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ON THE GLOBAL STRUCTURE OF A CLASS OF DYNAMIC SYSTEMS

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ABSTRACT. For a class of dynamic systems defined in the positive cone of $\mathbb{R}^n$ we prove the blow up of all solutions starting strictly above the unit point. Explicit conditions are obtained for blowing up in finite or infinite time, respectively. In 3 dimensions visualizations are presented for the numerically approximated interface separating the domains of attraction of 0 and ∞, respectively.

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1. INTRODUCTION

In this note we will consider dynamic systems

$$\dot{x} = f(x) \qquad (\dot{x} = \frac{d}{dt}x) \tag{1.1}$$

in the open positive cone $\mathbb{R}^n_+ = \{x = (x_i) \in \mathbb{R}^n | \; 0 < x_i \text{ for all } i = 1, \cdots, n\}$. The vector functions $f = (f_i)$ taken into account are

$$f_i(x) = \psi_i\Big(\prod_{j \neq i} |x_j|^{\alpha_{ij}} - |x_i|^{\gamma_i}\Big) \cdot g_i(x), \tag{1.2}$$

$\psi_i : \mathbb{R} \to \mathbb{R}$ being continuous, odd, strictly increasing, $\psi_i(0) = 0, g_i$ being continuous on a neighbourhood of the closed cone $\overline{\mathbb{R}^n_+}, g_i(x) \in \mathbb{R}^1_+$, and $\alpha_{ij} > 0$ for $i \neq j, \alpha_{ii} = 0, \gamma_i > 0$ denoting constants, where $\det(\delta_{ij}\gamma_j - \alpha_{ij}) \neq 0$.

The system (1.1) with f from (1.2) is governed by geometric properties of the matrix $\alpha = (\delta_{ij}\gamma_j - \alpha_{ij})$. Namely, if there exists a vector $A \in \mathbb{R}^n$, A fulfilling $\alpha \cdot A = \delta \in \mathbb{R}^n_+$ (or equivalently, α being M-matrix [Berman & Plemmons 1979]), then all solutions of (1.1) starting in $\mathbb{R}^n_+$ at $t = 0$ exist globally for all $t \geq 0$ and have the limit set $\{E\}, E = (1, \cdots, 1)^T$. Thus the unit point E (being the unique stationary point of (1.1) in $\mathbb{R}^n_+$) is globally attractive and asymptotically stable. We have proved this in [Rautmann 1999] using families of flow invariant rectangles contracting to $\{E\}$.

Otherwise, if α is not M-matrix and if we additionally assume continuous differentiability of ψ_i at 0 as well as of g_i at E for all i, then E will be unstable [Rautmann 2001].

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In the following we will always suppose that there exists a vector $A \in \mathbb{R}^n_+$ fulfilling $\alpha \cdot A = -\delta$ with $\delta \in \mathbb{R}^n_+$. Thus α will not be M-matrix. In this case families of flow invariant cones above E exist [Proposition 2.2, Corollary 2.1 below]. Our main result shows that the open cone above E in $\mathbb{R}^n_+$ belongs to the domain of attraction of ∞ (Theorem 2.1, Corollary 2.3 below). Beyond this, specializing ψ_i and g_i we will establish explicit conditions for blow up in finite or infinite time, respectively, of all solutions of (1.1) starting strictly above E (Theorem 3.1, Corollary 3.1 below).

Similar results hold for the dying out of solutions $x(t)$ of (1.1) starting at any $x(0) < E$ in $\mathbb{R}^n_+$. But, since such a solution possibly will reach the boundary $\partial\mathbb{R}^n_+$ in finite time, where some components f_i eventually are vanishing, in order to prove the flow invariance of rectangles having lower faces on $\partial\mathbb{R}^n_+$, we need an additional uniqueness condition for (1.1). Sufficient are local Lipschitz conditions for ψ_i in $\mathbb{R}$ and for g_i in a neighbourhood of $\overline{\mathbb{R}^n_+}$, together with the requirement $1 \leq \alpha_{ij}$ for $i \neq j, 1 \leq \gamma_i$ for all $i, j = 1, \cdots, n$, (Remarks 2.2 and 3.1 below). However in view of the restricted space for this article, in the following we will mainly concentrate on the blow up of solutions inside $\mathbb{R}^n_+$.

Until now we have studied the global behavior of solutions of (1.1) starting at $x(0) \in \mathbb{R}^n_+$, $x(0)$ being neither above nor below E, merely by numerical methods. In the last section of this paper, my coworker Robert Breitrück will present visualizations of his numerical results. The pictures show the interface separating the domains of attraction of 0 and ∞, respectively, for four different systems (1.1) in $\mathbb{R}^3_+$.

2. BLOW UP IN $Q_1^+ = \{x \in \mathbb{R}^n | E < x\}$

For any point $a = (a_i) \in \mathbb{R}^n_+$ we will denote by $Q_a = \{x \in \mathbb{R}^n | a \leq x\}$ the closed cone with lowest point a. The $n-1$-dimensional faces of Q_a are $Q_{a,i} = \{x \in Q_a | a_i = x_i\}$, $i = 1, \ldots, n$. Here and in the following we always use the partial order of $\mathbb{R}^n$ induced by $x \leq y \Leftrightarrow x_i \leq y_i$, $x < y \Leftrightarrow x_i < y_i$ for all $i = 1, \cdots, n$. We will also write $x \leq c$ if $x_i \leq c \in \mathbb{R}$ for all i.

Proposition 2.1: *Assume the continuous map $f = (f_i) : \mathbb{R}^n_+ \to \mathbb{R}^n$ fulfills the direction condition*

$$f_i|_{Q_{a,i}} > 0 \tag{2.1}$$

for all $i = 1, \ldots, n$. Then Q_a is flow invariant for the differential equation

$$\dot{x} = f(x) \qquad (\dot{x} = \frac{d}{dt}x)\,, \tag{2.2}$$

i.e. each solution $x(t)$ of (2.2) starting at any point $x(0) \in Q_a$ will remain in Q_a for all t of its right maximal interval $[0, T)$ of existence.

Note: Since (2.1) is excluding $f_i(x) = 0$ for each $x \in Q_{a,i}$, the flow invariance of Q_a does not require any uniqueness condition for (2.2).

Proof. By contradiction: Let $x(t)$ for $t \in [0, t_1]$ be a solution of (2.2) with $x(0) \in Q_a, x(t_1) \notin Q_a$. Consequently there exists $t^* = \sup\{t \in [0, t_1] \,|\, x(t) \in Q_a\} < t_1$, and we find $x(t^*) \in \partial Q_a = \bigcup_{i=1}^n Q_{a,i}$. We consider the set $J = \{i \,|\, x_i(t^*) = a_i\} \neq \emptyset$ of indices i and its complement $J' = \{1, \cdots, n\} - J$. Because of (2.1) we have $0 < \epsilon_1 = \min_{i \in J}\{f_i(x(t^*))\}$, in addition $0 < \epsilon_2 = \min_{j \in J'}\{x_j(t^*) - a_j\}$, since $x(t^*) \in \partial Q_a$. From the continuity of $x(t)$ and $f(x)$ we see that there exists $\delta \in (0, t_1 - t^*]$ such that (a) $\min_{i \in J}\{f_i(x(t))\} \geq \frac{\epsilon_1}{2}$ and (b) $\min_{j \in J'}\{x_j(t) - a_j\} \geq \frac{\epsilon_2}{2}$ hold for all $t \in [t^*, t^* + \delta]$. But then from (b) and the consequence of (a), namely $x_i(t) - a_i = \int_{t^*}^t f_i(x(\tau))d\tau > 0$ for all $t \in (t^*, t^* + \delta]$ and all $i \in J$ we conclude $x(t) \in Q_a$ for $t \in [t^*, t^* + \delta]$ which contradicts the definition of t^*. □

Proposition 2.2: *For all* $i = 1, \cdots, n, n \geq 2$, *assume* $f_i(x) = \psi_i\left(\prod_{j \neq i} |x_j|^{\alpha_{ij}} - |x_i|^{\gamma_i}\right) \cdot g_i(x)$, *where (1)* $\psi_i : \mathbb{R} \to \mathbb{R}$ *continuous, odd, strictly monotone increasing,* $\psi_i(0) = 0$, *(2)* $g_i : \mathbb{R}_+^n \to \mathbb{R}_+^1$ *continuous, (3)* $\alpha = (\delta_{ij}\gamma_j - \alpha_{ij}), \det \alpha \neq 0$, *(4)* $\alpha_{ii} = 0 < \alpha_{ij}$, $i \neq j$, $0 < \gamma_i$, *(5)* $\exists A = (A_i) \in \mathbb{R}_+^n$, $\alpha \cdot A = -\delta$, $\delta = (\delta_i) \in \mathbb{R}_+^n$.
Then, with $a_s^+ = (s^{A_i}) \in \mathbb{R}_+^n$, $1 < s$, *each set* $Q_s^+ = \{x \in \mathbb{R}_+^n \big| a_s^+ \leq x\}$ *is flow invariant with respect to (2.2).*

Proof. The first factor $F_i(x) = \psi_i(\prod_{j \neq i} |x_j|^{\alpha_{ij}} - |x_i|^{\gamma_i})$ in $f_i(x)$ fulfills $0 < b_s \leq F_i(a_s^+)$ with a bound b_s independent of i. From $F = (F_i)$ being quasimonotone increasing we see

$$b_s \leq F_i(a_s^+) \leq F_i(x), \tag{2.4}$$

thus $0 < f_i(x)$ for all $x \in Q_{s,i}^+$ and all $i = 1, \cdots, n$, because of our assumption $0 < g_i(x)$. Therefore the flow invariance of Q_s^+ follows by Proposition 2.1. □

Remark 2.1 *Under the sharper assumption (2.5)* $0 < c_0 \leq g_i(x)$ *with a constant* c_0, *from (2.4) we find*

$$0 < c_s \leq f_i(x) \text{ with } c_s = c_0 b_s \text{ for all } x \in Q_{s,i}^+ \text{ and all } i. \tag{2.6}$$

In the following we will turn to some subsets of $\mathbb{R}_+^n$ which have evident geometric relations to the cones $Q_s^+ = \{x \in \mathbb{R}_+^n | a_s^+ \leq x\}$ where $a_s^+ = (s^{A_i})$. Assume $s > 1$: For $\epsilon = (\epsilon_i) \in \mathbb{R}_+^n$, $\epsilon_i < s^{A_i} - 1$ we will consider the **ϵ-neighbourhood of Q_s^+:** $(Q_s^+)^\epsilon = \{x \in \mathbb{R}_+^n | s^{A_i} - \epsilon_i \leq x_i \text{ for all } i = 1, \cdots, n\}$, **$\epsilon$-retract of Q_s^+:**

$$(Q_s^+)^{-\epsilon} = \{x \in \mathbb{R}_+^n | s^{A_i} + \epsilon_i \leq x_i \text{ for all } i = 1, \cdots, n\},$$

η-cone near $Q_s^+ : \eta = (\eta_i) \in \mathbb{R}^n, |\eta_i| < s^{A_i} - 1,$

$$(Q_s^+)^\eta = \{x \in \mathbb{R}_+^n | s^{A_i} - \eta_i \leq x_i \text{ for all } i = 1, \cdots, n\},$$

and the **ϵ-neighbourhood of the i^{th} $n-1$-dimensional face $Q_{s,i}^+$ of** Q_s^+ :

$$(Q_{s,i}^+)^\epsilon = \{x \in (Q_s^+)^\epsilon | \; x_i \leq s^{A_i} + \epsilon_i\}.$$

Remark 2.2: *The rectangles* $Q_s^- = \{x \in \mathbb{R}^n | 0 \le x \le a_s^-\}$ *in* $\overline{\mathbb{R}^n}_+$ *with the upper corner* $a_s^- = (s^{-A_i}), s > 1$, *have the lower faces* $Q_{s,i}^{-;-} = \{x \in Q_s^- | x_i = 0\}$ *in addition to the upper faces* $Q_{s,i}^- = \{x \in Q_s^- | x_i = a_{s,i}^-\}$. *Because of* $F_i(0) = 0$, *the quasimonotonicity of* F_i *gives* $0 \le F_i(x)$, *thus* $0 \le f_i(x)$ *for* $x \in Q_{s,i}^{-;-}$. *Similarly from* $F_i(a_s^-) \le -b_s$ *we get* $f_i(x) < 0$ *for* $x \in Q_{s,i}^-$. *Therefore by inspection of the outer normals* N_x *in any point* x *of the* k*-dimensional edges of* $Q_s^-, k = 0, 1, \cdots, n-1$, *we see that the condition* $N_x \cdot f(x) \le 0$ *holds for the inner product of* N_x *with* $f(x)$ *in all points* $x \in \partial Q_s^-$. *Consequently, if we additionally require a local Lipschitz condition for* f *on a neighbourhood of* $\overline{\mathbb{R}^n_+}$, *we can apply Bony's theorem [Redheffer 1972] which gives us the flow invariance of* Q_s^- *for (2.2).*

Remark 2.3: *For all* $s > 1, \epsilon \in \mathbb{R}^n_+, \epsilon_i < s^{A_i} - 1$ *there exist* $\sigma > s$ *and* $\tau \in (1, s)$ *such that*
(a) $(Q_s^+)^{-\epsilon} \subset Q_\sigma^+ \subset Q_s^+$ *and* (b) $Q_\sigma^+ \subset (Q_s^+)^\epsilon \subset Q_\tau^+$ *hold.*
In the following, suppose additionally $(2b)\ 0 < c_0 \le g_i(x)$,

Then with the assumptions of the last Proposition 2.2, the following Corollaries hold:

Corollary 2.1: $\forall s > 1\ \exists\ \delta_0 > 0$, $\forall \epsilon = (\epsilon_i) \in \mathbb{R}^n_+$*:*

$\epsilon_i \le \delta_0 \Rightarrow f_{i|(Q_{s,i}^+)^\epsilon} \ge \frac{c_s}{2} > 0$, $\forall i$, *thus* $(Q_s^+)^\eta$ *being flow invariant for* (2.2), $\forall \eta \in \mathbb{R}^n$, $|\eta_j| \le \epsilon_j\ \forall_j$.

Proof. We take $\epsilon = (\epsilon_i)$, $0 < \epsilon_i \le \epsilon_0$. In order to find a positive lower bound for the restriction $f_i|(Q_{s,i}^+)^\epsilon$ of f_i we project each point $x \in (Q_{s,i}^+)^\epsilon$ on the point $x(i) \le x$, $x(i)$ having the coordinates $x(i)_i = x_i, x(i)_j = a_{s,j}^+ - \epsilon_j$ for $j \ne i$. From this we see $|x(i) - (a_s^+ - \epsilon)| \le 2\epsilon_i$, and the quasimonotonicity of $F = (F_i)$ gives $F_i(x(i)) \le F_i(x)$ for all i. Choosing ϵ_0 small enough such that

$$|F_i(a_s^+ + y) - F_i(a_s^+ + z)| \le \frac{b_s}{4} \tag{2.7}$$

for all $y, z \in \mathbb{R}^n, |y_i|, |z_i| \le 2\epsilon_0$, and all i, from (2.4) we find

$$\frac{b_s}{2} \le F_i(a_s^+) - |F_i(a_s^+) - F_i(a_s^+ - \epsilon)| - |F_i(a_s^+ - \epsilon) - F_i(x(i))| \le F_i(x),$$

therefore $\frac{c_s}{2} \le f_i(x)$ by (2.b) for all $x \in (Q_{s,i}^+)^\epsilon$.

Since each $n-1$-dimensional face $(Q_{s,i}^+)^\eta$ of any cone $(Q_s^+)^\eta$ near Q_s^+, $|\eta_i| \le \epsilon_i$ for all i, is subset of $(Q_{s,i}^+)^\epsilon$, the inequality $\frac{c_s}{2} \le f_i|_{(Q_{s,i}^+)^\eta}$ follows for all i. Recalling Proposition 2.1 we find the flow invariance for (2.2) of all cones $(Q_s^+)^\eta, |\eta_i| \le \epsilon_i$ for all i. □

Corollary 2.2: *Let* $x(t)$ *for* $t \in (0, \infty)$ *denote a solution of (2.2). Then* $\forall s > 1\ \ \exists$ $\delta(s) \in \mathbb{R}^n_+$, $\exists\ \tau > 0$, *such that*

$$x(t_0) \in (Q_s^+)^{\delta(s)} \Rightarrow x(t_0 + \tau) \in (Q_s^+)^{-\delta(s)}.$$

Proof. Let $x(t)$ for $t \in [0, \infty)$ denote a solution of (2.2). By definition of the cones near Q_s^+, the statement $x(t) \in (Q_s^+)^\eta$ means that for each $i = 1, \cdots, n$

$$\text{either (a)} \quad a_{s,i}^+ + \eta_i \leq x_i(t) \text{ or (b)} \quad a_{s,i}^+ - \eta_i \leq x_i(t) < a_{s,i}^+ + \eta_i$$

holds. Taking $\eta \in (0, \delta_0)$ we find from Corollary 2.1 that inequality (a) is flow invariant for (2.2): If it holds for any $t_0 \geq 0$ then it holds for all $t \geq t_0$, too. If additionally we require $2\eta \leq \tau \cdot \frac{c_s}{2}$, we conclude from (b) for $t = t_0 : a_{s,i}^+ + \eta_i \leq a_{s,i}^+ - \eta_i + \tau \frac{c_s}{2} \leq x_i(t_0 + \tau)$ for any $i = 1, \cdots, n$, thus $x(t_0 + \tau) \in (Q_s^+)^{-\eta}$. ☐

Theorem 2.1: *For all $i = 1, \ldots, n$, $n \geq 2$, assume $f_i(x) = \psi_i(\prod_{j \neq i} |x_j|^{\alpha_{ij}} - |x_i|^{\gamma_i}) \cdot g_i(x)$, where (1) $\psi_i : \mathbb{R} \to \mathbb{R}$ continuous, odd, strictly monotone increasing, $\psi_i(0) = 0$, (2) $g_i : \mathbb{R}_+^n \to \mathbb{R}_+^1$ continuous, (2 b) $0 < c_0 \leq g_i(x)$, (3) $\alpha = (\delta_{ij}\gamma_j - \alpha_{ij})$, $\det \alpha \neq 0$, (4) $\alpha_{ii} = 0 < \alpha_{ij}$, $i \neq j$, $0 < \gamma_i$, (5) $\exists\ A = (A_i) \in \mathbb{R}_+^n$, $\alpha \cdot A = -\delta$, $\delta = (\delta_i) \in \mathbb{R}_+^n$. Let $x(t)$ denote any solution of (2.2), $[0, T)$ being its right maximal existence interval, $E < x(0)$.*

Then either (i): With $t \to T \leq \infty$, $x(t)$ is entering each Q_s^+, or (ii): $s^* = \sup\{s > 1,\ \exists t = t_s, x(t_s) \in Q_s^+\} < \infty$ holds, and $|x(t_s)| \to \infty$ with $s \to s^*$ (thus $t_s \to T$).

An evident consequence of Theorem 2.1 is the following

Corollary 2.3: *The open cone $Q_1^+ = \{x = (x_i) \in \mathbb{R}_+^n \big| 1 < x_i, \forall_i\}$ belongs to the domain of attraction of ∞.*

Proof. of Theorem 2.1: The supremum s^* is well defined since $x(t) \in Q_s^+$ holds for some $t \geq 0$, $s > 1$ because of $x(0) > E$. (a) In case $s^* = \infty$ we have (i). (b) Otherwise in case $s^* < \infty$, there exist a strictly increasing sequence $(s_k) \uparrow s^*$ and a sequence $(t_k) \subset [0, T)$ with $x(t_k) \in Q_{s_k}^+$, where the sequence $(Q_{s_k}^+)$ is contracting to $Q_{s^*}^+$. (b1) If $(x(t_k))$ does not contain a bounded subsequence, we have (ii). (b2) Otherwise there would exist a bounded subsequence $(x(t_{k'})) \subset (x(t_k)), |x(t_{k'})| \leq M < \infty$. But then $(x(t_{k'}))$ contains a convergent subsequence $(x(t_{k''})) \longrightarrow \tilde{x}$, thus $\tilde{x} \in \partial Q_{s^*}^+ = \bigcup_{i=1}^n Q_{s^*,i}^+$. Without loss of generality we may assume that the related sequences $(t_k), (t_{k''})$ are monoton increasing, because each Q_s^+ is flow invariant for (2.2). (b21) In case $(t_{k''})$ being bounded there exists $t^* = \lim\limits_{k'' \to \infty} t_{k''} \in (0, T]$. Thus by the extension theorem for ordinary differential equations [Hartman 1964, Lemma 3.1], the solution $x(t)$ can be extended to $[0, t^*]$ with $x(t^*) = \tilde{x}$, and subsequently to $[0, t^* + \tau)$ for some $\tau > 0$ by the local existence theorem. However, as shown in Corollary 2.1, the vector $f(\tilde{x})$ is pointing strictly inwards to $\overset{\circ}{Q}{}_{s^*}^{+} = \bigcup_{s > s^*} Q_s^+$. From this we get $x(t) \in Q_s^+$ for some $s > s^*, t > t^*$ in contradiction to the definition of s^*. (b22) Otherwise the sequence $(t_{k''})$ is unbounded: We have $(t_{k''}) \uparrow \infty = T$. Because of $Q_{s_k}^+ \downarrow Q_{s^*}^+$, for each $\eta \in \mathbb{R}^n_+$ there exists a $k_\eta \in \mathbb{N}$ such that $Q^+_{s_{k''}} \subset (Q^+_{s^*})^\eta$ holds for all $k'' \geq k_\eta$. Choosing $\eta \in (0, \delta(s^*)]$ and recalling Corollary 2.2, from $x(t_{k''}) \in (Q^+_{s^*})^\eta$ we find $x(t_{k''} + \tau) \in (Q^+_{s^*})^{-\eta} \subset \bigcup_{s>s^*} Q^+_s$, which again contradicts the definition of s^*. $\square$

3. LOWER AND UPPER BOUNDS FOR SOLUTIONS IN Q^+_1

In order to find sub-as well as superfunctions $v(t)$ to any solution $x(t)$ of (2.2) with $x(0) > E$, recalling Theorem 2.1 we will represent the parameter $s > 1$ in $a^+_s = (s^{A_i})$ as a function $s = \varphi(t)$ of time $t \geq 0$. Suitable functions φ will be constructed by the methods of differential inequalities. The following systems are cooperative in the sense of [Hirsch 1982, Smith 1995].

Theorem 3.1: *For all $i = 1, \ldots, n, n \geq 2$, assume $f_i(x) = \left(\prod_{j \neq i} |x_j|^{\alpha_{ij}} - |x_i|^{\gamma_i}\right) \cdot |x_i|^{\zeta_i}$, where $\alpha = (\delta_{ij}\gamma_j - \alpha_{ij})$, $\det \alpha \neq 0$, $\alpha_{ii} = 0 < \alpha_{ij}, i \neq j$, $0 < \gamma_i, 0 \leq \zeta_i$, $\exists A = (A_i) \in \mathbb{R}^n_+$, $\alpha \cdot A = -\delta$, $\delta = (\delta_i) \in \mathbb{R}^n_+$.*

Then for any solution $x(t)$ of (2.2) with $x(0) \in Q^+_{s_0}$, $1 < s_0$, (1) each solution $\varphi(t) \in \mathbb{R}^1_+$ of

$$\dot{\varphi} \leq \varphi^{1+\delta_i + A_i(\gamma_i + \zeta_i - 1)} \cdot \frac{1 - \varphi^{-\delta_i}}{A_i}, \quad \forall i, \quad \varphi(0) \leq ((x_i(0))^{1/A_i}), \tag{3.1}$$

gives a lower bound $a^+(\varphi(t)) \leq x(t)$, and (2) each solution $\psi(t) \in \mathbb{R}^1_+$ of

$$\dot{\psi} \geq \psi^{1+\delta_i + A_i \cdot (\gamma_i + \zeta_i - 1)} \cdot \frac{1 - \varphi^{-\delta_i}}{A_i}, \quad \forall i, \quad \psi(0) \geq ((x_i(0))^{1/A_i}) \tag{3.2}$$

gives an upper bound $x(t) \leq a^+(\psi(t))$. The bounds hold on the right maximal interval, on which $x(t)$ and $\varphi(t)$ or $x(t)$ and $\psi(t)$ exist, respectively.

Proof. The definition

$$v(t) = (v_i) = ((\varphi(t))^{A_i}) \tag{3.3}$$

gives

$$\dot{v}_i = A_i \varphi^{A_i - 1} \cdot \dot{\varphi} \quad \text{and} \quad f_i(v) = \varphi^{A_i \cdot (\gamma_i + \zeta_i)} \cdot (\varphi^{\delta_i} - 1). \tag{3.4}$$

Since the direction field f in Theorem 3.1 is quasimonotone increasing and locally Lipschitz continuous in $\mathbb{R}^n_+$ we will get a subfunction $v(t) \leq x(t)$, if

$$\dot{v}_i \leq f_i(v) \quad \text{and} \quad v_i(0) \leq x_i(0) \tag{3.5}$$

holds for all $i = 1, \cdots, n$, c.p. [Walter 1970].

A short calculation shows that (3.4) and (3.5) are fulfilled, if we compute φ from the initial value problem

$$\dot{\varphi} = c_1 \varphi^{1+c_2}, \;\; \varphi(0) = c_0, \tag{3.6}$$

where

$$c_0 = \min_i\{(x_i(0))^{1/A_i}\},\ c_1 = \min_i\{\frac{1-\varphi(0)^{-\delta_i}}{A_i}\},$$
$$c_2 = \min_i\{\delta_i - A_i[1-(\gamma_i+\zeta_i)]\}. \tag{3.7}$$

The equality signs in (3.5), (3.6) and (3.7) are allowed, since the functions f_i in Theorem 3.1 obey a local Lipschitz conditions inside of $\mathbb{R}^n_+$, cp. [Walter1970]. The estimate $v(t) \leq x(t)$ will hold in the right maximal interval $[0, T)$, on which $v(t)$ and $x(t)$ both exist.

Similary, if we solve (3.7) but now with

$$c_0 = \max_i\{(x_i(0))^{1/A_i}\},\ \ c_1 = (\min_i\{A_i\})^{-1},$$
$$c_2 = \max_i\{\delta_i - A_i(1-(\gamma_1+\zeta_i))\}, \tag{3.8}$$

we get a superfunction $v(t) = (a_i^+(\varphi(t))) \geq x(t)$ on the right maximal interval $[0, T)$, on which $v(t)$ and $x(t)$ both exist. □

From the special size of the lower and upper bounds above we get

Corollary 3.1: *All solutions of (3.1) starting in Q_1^+ blow up to ∞ componentwise, each approximating ∞ (a) in finite time t, if $\frac{\delta_i}{A_i} > 1-\gamma_i-\zeta_i$, or (b) in infinite time only, if $\frac{\delta_i}{A_i} \leq 1-\gamma_i-\zeta_i$ holds for all $i = 1, \cdots, n$, respectively.*

Proof. Since the solution φ of (3.6) blows up to ∞, doing this in finite time if and only if $0 < c_2$, from (3.7) or (3.8), respectively, we find Corollary 3.1. □

Remark 3.1: *Quite similar to the proof of Theorem 3.1 we get upper bounds $v(t) = ((\varphi(t))^{-A_i})$ for a solution $x(t)$ of (2.2) in $Q_1^- = \{x \in \mathbb{R}^n | 0 \leq x < E\}$ if we compute φ from (3.6) with*

$$c_0 = \min_i\{(x_i(0))^{-\frac{1}{A_i}}\} - \epsilon_0,\ c_1 = \min_i\{\frac{1-\varphi(0)^{-\delta_i}}{A_i}\} - \epsilon_1,$$
$$c_2 = \min_i\{A_i(1-[\gamma_i+\zeta_i])\} - \epsilon_2 \tag{3.9}$$

where $0 < \epsilon_j$, and $v(t)$ will become a lower bound of $x(t)$, if we take φ from (3.6) with

$$c_0 = \max_i\{(x_i(0))^{-\frac{1}{A_i}}\} + \epsilon_0,\ \ c_1 = \max_i\{A_i^{-1}\},$$
$$c_2 = \max_i\{A_i(1-[\gamma_i+\zeta_i])\} + \epsilon_2. \tag{3.10}$$

The requirement $\epsilon_j > 0$ for all $j = 0, 1, 2$ is due to the fact that the direction field f in Theorem 3.1 possibly does not fulfill a uniqueness condition on $\partial\mathbb{R}^n_+$. Since $\varphi(t)$ from (3.4) blows up to ∞, approximating ∞ in finite time t if and only if $0 < c_2$, from (3.9) or (3.10), respectively, we conclude: All solutions $x(t)$ of (2.2) with $x(0) < E$,

f from Theorem 3.1, go to zero, each vanishing (a) in finite time, if $0 < 1 - \gamma_i - \zeta_i$, and (b) in infinite time only, if $1 - \gamma_i - \zeta_i < 0$ holds for all $i = 1, \cdots, n$, respectively.

4. VISUALIZATION OF SEPARATING INTERFACES IN $\mathbb{R}^3_+$

The results above relate to solutions $x(t)$ of (2.2) in $\mathbb{R}^n_+$ starting at any point $x(0) \in Q_1^+ \cup Q_1^-$. Beyond it in $\mathbb{R}^3_+$, numerical experiments show the interface S separating the domains of attraction of 0 and ∞, respectively, for solutions of (2.2) with any direction field from Theorem 3.1: We approximate S by subsets S_k of subcubes taken from successive subdivisions of a suitable cube $S_0 \subset \mathbb{R}^3_+$, where $E \in S_0$. Any subcube Q of the k^{th} subdivision is put into S_k if and only if Q contains starting points of solutions of (2.2) entering Q_1^+ as well as starting points of solutions entering Q_1^-.

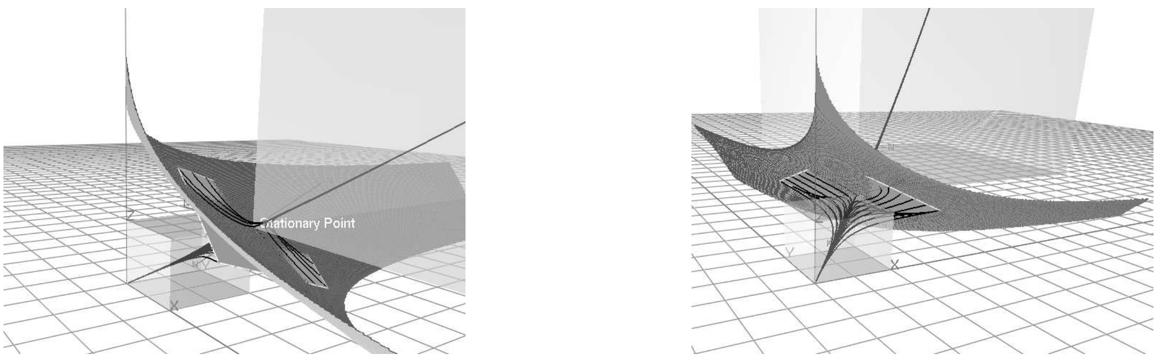

FIGURE 1. $\alpha_{12} = 0.2,\ \alpha_{13} = 3.6,\ \alpha_{21} = 2,\ \alpha_{23} = 2,\ \alpha_{31} = 2.2,\ \alpha_{32} = 2,\ \gamma_1 = \gamma_2 = \gamma_3 = 2,\ \zeta_1 = \zeta_2 = \zeta_3 = 0.3$

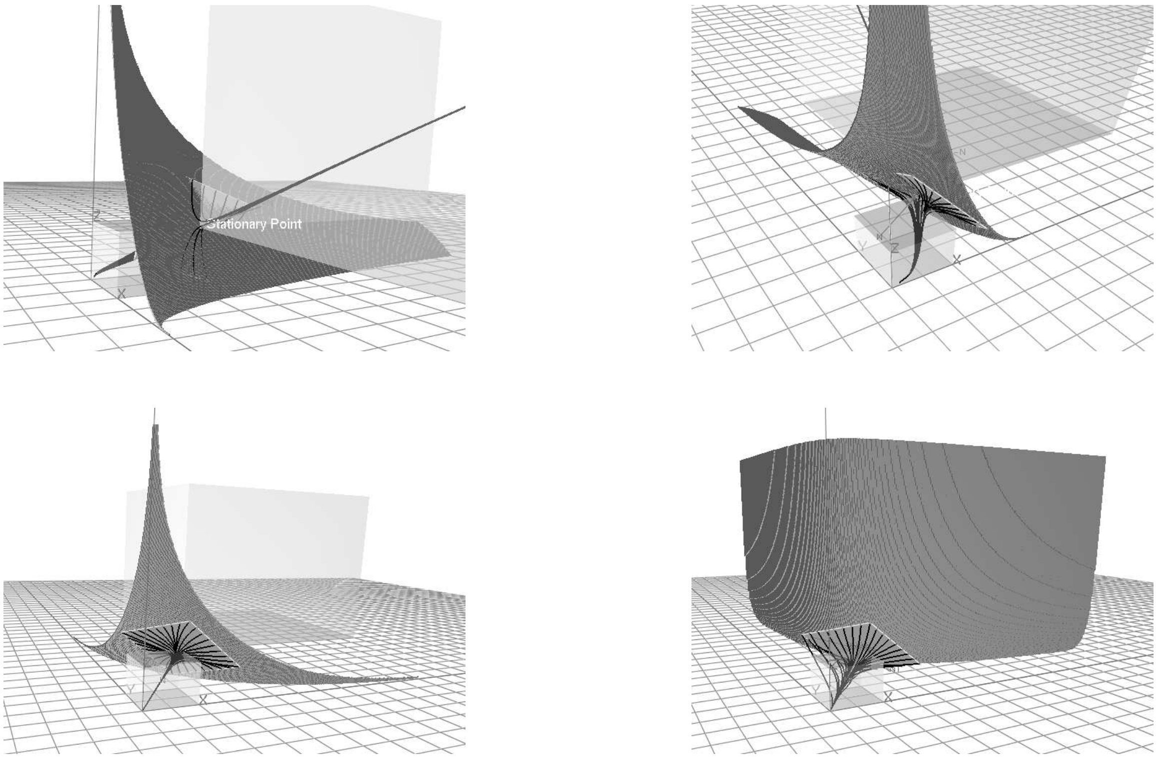

FIGURE 2. $\alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{23} = \alpha_{31} = \alpha_{32} = 1,\ \gamma_1 = \gamma_2 = \gamma_3 = 1,\ \zeta_1 = \zeta_2 = \zeta_3 = 0\ (left),\ \zeta_1 = \zeta_2 = \zeta_3 = 1\ (right)$

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