(\frac{1}{d_n}\right)<\infty.$$

Acknowledgment: *The author thanks the referee for his/her helpful suggestions.*

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