Dynamic Systems and Applications 16 (2007) 1-12

A LEFSCHETZ FIXED POINT THEOREM FOR ADMISSIBLE MAPS IN FRÉCHET SPACES

RAVI P. AGARWAL AND DONAL O'REGAN

Department of Mathematical Sciences, Florida Institute of Technology,
Melbourne, FL 32901–6975, U.S.A. agarwal@fit.edu
Department of Mathematics, National University of Ireland, Galway, Ireland.
donal.oregan@nuigalway.ie

ABSTRACT. The Lefschetz fixed point theorem is discussed for the admissible maps of Gorniewicz defined on admissible subsets of a Hausdorff topological space. Also using the projective limit approach we present new Lefschetz fixed point theorems for the admissible maps of Gorniewicz defined on PRLF's or CPRLF's.

Keywords and Phrases: Admissible maps, Hausdorff topological space, projective limit approach, Lefschetz fixed point theorem.

AMS Subject Classification: 47H10.

1. INTRODUCTION

This paper has two main sections. In Section 2 we present new Lefschetz fixed point theorems for the admissible maps of Gorniewicz defined on admissible (to be defined later) subsets of a Hausdorff topological space. In Section 3 we present some other Lefschetz fixed point theorems for the admissible maps of Gorniewicz between Fréchet spaces. Our maps will be defined on PRLF's or CPRLF's. These sets are natural in applications in the Fréchet space setting since they include pseudo–open sets. The theory in Section 3 is based on results in Section 2 and on viewing a Fréchet space as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in N}$ (here $N = \{1, 2, \dots\}$).

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii) p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $D(X, Y)$ be the set of all pairs $X \stackrel{p}{\Leftarrow} \Gamma \stackrel{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q). Given two diagrams

Received August 30, 2006

(p,q) and (p',q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\, p^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : \ f^{(n)}(x) = 0 \ \text{ for some } \ n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\, \tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q\, Tr\,(f_q).$$

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$.

With Čech homology functor extended to a category of morphisms (see [7 pp. 364]) we have the following well known results (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

we define the induced map

$$H\,(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$).

Recall the following result [6 pp. 227].

Theorem 1.1. *If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then*

$$(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star.$$

Two morphisms $\phi,\ \psi \in M(X,Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0,1], Y)$ such that $\chi(x,0) = \phi(x)$, $\chi(x,1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0,\ i_1 : X \to X \times [0,1]$ are defined by $i_0(x) = (x,0)$, $i_1(x) = (x,1)$).

Recall the following result [6 pp. 231].

Theorem 1.2. *If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.*

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p,q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p,q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map and
(ii) $q\left(p^{-1}(x)\right) \subset \phi(x)$ for any $x \in X$.

Definition 1.1. A upper semicontinuous compact map $\phi : X \to Y$ is said to be admissible (and we write $\phi \in Ad(X,Y)$) provided there exists a selected pair (p,q) of ϕ.

Definition 1.2. A map $\phi \in Ad(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

If $\phi : X \to X$ is a Lefschetz map, we define the Lefschetz set $\mathbf{\Lambda}\left(\phi\right)$ (or $\mathbf{\Lambda}_X\left(\phi\right)$) by

$$\mathbf{\Lambda}\left(\phi\right) = \left\{ \Lambda(q_\star\, p_\star^{-1}) :\ (p,q) \subset \phi \right\}.$$

Definition 1.3. A Hausdorff topological space X is said to be a Lefschetz space provided every $\phi \in Ad(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies ϕ has a fixed point.

Now let I be a directed set with order $\le$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \le \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha :\ x_\alpha = \pi_{\alpha,\beta}(x_\beta)\ \forall\, \alpha,\ \beta \in I,\ \alpha \le \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [8 pp. 439] $\bigcap_{\alpha \in I} E_\alpha$.)

2. FIXED POINT THEORY

We begin with Hausdorff topological vector spaces. Some of the ideas in this section were motivated from [1, 3, 5].

For our first result we assume X is a subset of a Hausdorff topological vector space E. We say X is NES admissible if for every compact subset K of X and every neighborhood V of zero there exists a continuous function $h_V : K \to E$ such that

(i) $x - h_V(x) \in V$ for all $x \in K$;
(ii) $h_V(K)$ is contained in a subset C of X with C a Lefschetz space; and
(iii) h_V and $i : K \hookrightarrow X$ are homotopic.

Theorem 2.1. *Let E be a Hausdorff topological vector space and let $X \subseteq E$ be NES admissible. If $\phi \in Ad(X, X)$ then*

(i) *ϕ is a Lefschetz map and*
(ii) *if $\mathbf{\Lambda}(\phi) \neq \{0\}$ then ϕ has a fixed point*

i.e. X is a Lefschetz space.

Proof. Let $p, q : \Gamma \to X$ be a pair of maps with $(p, q) \subset \phi$ and let K denote a compact set in which $\phi(X)$ is included. Next let $\mathcal{N}$ be a fundamental system of neighborhoods of the origin 0 in E and $V \in \mathcal{N}$. Now there exists a continuous function $h_V : K \to E$ and a $C \subseteq X$, C a Lefschetz space with $x - h_V(x) \in V$ for all $x \in K$, $h_V(K) \subseteq C$ and $h_V \sim i$ where $i : K \hookrightarrow C$.

Now let $q_V = h_V\, q : \Gamma \to X$. Notice q_V is a compact map, $q_V(\Gamma) \subseteq C$ and $q \sim q_V$; we know since $h_V \sim i$ that there exists a map $\chi : K \times [0,1] \to X$ with $\chi(x, 0) = h_V(x)$ and $\chi(x, 1) = i(x)$, and now let $\Phi(x, t) = \chi(q(x), t)$ for $(x, t) \in \Gamma \times [0, 1]$ (note p is surjective so $p^{-1}(X) = \Gamma$ and so $q : \Gamma \to K$) and note $\Phi(x, 0) = h_V\, q(x) = q_V(x)$ and $\Phi(x, 1) = i(q(x)) = q(x)$.

Let

$$p_V : p^{-1}(C) \to C,\ \overline{q_V} : p^{-1}(C) \to C,\ q'_V : \Gamma \to C$$

denote contractions of the appropriate maps (see also (1.1) on [3 pp. 214]). Note Theorem 1.1 and Theorem 1.2 imply

$$i_\star\, (q'_V)_\star\, p_\star^{-1} = (i\, q'_V)_\star\, p_\star^{-1} = (q_V)_\star\, p_\star^{-1} = q_\star\, p_\star^{-1}$$

since $q \sim q_V$. Also it is easy to see that $(q'_V)_\star\, p_\star^{-1}\, i_\star = (\overline{q_V})_\star\, (p_V)_\star^{-1}$. Notice q_V is a compact map. Lets look at the map $\psi : C \to C$ given by $\psi = \overline{q_V}\, p_V^{-1}$. Notice ψ is an admissible map and hence $(p_V, \overline{q_V}) \subset \psi$. Since C is a Lefschetz space we have that $(\overline{q_V})_\star\, (p_V)_\star^{-1}$ is a Leray endomorphism. Now [3 pp. 214 (see (1.3))] guarantees that $q_\star\, p_\star^{-1}$ is a Leray endomorphism and $\Lambda(q_\star\, p_\star^{-1}) = \Lambda((\overline{q_V})_\star\, (p_V)_\star^{-1})$.

Next assume $\mathbf{\Lambda}(\phi) \neq \{0\}$. Then there exists $(p, q) \subset \phi$ with $\Lambda(q_\star\, p_\star^{-1}) \neq 0$. Also there exists $q_V,\, p_V,\, \overline{q_V}$ as described above with $\Lambda((\overline{q_V})_\star\,(p_V)_\star^{-1}) = \Lambda(q_\star\, p_\star^{-1}) \neq 0$. Thus $\Lambda((\overline{q_V})_\star\,(p_V)_\star^{-1}) \neq 0$ so since C is a Lefschetz space there exists $x_V \in C$ with $x_V \in \overline{q_V}\, p_V^{-1}(x_V)$. Now $x_V = h_V(y_V)$ for some $y_V \in q\, p^{-1}(x_V)$. Now since $q\, p^{-1}(x_V) \in K$ (note $q : \Gamma \to K$) from (i) in the definition above we have $y_V - h_V(y_V) \in V$. Thus $y_V - x_V \in V$. Now since K is compact we may assume without loss of generality that there exists x with $y_V \to x$. Also since $y_V - x_V \in V$ we have $x_V \to x$. This together with $y_V \in q\, p^{-1}(x_V)$ and the upper semicontinuity of $q\, p^{-1}$ (see [5 pp.26]) implies $x \in q\, p^{-1}(x) \subset \phi(x)$ and the proof is complete. ∎

Let X be a subset of a Hausdorff topological vector space E. Let V be a neighborhood of the origin 0 in E. X is said to be NES admissible V-dominated if there exists a NES admissible space X_V and two continuous maps $r_V : X_V \to X$, $s_V : X \to X_V$ such that $x - r_V\, s_V(x) \in V$ for all $x \in X$ and also that $r_V\, s_V \sim Id_X$. X is said to be almost NES admissible dominated if X is NES admissible V-dominated for every neighborhood V of the origin 0 in E.

Essentially the same reasoning as in [3 pp. 219 (see (5.6)] and the ideas in Theorem 2.1 above yields the following result.

Theorem 2.2. *Let X be a subset of a Hausdorff topological vector space E. Also assume X is almost NES admissible dominated. If $\phi \in Ad(X, X)$ then*

(i) *ϕ is a Lefschetz map and*
(ii) *if $\mathbf{\Lambda}(\phi) \neq \{0\}$ then ϕ has a fixed point*

i.e. X is a Lefschetz space.

Next we extend Theorems 2.1 and 2.2 to the case of Hausdorff topological spaces. First we gather together some well known preliminaries. For a subset K of a topological space X, we denote by $Cov_X(K)$ the set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Two multivalued maps $\phi,\, \psi : X \to Y$ are said to be α–close (here and $\alpha \in Cov(Y)$) if for any $x \in X$ there exists $U_x \in \alpha$ such that $\phi(x) \cap U_x \neq \emptyset$ and $\psi(x) \cap U_x \neq \emptyset$. Given a multivalued map $\phi : X \to X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α–fixed point of ϕ if there exists a member $U \in \alpha$ such that $x \in U$ and $\phi(x) \cap U \neq \emptyset$.

The following result can be found in [2 pp. 297].

Theorem 2.3. *Let X be a topological space and $\Phi : X \to C(X)$ a upper semicontinuous map (here $C(X)$ denotes the family of nonempty closed subsets of X). Suppose there exists a cofinal family of coverings $\theta \subseteq Cov_X(\overline{\Phi(X)})$ such that Φ has an α–fixed point for every $\alpha \in \theta$. Then Φ has a fixed point.*

Remark 2.1. From Theorem 2.3 in proving the existence of fixed points in uniform spaces for continuous compact maps it suffices [2 pp. 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : \ x \in A\}$ where U is a member of the uniformity [9 pp. 199] so such refinements form a cofinal family of open covers). For convenience in this paper we will apply Theorem 2.3 only when the space is uniform.

Let X be a subset of a Hausdorff topological space and let X be a uniform space. Then X is said to be Schauder NES admissible if for every compact subset K of X and every open covering $\alpha \in Cov_X(K)$ there exists a continuous function $\pi_\alpha : K \to E$ such that

(i) π_α and $i : K \hookrightarrow X$ are α–close;
(ii) $\pi_\alpha(K)$ is contained in a subset C of X with C a Lefschetz space; and
(iii) π_α and $i : K \hookrightarrow X$ are homotopic.

Theorem 2.4. *Let X be a subset of a Hausdorff topological space and let X be a uniform space. Also suppose X is Schauder NES admissible. If $\phi \in Ad(X,X)$ then*

(i) *ϕ is a Lefschetz map and*
(ii) *if $\mathbf{\Lambda}(\phi) \neq \{0\}$ then ϕ has a fixed point*

i.e. X is a Lefschetz space.

Proof. Let $p, q : \Gamma \to X$ be a pair of maps with $(p,q) \subset \phi$ and let K denote a compact set in which $\phi(X)$ is included. Also let $\alpha \in Cov_X(K)$. Then there exists a continuous function $\pi_\alpha : K \to E$, a subset C of X, C a Lefschetz space, $\pi_\alpha(K) \subseteq C$, π_α and $i : K \hookrightarrow X$ are α–close and $\pi_\alpha \sim i$. Let $q_\alpha = \pi_\alpha \, q : \Gamma \to X$. Notice as in Theorem 2.1, q_α is a compact map, $q_\alpha(\Gamma) \subseteq C$ and $q \sim q_\alpha$. Let

$$p_\alpha : p^{-1}(C) \to C, \ \overline{q_\alpha} : p^{-1}(C) \to C, \ q'_\alpha : \Gamma \to C$$

denote contractions of the appropriate maps and as in Theorem 2.1 we have

$$i_\star \, (q'_\alpha)_\star \, p_\star^{-1} = q_\star \, p_\star^{-1} \ \text{ and } \ (q'_\alpha)_\star \, p_\star^{-1} \, i_\star = (\overline{q_\alpha})_\star \, (p_\alpha)_\star^{-1}.$$

Lets look at the map $\psi : C \to C$ given by $\psi = \overline{q_\alpha} \, p_\alpha^{-1}$. Notice ψ is an admissible map and hence $(p_\alpha, \overline{q_\alpha}) \subset \psi$. Since C is a Lefschetz space we have that $(\overline{q_\alpha})_\star \, (p_\alpha)_\star^{-1}$ is a Leray endomorphism. Now [3 pp. 214 (see (1.3))] guarantees that $q_\star \, p_\star^{-1}$ is a Leray endomorphism and $\Lambda(q_\star \, p_\star^{-1}) = \Lambda((\overline{q_\alpha})_\star \, (p_\alpha)_\star^{-1})$.

Next assume $\mathbf{\Lambda}\,(\phi) \neq \{0\}$. Then there exists $(p,q) \subset \phi$ with $\Lambda(q_\star \, p_\star^{-1}) \neq 0$. Also there exists q_α, p_α, $\overline{q_\alpha}$ as described above with $\Lambda((\overline{q_\alpha})_\star \, (p_\alpha)_\star^{-1}) = \Lambda(q_\star \, p_\star^{-1}) \neq 0$. Since C is a Lefschetz space there exists $x_\alpha \in C$ with $x_\alpha \in \overline{q_\alpha} \, p_\alpha^{-1}\,(x_\alpha)$. Now since π_α and i are α–close we have that x_α is an α–fixed point of ϕ (note $x_\alpha = \pi_\alpha\,(y_\alpha)$ and $y_\alpha \in q \, p^{-1}\,(x_\alpha) \subset \phi(x_\alpha)$ so there exists $U_\alpha \in \alpha$ with $(x_\alpha =)\pi_\alpha\,(y_\alpha) \in U_\alpha$ and

$y_\alpha \in U_\alpha$ i.e. $x_\alpha \in U_\alpha$ and $y_\alpha \in U_\alpha$ i.e. $x_\alpha \in U_\alpha$ and $\phi(x_\alpha) \cap U_\alpha \neq \emptyset$ since $y_\alpha \in U_\alpha$ and $y_\alpha \in \phi(x_\alpha)$). The result now follows from Theorem 2.3 (with Remark 2.1). ∎

Let X be a Hausdorff topological space and let $\alpha \in Cov(X)$. X is said to be Schauder NES admissible α-dominated if there exists a Schauder NES admissible space X_α and two continuous functions $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha\, s_\alpha : X \to X$ and $i : X \to X$ are α–close and also that $r_\alpha\, s_\alpha \sim Id_X$. X is said to be almost Schauder NES admissible dominated if X is Schauder NES admissible α-dominated for every $\alpha \in Cov(X)$.

The same reasoning as in [3 pp. 219 (see (5.6)] establishes the following result.

Theorem 2.5. *Let X be a uniform space and let X be almost Schauder NES admissible dominated. If $\phi \in Ad(X,X)$ then*

(i) *ϕ is a Lefschetz map and*
(ii) *if $\mathbf{\Lambda}(\phi) \neq \{0\}$ then ϕ has a fixed point*

i.e. X is a Lefschetz space.

3. FIXED POINT THEORY IN FRÉCHET SPACES

Let $E = (E, \{|\cdot|_n\}_{n\in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \cdots\cdots \quad \text{for every } x \in E. \tag{3.1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x-y|_n = 0. \tag{3.2}$$

We denote by $\mathbf{E}^n = (E\,/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (3.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (3.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m\,/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \ \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{3.3}$$

Remark 3.1. (i) For convenience the norm on E_n is denoted by $|\cdot|_n$.

(ii) Usually in applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(iii) Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessarily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0, \infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0, n]$ and $E_n = C[0, n]$.

Finally we assume

$$E_1 \supseteq E_2 \supseteq \cdots\cdots \quad \text{and for each } n \in N,\ |x|_n \le |x|_{n+1}\ \forall\ x \in E_{n+1}. \tag{3.4}$$

Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [8]) denote the projective limit of $\{E_n\}_{n \in N}$ (note $\pi_{n,m} = j_n\, \mu_{n,m}\, j_m^{-1} : E_m \to E_n$ for $m \ge n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\, \mu_n(X)$, and we let $\overline{X_n}$ and ∂X_n denote respectively the closure and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by [4]

$$pseudo - int\,(X) = \{x \in X :\ j_n\, \mu_n(x) \in \overline{X_n} \setminus \partial X_n \ \text{ for every }\ n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int\,(X)$.

Let E and E_n be as described above. Some of the ideas in this section were motivated from [10].

Definition 3.1. A set $A \subseteq E$ is said to be PRLS if for each $n \in N$, $A_n \equiv j_n\, \mu_n\,(A)$ is a Lefschetz space.

Definition 3.2. A set $A \subseteq E$ is said to be CPRLS if for each $n \in N$, $\overline{A_n}$ is a Lefschetz space.

Example 3.1. Let A be pseudo-open. Then A is a PRLS.

To see this fix $n \in N$. We now show

$$A_n \ \text{ is a open subset of }\ E_n.$$

First notice $A_n \subseteq \overline{A_n} \setminus \partial A_n$ since if $y \in A_n$ then there exists $x \in A$ with $y = j_n \mu_n(x)$ 0and this together with $A = pseudo - int\, A$ yields $j_n \mu_n(x) \in \overline{A_n} \setminus \partial A_n$ i.e. $y \in \overline{A_n} \setminus \partial A_n$. In addition notice

$$\overline{A_n} \setminus \partial A_n = (int\, A_n \cup \partial A_n) \setminus \partial A_n = int\, A_n \setminus \partial A_n = int\, A_n$$

since $int\, A_n \cap \partial A_n = \emptyset$. Consequently

$$A_n \subseteq \overline{A_n} \setminus \partial A_n = int\, A_n, \quad \text{so} \quad A_n = int\, A_n.$$

As a result A_n is open in E_n. Thus A_n is a Lefschetz space [5 pp. 41 (see (3.1)], so A is a PRLS.

Our first result is for Volterra type operators.

Theorem 3.1. *Let E and E_n be as described above, $C \subseteq E$ is an PRLS and $F : C \to 2^E$ and for each $n \in N$ assume $F : C_n \to 2^{E_n}$. Suppose the following conditions are satisfied:*

$$\text{for each } \ n \in N, \ F \in Ad(C_n, C_n) \tag{3.5}$$

$$\text{for each } \ n \in N, \ \mathbf{\Lambda}_{C_n}(F) \neq \{0\} \tag{3.6}$$

and

$$\begin{cases} \text{for each } \ n \in \{2, 3, \ldots.\} \ \text{ if } \ y \in C_n \ \text{ solves } \ y \in F\,y \ \text{ in } \ E_n \\ \text{then } \ y \in C_k \ \text{ for } \ k \in \{1, \ldots, n-1\}. \end{cases} \tag{3.7}$$

Then F has a fixed point in E.

Proof. Fix $n \in N$. Now there exists $y_n \in C_n$ with $y_n \in F\,y_n$. Lets look at $\{y_n\}_{n\in N}$. Notice $y_1 \in C_1$ and $y_k \in C_1$ for $k \in N\backslash\{1\}$ from (3.7). As a result $y_n \in C_1$ for $n \in N$, $y_n \in F\,y_n$ in E_n together with (3.5) implies there is a subsequence $N_1^\star$ of N and a $z_1 \in C_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $y_n \in C_2$ for $n \in N_1$ together with (3.5) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in C_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note from (3.4) that $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \ldots\ldots, \qquad N_k^\star \subseteq \{k, k+1, \ldots\}$$

and $z_k \in C_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \ldots\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F\,y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with the fact that $F : C_k \to 2_k^E$ is upper semicontinuous (note $y_n \in C_k$ for $n \in N_k$) implies $y \in F\,y$ in E_k. We can do this for each $k \in N$ so as a result we have $y \in F\,y$ in E. ∎

Essentially the same reasoning as in Theorem 3.1 yields the following result.

Theorem 3.2. *Let E and E_n be as described above, $C \subseteq E$ is an CPRLS and $F : C \to 2^E$ and for each $n \in N$ assume $F : \overline{C_n} \to 2^{E_n}$. Suppose the following conditions are satisfied:*

$$\text{for each } \ n \in N, \ F \in Ad(\overline{C_n}, \overline{C_n}) \tag{3.8}$$

$$\text{for each } \ n \in N, \ \mathbf{\Lambda}_{\overline{C_n}}(F) \neq \{0\} \tag{3.9}$$

and

$$\begin{cases} \text{for each } \ n \in \{2, 3, \ldots.\} \ \text{ if } \ y \in \overline{C_n} \ \text{ solves } \ y \in F\,y \ \text{ in } \ E_n \\ \text{then } \ y \in \overline{C_k} \ \text{ for } \ k \in \{1, \ldots, n-1\}. \end{cases} \tag{3.10}$$

Then F has a fixed point in E.

Our next result was motivated by Urysohn type operators. In this case the map F_n will be related to F by the closure property (3.15).

Theorem 3.3. *Let E and E_n be as described in the beginning of Section 3, $C \subseteq E$ is an PRLS and $F : C \to 2^E$. Also for each $n \in N$ assume there exists $F_n : C_n \to 2^{E_n}$. Suppose the following conditions are satisfied:*

$$C_1 \supseteq C_2 \supseteq \cdots\cdots \tag{3.11}$$

$$\text{for each } \ n \in N, \ F_n \in Ad(C_n, C_n) \tag{3.12}$$

$$\text{for each } \ n \in N, \ \mathbf{\Lambda}_{C_n}(F) \neq \{0\} \tag{3.13}$$

$$\begin{cases} \text{for each } \ n \in N, \ \text{ the map } \ \mathcal{K}_n : C_n \to 2^{E_n}, \ \text{ given by} \\ \mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \ \ \text{(see Remark 3.2), is compact} \end{cases} \tag{3.14}$$

and

$$\begin{cases} \text{if there exists a } \ w \in E \ \text{ and a sequence } \ \{y_n\}_{n\in N} \\ \text{with } \ y_n \in C_n \ \text{ and } \ y_n \in F_n\, y_n \ \text{ in } \ E_n \ \text{ such that} \\ \text{for every } \ k \in N \ \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \dots\} \ \text{ of } \ N \ \text{ with } \ y_n \to w \ \text{ in } \ E_k \\ \text{as } \ n \to \infty \ \text{ in } \ S, \ \text{ then } \ w \in F\, w \ \text{ in } \ E. \end{cases} \tag{3.15}$$

Then F has a fixed point in E.

Remark 3.2. The definition of $\mathcal{K}_n$ is as follows. If $y \in C_n$ and $y \notin C_{n+1}$ then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in C_{n+1}$ and $y \notin C_{n+2}$ then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in N$. Now there exists $y_n \in C_n$ with $y_n \in F_n\, y_n$ in E_n. Lets look at $\{y_n\}_{n\in N}$. Note from (3.11) and the fact that $|x|_1 \leq |x|_n$ for all $x \in E_n$ that $y \in C_1$ and $y_n \in \mathcal{K}_1(y_n)$ in E_1 for each $n \in N$. Now $\mathcal{K}_1 : C_1 \to 2^{E_1}$ compact guarantees that there exists a subsequence $N_1^\star$ of N and a $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Look at $\{y_n\}_{n\in N_1}$. Also there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in E_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note $z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1^\star$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \cdots\cdots, \qquad N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $z_{k+1} = z_k$ in E_k for $k \in N$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Let $y = z_k$ in E_k. Notice y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F_n\, y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k) together with (3.15) implies $y \in F\, y$ in E. ∎

Similarly we have the following result.

Theorem 3.4. *Let E and E_n be as described in the beginning of Section* 3*, $C \subseteq E$ is an CPRLS and $F : C \to 2^E$. Also for each $n \in N$ assume there exists $F_n : \overline{C_n} \to 2^{E_n}$. Suppose the following conditions are satisfied:*

(3.16) $$\overline{C_1} \supseteq \overline{C_2} \supseteq \cdots\cdots$$

(3.17) $$\text{for each } n \in N,\ F_n \in Ad(\overline{C_n}, \overline{C_n})$$

(3.18) $$\text{for each } n \in N,\ \mathbf{\Lambda}_{\overline{C_n}}(F) \neq \{0\}$$

(3.19) $$\begin{cases} \text{for each } n \in N, \text{ the map } \mathcal{K}_n : \overline{C_n} \to 2^{E_n}, \text{ given by} \\ \mathcal{K}_n(y) = \cup_{m=n}^{\infty} F_m(y) \text{ is compact} \end{cases}$$

and

(3.20) $$\begin{cases} \text{if there exists a } w \in E \text{ and a sequence } \{y_n\}_{n \in N} \\ \text{with } y_n \in \overline{C_n} \text{ and } y_n \in F_n\, y_n \text{ in } E_n \text{ such that} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \dots\} \text{ of } N \text{ with } y_n \to w \text{ in } E_k \\ \text{as } n \to \infty \text{ in } S, \text{ then } w \in F\, w \text{ in } E. \end{cases}$$

Then F has a fixed point in E.

REFERENCES

1. R.P. Agarwal and D. O'Regan, A note on the Lefschetz fixed point theorem for admissible spaces, *Bull. Korean Math. Soc.*, **42**(2005), 307–313.
2. H. Ben-El-Mechaiekh, Spaces and maps approximation and fixed points, *Jour. Computational and Appl. Mathematics*, **113**(2000), 283–308.
3. G. Fournier and L. Gorniewicz, The Lefschetz fixed point theorem for multi-valued maps of non-metrizable spaces, *Fundamenta Mathematicae*, **92**(1976), 213–222.
4. M. Frigon, Fixed point results for compact maps on closed subsets of Fréchet spaces and applications to differential and integral equations, *Bull. Soc. Math. Belgique*, **9**(2002), 23–37.
5. L. Gorniewicz, Homological methods in fixed point theory of multivalued maps, *Dissertationes Mathematicae*, **129**(1976), 1-66.
6. L. Gorniewicz, Topological fixed point theory of multivalued mappings, *Kluwer Academic Publishers*, Dordrecht, 1999.

7. L. Gorniewicz and A. Granas, Some general theorems in coincidence theory, *J. Math. Pures et Appl.,* **60**(1981), 361–373.
8. L.V. Kantorovich and G.P. Akilov, Functional analysis in normed spaces, *Pergamon Press*, Oxford, 1964.
9. J.L. Kelley, General Topology, *D. Van Nostrand Reinhold Co.*, New York, 1955.
10. D. O'Regan, A Lefschetz fixed point theorem in Fréchet spaces for multivalued maps using the projective limit approach, to appear.