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BACKWARD STOCHASTIC DIFFERENTIAL INCLUSIONS

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ABSTRACT. Existence of solutions to backward stochastic differential inclusions is considered. The paper contains the basic notions dealing with backward stochastic differential inclusions.

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1. INTRODUCTION

Given measurable set-valued mappings $F : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : \mathbb{R}^m \to Cl(\mathbb{R}^m)$ by a backward stochastic differential inclusion $BSDI(F,H)$ we mean relations

$$\begin{cases} x_s \in E\left[x_t + \int_s^t F(\tau, x_\tau)d\tau|\mathcal{F}_s\right] \\ x_T \in H(x_T) \end{cases} \tag{1.1}$$

that have to be satisfied a.s. for every $0 \le s \le t \le T$ by a cádlág process $x = (x_t)_{0 \le t \le T}$ defined on a complete filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, P, \mathbb{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual hypothesis (see [14]). $E[x_t + \int_s^t F(\tau, x_\tau)d\tau|\mathcal{F}_s]$ denotes the set-valued conditional expectation (see [4], [5]) of the set-valued mapping $\Omega \ni \omega \longrightarrow x_t(\omega) + \int_s^t F(\tau, x_\tau(\omega))d\tau \subset \mathbb{R}^m$ with respect to the sub-σ-algebra $\mathcal{F}_s \subset \mathcal{F}$. If $\mathcal{P}_{\mathbb{F}}$ is given then x, satisfying conditions presented above, is said to be a strong solution to $BSDI(F,H)$. In a general case we can look for systems $(\mathcal{P}_{\mathbb{F}}, x)$ satisfying conditions (1). Such systems are said to be weak solutions to $BSDI(F,H)$. It is clear that if x is a strong solution to $BSDI(F,H)$ on $\mathcal{P}_{\mathbb{F}}$, then a pair $(\mathcal{P}_{\mathbb{F}}, x)$ is its weak solution. Backward stochastic differential inclusions can be treated as some generalizations of backward stochastic differential equations of the form

$$x_t = E\left[h(x) + \int_t^T f(\tau, x_\tau, z_\tau)d\tau|\mathcal{F}_t\right] \quad a.s. \tag{1.2}$$

where the tripet (h, f, z) is called the data set of such equation (see [2], [3], [7], [13]). Usually, if we consider strong solutions to (1.2) apart from (h, f, z), a probability space $\mathcal{P} = (\Omega, \mathcal{F}, P)$ is also given and a filtration $\mathbb{F}$ is defined by a process z by taking $\mathbb{F}^z = (\mathcal{F}_t^z)_{0 \le t \le T}$, where $(\mathcal{F}_t^z)_{0 \le t \le T}$ is the smallest filtration satisfying the usual

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conditions and such that z is $\mathbb{F}^z$-adapted. Process z is called the driving process. In practical applications the driving process z is taken as a d-dimensional Brownian motion or it is a strong solution to a forward stochastic differential equation. In the case of weak solutions to (1.2) apart from h and f a probability measure μ on the space $D(\mathbb{R}^d)$ of d-dimensional cádlág functions on $[0,T]$ is given and its weak solution with an initial distribution μ is defined as a system $(\mathcal{P}_{\mathbb{F}}, x, z)$ satisfying (1.2) and such that $Pz^{-1} = \mu$ and every $\mathbb{F}^z$-martingale is also $\mathbb{F}$ - martingale. Let us observe that in particular for a given weak solution $(\mathcal{P}_{\mathbb{F}}, x)$ to $BSDI(F,H)$ with $H(x) = \{h(x)\}$ and $F(t,x) = \{f(t,x,z) : z \in \mathcal{Z}\}$ for $(t,x) \in [0,T] \times \mathbb{R}^m$, where f and h are given measurable functions and $\mathcal{Z}$ is a nonempty compact subset of the space $D(\mathbb{R}^d)$, there exists (see [8], Th. II.3.12) a measurable $\mathbb{F}$-adapted stochastic process $(z_t)_{0\le t\le T}$ with values in $\mathcal{Z}$ such that

$$x_t = E\left[h(x) + \int_t^T f(\tau, x_\tau, z_\tau)d\tau|\mathcal{F}_t\right] \quad \text{a.s.} \tag{1.3}$$

For given probability masures μ_0 and μ_T on $\mathbb{R}^m$, we can look for a weak solution $(\mathcal{P}_{\mathbb{F}}, x)$ to $BSDI(F,H)$ such that $Px_0^{-1} = \mu_0$ and $Px_T^{-1} = \mu_T$. If F and H are such as above then there exists a measurable and $\mathbb{F}$-adapted stochastic process $(z_t)_{0\le t\le T}$ such that (1.3) is satisfied and such that $E[h(x) + \int_0^T f(\tau, x_\tau, z_\tau)d\tau] = \int_{\mathbb{R}^m} ud\mu_0$. If $f(t,x,z) = f(t,x) + g(z)$, with $g \in C(D(\mathbb{R}^d), \mathbb{R}^m)$, then

$$\int_0^T \int_{D(\mathbb{R}^d)} g(v)d\lambda_\tau d\tau = \int_{\mathbb{R}^m} ud\mu_0 - \int_{\mathbb{R}^m} h(u)d\mu_T - E\int_0^T f(\tau, x_\tau)d\tau$$

where $\lambda_t = Pz_t^{-1}$ for $t \in [0,T]$. In some special case weak solutions to $BSDI(F,H)$ describe a class of recursive utilities under uncertainty (see [7]). To verify that suppose $(\mathcal{P}_{\mathbb{F}}, x)$ is a weak solution to $BSDI(F,H)$ with $H(x) = \{h(x)\}$ and $F(t,x) = \{f(t,x,c,z) : (c,z) \in \mathcal{C} \times \mathcal{Z}\}$, where h and f are measurable functions and $\mathcal{C}$, $\mathcal{Z}$ are nonempty compact subsets of $C([0,T], \mathbb{R}^+)$ and $D(\mathbb{R}^m)$, respectively. Similarly as above we can select a pair of measurable $\mathbb{F}$-adapted stochastic processes $(c_t)_{0\le t\le T}$ and $(z_t)_{0\le t\le T}$ with values at $\mathcal{C}$ and $\mathcal{Z}$, respectively and such that

$$x_t = E\left[h(x) + \int_t^T f(\tau, x_\tau, c_\tau, z_\tau)d\tau|\mathcal{F}_t\right] \quad \text{a.s.} \tag{1.4}$$

for $0 \le t \le T$. In such a case (1.4) describes some class of recursive utilities under uncertainty, where $(c_t(s,\cdot))_{0\le s\le T}$ denotes for fixed $t \in [0,T]$ the future consumption. Let us observe that in some special case a strong solution x to $BSDI(F,H)$ on a filtered probability space $\mathcal{P}_{\mathbb{F}}$ with the "constant" filtration $\mathbb{F} = (\mathcal{F})$ is a solution to a backward random inclusion $-x_t' \in \mathrm{co}F(t,x_t)$ with a terminal condition $x_T \in H(x_T)$ a.s. for a.e. $t \in [0,T]$. As usual $\mathrm{co}F(t,x_t)$ denotes the convex hull of the set $F(t,x_t)$. Throughout the paper we assume that $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}\, P, \mathbb{F})$ is a complete filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual hypotheses. Given

$\mathcal{P}_{\mathbb{F}}$ we denote by $\mathbb{D}(\mathbb{F},\mathbb{R}^m)$ the space of all m-dimensional $\mathbb{F}$-adapted cádlág prcesses on $\mathcal{P}_{\mathbb{F}}$ and by $\mathcal{S}^2(\mathbb{F},\mathbb{R}^m)$ the set of all m-dimensional $\mathbb{F}$-semimartingales x such that $\|x\|_{\mathcal{S}^2} = E[\sup_{s\in[0,T]}|x_s|^2] < \infty$. We have $\mathcal{S}^2(\mathbb{F},\mathbb{R}^m) \subset \mathbb{D}(\mathbb{F},\mathbb{R}^m)$. It can be proved (see [14], Th. IV2.1, Th. V.2.2) that $(\mathcal{S}^2(\mathbb{F},\mathbb{R}^m), \|\cdot\|_{\mathcal{S}^2})$ is a Banach space. The present paper is mainly devoted to properties of solutions set of weak continuous solutions to $BSDI(F,H)$. It is organized as follows. Section 2 contains some properties of the set-valued conditional expectations of set-valued integrals. In Section 3 some measurable selection theorems are given. Existence theorems to $BSDI(F,H)$ are given in Section 4 and Section 5. Finally, in Section 6 a weak compactness of the set $\mathcal{X}(F,H)$ of all continuous weak solutions to $BSDI(F,H)$ is proved.

2. CONDITIONAL EXPECTION OF SET-VALUED INTEGRALS

Let $(\Omega,\mathcal{F},P)$ be a probability space and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Given an $\mathcal{F}$-measurable set-valued mapping $\Phi : \Omega \to Cl(\mathbb{R}^m)$ with a nonempty set $S(\Phi)$ of all its $\mathcal{F}$-measurable and integrable selectors there exists (see [4]) an unique (in the a.s. sense) $\mathcal{G}$-measurable set-valued mapping $E[\Phi|\mathcal{G}]$ satisfying

$$S(E[\Phi|\mathcal{G}]) = cl_L\{E[\varphi|\mathcal{G}] : \varphi \in S(\Phi)\} \tag{2.1}$$

where cl_L denotes the closure operation in $L(\Omega,\mathcal{G},\mathbb{R}^m)$. We call $E[\Phi|\mathcal{G}]$ the multivalued conditional expectation of Φ relative to $\mathcal{G}$. This conditional expectation has properties similar to those of the usual ones. For example, we have $\int_A E[\Phi|\mathcal{G}]dP = \int_A \Phi dP$ for every $A \in \mathcal{G}$, where integrals are understood in the Aumann's sense (see [5], Prop. 6.8). It can be proved (see [5], Prop. 6.2) that for given measurable and integrably bounded set-valued mappings $\Phi,\Psi : [0,T]\times\Omega \to Cl(\mathbb{R}^m)$ one has $Eh(E[\Phi|\mathcal{G}], E[\Psi|\mathcal{G}]) \le Eh(\Phi,\Psi)$, where h is the Hausdorff metric on $Cl(\mathbb{R}^m)$. Let $G : [0,T]\times\Omega \to Cl(\mathbb{R}^m)$ be measurable and integrably bounded, i.e. there is $m \in L([0,T]\times\Omega,\mathbb{R}_+)$ such that $\|G(t,x)\| \le m(t,\omega)$, a.e., where $\mathbb{R}_+ = [0,\infty)$ and $\|G(t,\omega)\| = \sup\{|g| : g \in G(t,\omega)\}$. As usual we denote by $S(G)$ a set of all integrable selectors for G. We have $S(G) = \{g \in L([0,T]\times\Omega,\mathbb{R}^m) : g(t,\omega) \in G(t,\omega) \text{ a.e.}\}$. It is easy to verify (see [8]) that $S(G)$ is nonempty and decomposable, i.e. that for every $f,g \in S(G)$ and $E \in \beta_T\otimes\mathcal{F}$ one has $1\!\!1_E f + 1\!\!1_{E^\sim} g \in S(G)$, where β_T denotes the Borel σ-algebra of $[0,T]$ and $E^\sim$ is the complement of E. In particular, if $G(t,\omega)$ are convex subsets of $\mathbb{R}^m$ for $(t,\omega)\in[0,T]\times\Omega$, the set $S(G)$ is a convex weakly compact subset of $L([0,T]\times\Omega,\mathbb{R}^m)$. Then it is also a closed subset of this space. For the given above G we can define an Aumann integral $\Phi(\omega) = \int_0^T G(t,\omega)dt$ depending on a parameter $\omega\in\Omega$. By Aumann's theorem (see [8], Th. II.3.20) $\int_0^T G(t,\omega)dt$ is a nonempty, convex compact subset of $\mathbb{R}^m$ for every $\omega\in\Omega$. Furthermore, $\int_0^T G(t,\omega)dt = \int_0^T co\,G(t,\omega)dt$ for $\omega\in\Omega$. Hence and ([8],Th. II.3.21) we obtain the following result.

Proposition 2.1. *Let $G : [0,T]\times\Omega \to Cl(\mathbb{R}^m)$ be measurable and integrably bounded. Then a set-valued mapping $\Phi : \Omega \to \mathrm{Conv}(\mathbb{R}^m)$ defined by $\Phi(\omega) = \int_0^T G(t,\omega)dt$ for $\omega \in \Omega$ is measurable.*

Proof. By virtue of ([8],Th. II.3.8) it is enough only to verify that the function $\Omega \ni \omega \to s(p,\Phi(\omega)) \in \mathbb{R}$ is measurable for every $p \in \mathbb{R}^n$, where $s(\cdot, A)$ denotes a support function of $A \in Cl(\mathbb{R}^m)$. By the measurability of G and its integrably boundedness a function $[0,T] \times \Omega \ni (t,\omega) \to s(p, G(t,\omega)) \subset \mathbb{R}$ is measurable for every $p \in \mathbb{R}^m$ (see [8], Remark II.3.5). By virtue of ([8], Th. II.3.21) for every $p \in \mathbb{R}^m$ one has $s(p,\Phi(\omega)) = \int_0^T s(p, G(t,\omega))dt$ for $\omega \in \Omega$. Hence the measurability of the function $\Omega \ni \omega \to s(p,\Phi(\omega)) \in \mathbb{R}$ follows for every $p \in \mathbb{R}^m$. Therefore Φ is $\mathcal{F}$-measurable. $\square$

Proposition 2.2. *Let $G : [0,T]\times\Omega \to Cl(\mathbb{R}^m)$ be measurable and integrably bounded and let $\Phi(\omega) = \int_0^T G(t,\omega)dt$ for $\omega \in \Omega$. Then $S(\Phi)$ is a nonempty convex weakly compact subset of $L(\Omega,\mathcal{F},\mathbb{R}^m)$. Furthermore, $\varphi \in S(\Phi)$ if and only if there is $g \in S(co\, G)$ such that $\varphi(\omega) = \int_0^T g(t,\omega)dt$ for a.e. $\omega \in \Omega$.*

Proof. By Proposition 2.1, Φ is $\mathcal{F}$-measurable. It is also integrably bounded, because $\|\Phi(\omega)\| \le \int_0^T m(t,\omega)dt$ for a.e. $\omega \in \Omega$. Therefore (see [8], Th. III.2.3) $S(\Phi)$ is a nonempty convex weakly compact subset of $L(\Omega,\mathcal{F},\mathbb{R}^m)$. For every $g \in S(co\, G)$ a function $\varphi(\omega) = \int_0^T g(t,\omega)dt$ is a measurable selector for Φ, because of ([8], Th. II.3.20) we have $\Phi(\omega) = \int_0^T co\ G(t,\omega)dt$ for $\omega \in \Omega$. It is also integrably bounded, because $|\varphi(\omega)| \le \int_0^T m(t,\omega)dt$ for a.e. $\omega \in \Omega$. Then $\varphi \in S(\Phi)$ for every $g \in S(co\, G)$. Assume now $\varphi \in S(\Phi)$. Then for every $A \in \mathcal{F}$ one has $E_A\varphi \in E_A\Phi$, where $E_A\varphi = \int_A \varphi dP$ and $E_A\Phi = \int_A \Phi dP$. Let $\varepsilon > 0$ be given and select a measurable partition $(A_n^\varepsilon)_{n=1}^{N_\varepsilon}$ of Ω such that $E_{A_n^\varepsilon}\int_0^T m(t,\cdot)dt < \varepsilon/2^{n+1}$. For every $n = 1,\dots,N_\varepsilon$ there is a $g_n^\varepsilon \in S(G)$ such that $E_{A_n^\varepsilon}\varphi = E_{A_n^\varepsilon}\int_0^T g_n^\varepsilon(t,\cdot)dt$. Let $g^\varepsilon = \sum_{n=1}^{N_\varepsilon} \mathbb{1}_{A_n^\varepsilon} g_n^\varepsilon$. By the decomposability of $S(G)$ one has $g^\varepsilon \in S(G)$. We have $g^\varepsilon \in S(co\ G)$ because $S(G) \subset S(co\ G)$. Taking a sequence $(\varepsilon_k)_{k=1}^\infty$ of positive numbers $\varepsilon_k > 0$ such that $\varepsilon_k \to 0$ as $k \to \infty$ we can select $g \in S(co\, G)$ and a subsequence, denoted again by $(g^{\varepsilon_k})_{k=1}^\infty$, of $(g^{\varepsilon_k})_{k=1}^\infty$ weakly converging to g in $L([0,T]\times\Omega,\mathbb{R}^n)$, because $S(co\ G)$ is a weakly compact subset of $L([0,T]\times\Omega,\mathbb{R}^n)$. For every $A \in \mathcal{F}$ and $k = 1,2,\dots$ there is a subset $\{n_1,\dots,n_p\}$ of $\{1,\dots,N_{\varepsilon_k}\}$ such that $A \cap A_{n_i}^{\varepsilon_k} \ne \emptyset$ for $i = 1,2,\dots,p$ and $A \cap A_r = \emptyset$ for $r \in \{1,2,\dots,N_{\varepsilon_k}\} \setminus \{n_1,\dots,n_p\}$. Therefore

$$
\begin{aligned}
\left|E_A\varphi - E_A\int_0^T g^{\varepsilon_k}(t,\cdot)dt\right| &\le \sum_{n=1}^{N_{\varepsilon_k}} \left|E_{A\cap A_n^{\varepsilon_k}}\varphi - E_{A\cap A_n^{\varepsilon_k}}\int_0^T g_n^{\varepsilon_k}(t,\cdot)dt\right| \\
&= \sum_{i=1}^{p} \left|E_{A\cap A_{n_i}^{\varepsilon_k}}\varphi - E_{A\cap A_{n_i}^{\varepsilon_k}}\int_0^T g_n^{\varepsilon_k}(t,\cdot)dt\right| \\
&\le 2\sum_{i=1}^{p} E_{A_{n_i}^{\varepsilon_k}}\int_0^T m(t,\cdot)dt \le \varepsilon_k
\end{aligned}
$$

for every $k = 1, 2, \dots$. On the other hand for every $A \in \mathcal{F}$ we also have

$$\left|E_A\varphi - E_A \int_0^T g(t,\cdot)dt\right| \le$$

$$\le \left|E_A\varphi - E_A \int_0^T g^{\varepsilon_k}(t,\cdot)dt\right| + \left|E_A \int_0^T g^{\varepsilon_k}(t,\cdot)dt - E_A \int_0^T g(t,\cdot)dt\right|$$

$$\le \varepsilon_k + \left|E_A \int_0^T g^{\varepsilon_k}(t,\cdot)dt - E_A \int_0^T g(t,\cdot)dt\right|$$

for $k = 1, 2, \dots$. Hence it follows that $E_A\varphi = E_A \int_0^T g(t,\cdot)dt$ for every $A \in \mathcal{F}$, because $\varepsilon_k \to 0$ and $|E_A \int_0^T g^{\varepsilon_k}(t,\cdot)dt - E_A \int_0^T g(t,\cdot)dt| \to 0$ as $k \to \infty$. Therefore $\varphi(\omega) = \int_0^T g(t,\cdot)dt$ for a.e. $\omega \in \Omega$. □

Corollary 2.3. *If $G : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ is measurable and integrably bounded then*

$$S\left(\int_0^T G(t,\cdot)dt\right) = \left\{\int_0^T g(t,\cdot)dt : g \in S(co\ G)\right\}.$$

Corollary 2.4. *If $G : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ is measurable and integrably bounded and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ then*

$$S\left(E\left[\int_0^T G(t,\cdot)dt|\mathcal{G}\right]\right) = \left\{E\left[\int_0^T g(t,\cdot)dt|\mathcal{G}\right] : g \in S(co\ G)\right\}.$$

Proof. It is enough only to see that the set $\mathcal{H} = \{E[\int_0^T g(t,\cdot)dt|\mathcal{G}] : g \in S(co\ G)\}$ is a closed subset of $L(\Omega, \mathcal{G}, \mathbb{R}^m)$. By the properties of conditional expectations and the properties of the set $S(co\ G)$ it follows that $\mathcal{H}$ is a convex weakly compact subset of $L(\Omega, \mathcal{G}, \mathbb{R}^m)$. Therefore $\mathcal{H}$ is a closed subset of $L(\Omega, \mathcal{G}, \mathbb{R}^m)$. □

3. MEASURABLE SELECTION THEOREMS

Let $x = (x_t)_{0 \le t \le T}$ be an $\mathbb{F}$-adapted m-dimensional cádlág process on $\mathcal{P}_{\mathbb{F}}$. Given a measurable and uniformly integrably bounded multivalued mapping $F : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ we denote by $F \circ x$ a set-valued mapping defined on $[0,T] \times \Omega$ by setting $(F \circ x)(t,\omega) = F(t, x_t(\omega))$ for $(t,\omega) \in [0,T] \times \Omega$. It is clear that $F \circ x$ is measurable and $\mathbb{F}$-adapted, i.e., it is $\beta_T \otimes \mathcal{F}$-measurable and such that for every fixed $t \in [0,T]$ a mapping $\Omega \ni \omega \to (F \circ x)(t,\omega) \subset \mathbb{R}^m$ is $\mathcal{F}_t$-measurable. In what follows we shall denote by $S_{\mathbb{F}}(F \circ x)$ a set of all measurable and $\mathbb{F}$-adapted selectors for $F \circ x$. Let us observe that $F \circ x$ is measurable and $\mathbb{F}$-adapted if and only if it is $\Sigma_{\mathbb{F}}$-measurable, where $\Sigma_{\mathbb{F}} = \{A \in \beta_T \otimes \mathcal{F} : A_t \in \mathcal{F}_t \text{ for } 0 \le t \le T\}$ and A_t denotes a section of a set $A \in \beta_T \otimes \mathcal{F}$ at $t \in [0,T]$. Therefore, immediately from Kuratowski and Ryll-Nardzewski measurable selection theorem (see [8], Th. II.3.10) it follows that for the given above F and x the set $S_{\mathbb{F}}(F \circ x)$ is nonempty. In the general case we shall also denote by $S_{\mathbb{F}}(G)$ the set of all measurable and $\mathbb{F}$-adapted selectors for a given measurable $\mathbb{F}$-adapted and integrably bounded set-valued mapping $G : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$. Similarly as above we can verify that $S_{\mathbb{F}}(co\ G)$ is a nonempty convex and weakly compact subset of $L([0,T] \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^m)$. We shall prove the following measurable selection theorem.

Theorem 3.1. *Let $G : [0,T] \times \Omega \to Cl(\mathbb{R}^m)$ be a measurable $\mathbb{F}$-adapted and integrably bounded set-valued mapping. Assume $x = (x_t)_{0 \le t \le T}$ is an m-dimensional measurable process on $\mathcal{P}_{\mathbb{F}}$ such that $E|x_T| < \infty$. Then*

$$x_s \in E\left[x_t + \int_s^t G(\tau, \cdot) d\tau | \mathcal{F}_s\right] \qquad a.s. \tag{3.1}$$

for every $0 \le s \le t \le T$ if and only if there is $g \in S_{\mathbb{F}}(co\ G)$ such that

$$x_t = E\left[x_T + \int_t^T g(\tau, \cdot) d\tau | \mathcal{F}_t\right] \qquad a.s. \tag{3.2}$$

for every $0 \le t \le T$.

Proof. Suppose there is $g \in S_{\mathbb{F}}(co\ G)$ such that (3.2) is satisfied. Then for every $0 \le s \le t \le T$ one has

$$x_s = E\left[x_T + \int_s^T g(\tau, \cdot) d\tau | \mathcal{F}_s\right] = E\left[\int_s^t g(\tau, \cdot) d\tau | \mathcal{F}_s\right]$$

$$+E\left[x_T + \int_t^T g(\tau, \cdot) d\tau | \mathcal{F}_s\right]$$

and

$$E[x_t|\mathcal{F}_s] = E\left[x_T + \int_t^T g(\tau, \cdot) d\tau | \mathcal{F}_s\right] \quad \text{a.s.}$$

Therefore

$$x_s = E\left[x_t + \int_s^t g(\tau, \cdot) d\tau | \mathcal{F}_s\right] \quad \text{a.s.}$$

for $0 \le s \le t \le T$. Hence by Corollary 2.4 it follows that

$$x_s \in S\left(E\left[x_t + \int_s^t G(\tau, \cdot) d\tau | \mathcal{F}_s\right]\right)$$

for $0 \le s \le t \le T$. Therefore, (3.1) is satisfied a.s. for $0 \le s \le t \le T$. Assume that (3.1) is satisfied for every $0 \le s \le t \le T$ a.s. and let $m \in L([0,T] \times \Omega, \mathbb{R}_+)$ be such that $\|G(t,\omega)\| \le m(t,\omega)$ for a.e. $(t,\omega) \in [0,T] \times \Omega$. For every $0 \le t \le T$ one has $E|x_t| \le E|x_T| + E\int_0^T m(t, \cdot) dt < \infty$. By virtue of Corollary 2.4 x is $\mathbb{F}$-adapted. Let $\eta > 0$ be arbitrarily fixed and select $\delta > 0$ such that $\delta < T$ and $\sup_{0 \le t \le T-\delta} E\int_t^{t+\delta} m(\tau, \cdot) d\tau < \eta/2$. For fixed $t \in [0, T-\delta]$ and $t \le \tau \le t+\delta$ we have $x_t \in E[x_\tau + \int_t^\tau G(s, \cdot) ds|\mathcal{F}_t]$ a.s. Therefore, for every $A \in \mathcal{F}_t$ we get $E_A(x_t - x_\tau) \in E_A\int_t^\tau G(s, \cdot) ds$. Then $|E_A(x_t - x_\tau)| \le E_A\int_t^\tau \|G(s, \cdot)\| ds \le E\int_t^{t+\delta} m(s, \cdot) ds < \eta/2$ for every $0 \le t \le T-\delta$ and $A \in \mathcal{F}_t$. Therefore, $\sup_{t \le \tau \le t+\delta} |E_A(x_t - x_\tau)| \le \eta/2$ for every $A \in \mathcal{F}_t$ and fixed $0 \le t \le T-\delta$. Let $\tau_0 = 0$, $\tau_1 = \delta, \ldots, \tau_{N-1} = (N-1)\delta < T \le N\delta$.

Immediately from (3.1) and Corollary 2.4 it follows that for every $i = 1, 2, \dots, N-1$ there is $g_i^\eta \in S_{\mathbb{F}}(co\, G)$ such that

$$E\left|x_{\tau_{i-1}} - E\left[x_{\tau_i} + \int_{\tau_{i-1}}^{\tau_i} g_i^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{i-1}}\right]\right| = 0.$$

Furthermore, there is $g_N^\eta \in S_{\mathbb{F}}(co\, G)$ such that

$$E\left|x_{\tau_{N-1}} - E\left[x_T + \int_{\tau_{N-1}}^{T} g_N^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\right]\right| = 0.$$

Define $g^\eta = \sum_{i=1}^{N-1} \mathbb{1}_{[\tau_{i-1},\tau_i)} g_i^\eta + \mathbb{1}_{[\tau_{N-1},T]} g_N^\eta$. By the decomposability of $S_{\mathbb{F}}(co\, G)$ we have $g^\eta \in S_{\mathbb{F}}(co\, G)$. For fixed $t \in [0,T]$ there is $p \in \{1, 2, \dots, N-1\}$ or $p = N$ such that $t \in [\tau_{p-1}, \tau_p)$ or $t \in [\tau_{N-1}, T]$. Let $t \in [\tau_{p-1}, \tau_p)$ with $1 \le p \le N-1$. For every $A \in \mathcal{F}_t$ one has

$$\begin{aligned}
&\left|E_A\left(x_t - E\left[x_T + \int_t^T g^\eta(s,\cdot)ds\mathcal{F}_t\right]\right)\right| \le \\
&\le |E_A(x_t - x_{\tau_p})| + E\left|x_{\tau_p} - E\left[x_{\tau_{p+1}} + \int_{\tau_p}^{\tau_{p+1}} g^\eta(s,\cdot)d\tau|\mathcal{F}_{\tau_p}\right]\right| \\
&+ |E_A(E[x_{\tau_{p+1}}|\mathcal{F}_{\tau_p}] - x_{\tau_{p+1}})| + E\left|\int_t^{\tau_p} g^\eta(s,\cdot)ds\right| + \\
&+ \left|E_A\left(E\left[\int_{\tau_p}^{\tau_{p+1}} g^\eta(s,\cdot)ds|\mathcal{F}_{\tau_p}\right] - E\left[\int_{\tau_p}^{\tau_{p+1}} g^\eta(s,\cdot)d\tau|\mathcal{F}_t\right]\right)\right| + \dots + \\
&+ E\left|x_{\tau_{N-1}} - E\left[x_T + \int_{\tau_{N-1}}^T g^\eta(s,\cdot)d\tau|\mathcal{F}_{\tau_{N-1}}\right]\right| \\
&+ |E_A(E[x_{\tau_{N-1}}|\mathcal{F}_{\tau_{N-1}}] - x_{\tau_{N-1}})| + E_A\left(E\left[\int_{\tau_{N-1}}^T g^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\right] - \right. \\
&\left. -E\left[\int_{\tau_{N-1}}^T g^\eta(s,\cdot)ds|\mathcal{F}_t\right]\right)\Bigg| \le \sup_{t\le\tau\le t+\delta} |E_A(x_t - x_\tau)| + E\int_t^{t+\delta} m(s,\cdot)ds + \\
&+ \sum_{i=p}^{N-2} E\left|x_{\tau_i} - E\left[x_{\tau_{i+1}} + \int_{\tau_i}^{\tau_{i+1}} g^\eta(s,\cdot)ds|\mathcal{F}_{\tau_i}\right]\right| \\
&+ E\left|x_{\tau_{N-1}} - E\left[x_T + \int_{\tau_{N-1}}^T g^\eta(s,\cdot)d\tau|\mathcal{F}_{\tau_{N-1}}\right]\right| \\
&+ \sum_{i=p}^{N-2} |E_A(E[x_{\tau_{i+1}}|\mathcal{F}_{\tau_i}] - x_{\tau_{i+1}})| + \sum_{i=p}^{N-2}\left|E_A\left(E\left[\int_{\tau_i}^{\tau_{i+1}} g^\eta(s,\cdot)ds|\mathcal{F}_{\tau_i}\right] - \right.\right. \\
&\left.\left. -E\left[\int_{\tau_i}^{\tau_{i+1}} g^\eta(s,\cdot)ds|\mathcal{F}_t\right]\right)\right| \\
&+ \left|E_A\left(E\left[\int_{\tau_{N-1}}^T g^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\right] - E\left[\int_{\tau_{N-1}}^T g^\eta(s,\cdot)ds|\mathcal{F}_t\right]\right)\right|.
\end{aligned}$$

But $\mathcal{F}_t \subset \mathcal{F}_{\tau_i}$ for $i = p, p+1, \dots, N-1$. Then for $A \in \mathcal{F}_t$ one has

$$\sum_{i=p}^{N-2} |E_A(E[x_{\tau_{i+1}}|\mathcal{F}_{\tau_i}] - x_{\tau_{i+1}})| = 0,$$

$$\sum_{i=p}^{N-2} \left| E_A\biggl(E\left[\int_{\tau_i}^{\tau_{i+1}} g^\eta(s,\cdot)ds|\mathcal{F}_{\tau_i}\right] - E\left[\int_{\tau_i}^{\tau_{i+1}} g^\eta(s,\cdot)ds|\mathcal{F}_t\right]\biggr)\right| = 0$$

and

$$\left| E_A\biggl(E\biggl[\int_{\tau_{N-1}}^{T} g^\eta(s,\cdot)ds|\mathcal{F}_{\tau_{N-1}}\biggr] - E\biggl[\int_{\tau_{N-1}}^{T} g^\eta(s,\cdot)ds|\mathcal{F}_t\biggr]\biggr)\right| = 0.$$

Hence it follows

$$\left| E_A\left(x_t - E\left[x_T + \int_t^T g^\eta(s,\cdot)ds|\mathcal{F}_t\right]\right)\right| \leq \eta$$

for fixed $0 \leq t \leq T$ and $A \in \mathcal{F}_t$. Let $(\eta_j)_{j=1}^\infty$ be a sequence of positive numbers converging to zero. For every $j = 1, 2, \dots$ we can select $g^{\eta_j} \in S_{\mathbb{F}}(co\ G)$ such that (3.2) is satisfied with $\eta = \eta_j$. By the weak compactness of $S_{\mathbb{F}}(co\ G)$ there are $g \in S_{\mathbb{F}}(co\ G)$ and a subsequence $(g^{\eta_k})_{k=1}^\infty$ of $g^{\eta_j})_{j=1}^\infty$ weakly converging to g in $L([0,T]\times\Omega, \Sigma_{\mathbb{F}}, \mathbb{R})$. Then for every $A \in \mathcal{F}_t \subset \mathcal{F}$ one has $\lim_{k\to\infty} E_A \int_t^T g^{\eta_k}(s,\cdot)ds = E_A \int_t^T g(s,\cdot)ds$. On the other hand for every fixed $t \in [0,T]$ and $A \in \mathcal{F}_t$ we have

$$\begin{aligned}
&\left| E_A\biggl(x_t - E\biggl[x_T + \int_t^T g(s,\cdot)ds|\mathcal{F}_t\biggr]\biggr)\right| \leq \\
&\leq \left| E_A\biggl(x_t - E\biggl[x_T + \int_t^T g^{\eta_k}(s,\cdot)ds|\mathcal{F}_t\biggr]\biggr)\right| + \\
&+ \left| E_A\biggl(E\biggl[\int_t^T g^{\eta_k}(s,\cdot)ds|\mathcal{F}_t\biggr] - E\biggl[+\int_t^T g(s,\cdot)ds|\mathcal{F}_t\biggr]\biggr)\right| \leq \\
&\leq \eta_k + \left| E_A \int_t^T g^{\eta_k}(s,\cdot)ds - E_A \int_t^T g(s,\cdot)ds\right|
\end{aligned}$$

for $k = 1, 2, \dots$. Therefore

$$E_A\biggl(x_t - E\biggl[x_T + \int_t^T g(s,\cdot)ds|\mathcal{F}_t\biggr]\biggr) = 0$$

for every $A \in \mathcal{F}_t$ and fixed $0 \leq t \leq T$. But x_t and $E[x_T + \int_t^T g(s,\cdot)ds|\mathcal{F}_t]$ are $\mathcal{F}_t$-measurable. Then $x_t = E[x_T + \int_t^T g(s,\cdot)ds|\mathcal{F}_t]$ for $0 \leq t \leq T$ with (P.1). □

Corollary 3.2. *Let $G : [0,T]\times\Omega \to Cl(\mathbb{R}^m)$ be measurable $\mathbb{F}$-adapted and square integrably bounded. If $x = (x_t)_{0\leq t\leq T}$ is mesurable, satisfies (3.1) a.s. for every $0 \leq s \leq t \leq T$ and $E|x_T|^2 < \infty$ then $x \in \mathcal{S}^2(\mathbb{F}, \mathbb{R}^m)$ and $x_t = x_0 + M_t + A_t$, where $M_t = E[x_T + \int_0^T g_\tau d\tau|\mathcal{F}_t] - E[x_T + \int_0^T g_\tau d\tau|\mathcal{F}_0]$ and $A_t = -\int_0^t g_\tau d\tau$ for $0 \leq t \leq T$ with $g \in S_{\mathbb{F}}(coG)$ such that $x_t = E[x_T + \int_t^T g_\tau d\tau|\mathcal{F}_t]$ a.s. for $0 \leq t \leq T$.*

Proof. The result follows immediately from the representation $x_t = E[x_T + \int_t^T g_\tau d\tau|\mathcal{F}_t]$ given in Theorem 3.1 (see [3], Lemma 1.1). □

In what follows we shall assume that $F : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : \mathbb{R}^m \to Cl(\mathbb{R}^m)$ satisfy the following conditions (A):

(i) *F is measurable and uniformly square integrably bounded by a function $m \in L^2([0,T],\mathbb{R}_+)$,*

(ii) *H is measurable and bounded by a number $L > 0$,*

(iii) *$F(t,\cdot)$ is Lipschitz continuous, i.e. there is $k \in L^2([0,T],\mathbb{R}_+)$ such that $h(F(t,x_1),F(t,x_2)) \leq k(t)|x_1 - x_2|$ for a.e. $t \in [0,T]$ and $x_1, x_2 \in \mathbb{R}^m$, where h is the Hausdorff metric on $Cl(\mathbb{R}^m)$,*

(iv) *there is a random variable $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{R}^m)$ such that $\xi \in H(\xi)$ a.s.*

We shall prove now that conditions (A) imply the existence of some special sequence of successive approximations for BSDI(F,H).

Theorem 3.3. *Let $F : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : \mathbb{R}^m \to Cl(\mathbb{R}^m)$ satisfy conditions (A). There exists a sequence $(x^n)_{n=1}^\infty$ of $\mathcal{S}^2(\mathbb{F},\mathbb{R}^m)$ defined by $x_t^n = E[\xi + \int_t^T f_\tau^{n-1} d\tau|\mathcal{F}_t]$ a.s. with $f^{n-1} \in S_\mathbb{F}(F \circ x^{n-1})$ for $n = 1,2,\ldots$ and $0 \leq t \leq T$ such that $x_s^n \in E[x_t^n + \int_s^t F(\tau, x_\tau^{n-1})d\tau|\mathcal{F}_s]$ a.s. and $E\sup_{t\leq u\leq T}|x_u^{n+1} - x_u^n|^2 \leq 4E(\int_t^T k(\tau)\sup_{\tau\leq s\leq T}|x_s^n - x_s^{n-1}|d\tau)^2$ for $n = 1,2\ldots$ and $0 \leq s \leq t \leq T$.*

Proof. Let us observe (see [8], Th. II.3.13) that for every m-dimensional measurable and $\mathbb{F}$-adapted processes x and y on $\mathcal{P}_\mathbb{F}$ and every $f^x \in S_\mathbb{F}(F\circ x)$ there is $f^y \in S_\mathbb{F}(F\circ y)$ such that $|f_t^x(\omega) - f_t^y(\omega)| = \text{dist}(f_t^x(\omega),\, F(t,y_t(\omega)) \leq h(F(t,x_t(\omega)),\, F(t,y_t(\omega)) \leq k(t)|x_t(\omega) - y_t(\omega)|$ for a.e. $t \in [0,T]$ and $\omega \in \Omega$. Furthermore, by properties of H there is $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{R}^m)$ such that $\xi \in H(\xi)$ a.s. Let $(x_t^0)_{0\leq t\leq T}$ be an m-dimensional measurable $\mathbb{F}$-adapted process on $\mathcal{P}_\mathbb{F}$ such that $x_T^0 = \xi$ a.s. and let $f^0 \in S_\mathbb{F}(F \circ x^0)$. Define $x_t^1 = E[\xi + \int_t^T f_\tau^0 d\tau|\mathcal{F}_t]$ a.s. for $0 \leq t \leq T$. By Corollary 3.2 we have $x^1 \in \mathcal{S}^2(\mathbb{F},\mathbb{R}^m)$. Select now $f^1 \in S_\mathbb{F}(F\circ x^1)$ such that $|f_t^1 - f_t^0| = \text{dist}(f_t^1, F(t,x_t^0))$ for a.e. $0 \leq t \leq T$ with (P.1). Then $|f_t^1 - f_t^0| \leq k(t)|x_t^1 - x_t^0|$ a.s. for a.e. $0 \leq t \leq T$. Define $x_t^2 = E[\xi + \int_t^T f_\tau^1 d\tau|\mathcal{F}_t]$ a.s. for $0 \leq t \leq T$. We have $x^2 \in \mathcal{S}^2(\mathbb{F},\mathbb{R}^m)$. Continuing the above procedure we can define $x_t^{n+1} = E[\xi + \int_t^T f_\tau^n d\tau|\mathcal{F}_t]$ a.s. for $0 \leq t \leq T$ with $f^n \in S_\mathbb{F}(F \circ x^n)$ such that $|f_t^n - f_t^{n-1}| \leq k(t)|x_t^n - x_t^{n-1}|$ a.s. for a.e. $0 \leq t \leq T$ and $n = 2,3,\ldots$. By Corollary 3.2 we also have $x^n \in \mathcal{S}^2(\mathbb{F},\mathbb{R}^m)$. Hence it follows

$$\begin{aligned} |x_t^{n+1} - x_t^n| &\leq E\left[\int_t^T |f_\tau^n - f_\tau^{n-1}|d\tau|\mathcal{F}_t\right] \\ &\leq E\left[\int_t^T k(\tau)\sup_{\tau\leq s\leq T}|x_s^n - x_s^{n-1}||\mathcal{F}_t\right] \end{aligned}$$

a.s. for $0 \le t \le T$. Therefore,

$$\begin{aligned}
&\sup_{t\le u\le T} |x_u^{n+1} - x_u^n| \\
&\le \sup_{t\le u\le T} E\left[\int_u^T k(\tau) \sup_{\tau\le s\le T} |x_s^n - x_s^{n-1}| d\tau |\mathcal{F}_u\right] \le \\
&\le \sup_{t\le u\le T} E\left[\int_t^T k(\tau) \sup_{\tau\le s\le T} |x_s^n - x_s^{n-1}| d\tau |\mathcal{F}_u\right]
\end{aligned}$$

a.s. for $0 \le t \le T$ and $n = 1, 2, \ldots$. By Doob's inequality, we obtain

$$\begin{aligned}
&E\left(\sup_{t\le u\le T} E\left[\int_t^T k(\tau) \sup_{\tau\le s\le T} |x_s^n - x_s^{n-1}| d\tau |\mathcal{F}_u\right]\right)^2 \le \\
&\le 4E\left(\int_t^T k(\tau) \sup_{\tau\le s\le T} |x_s^n - x_s^{n-1}| d\tau\right)^2
\end{aligned}$$

for $0 \le t \le T$. Therefore, for every $n = 1, 2, \ldots$ and $0 \le t \le T$ we have

$$E \sup_{t\le u\le T} |x_u^{n+1} - x_u^n|^2 \le\le 4E\left(\int_t^T k(\tau) \sup_{\tau\le t\le T} |x_s^n - x_s^{n-1}| d\tau\right)^2.$$

□

4. EXISTENCE OF STRONG SOLUTIONS

We shall prove that if F and H satisfy conditions (A) then BSDI(F,H) possesses at least one strong solution. Let us observe that immediately from Corollary 3.2 it follows that every strong solution to BSDI(F,H) on $\mathcal{P}_{\mathbb{F}}$ belongs to $\mathcal{S}^2(\mathbb{F}, \mathbb{R}^m)$. Immediately from the properties of multivalued conditional expections (see [5], Prop. 6.2.) the following result follows.

Proposition 4.1. *Let F satisfies conditions (A). Then for every $x, y \in \mathcal{S}^2(\mathbb{F}, \mathbb{R}^m)$ one has*

$$Eh\left(E\left[\int_s^t F(\tau, x_\tau)\, d\tau |\mathcal{F}_s\right], E\left[\int_s^t F(\tau, y_\tau) d\tau |\mathcal{F}_s\right]\right) \le \int_s^t k(\tau) E|x_\tau - y_\tau| d\tau$$

for every $0 \le s \le t \le T$, where h is the Hausdorff metric on $Cl(\mathbb{R}^m)$.

We can prove now the following existence theorem.

Theorem 4.2. *Let $\mathcal{P}_{\mathbb{F}}$ be given. If $F : [0, T] \times \mathbb{R}^n \to Cl(\mathbb{R}^n)$ and $H : \mathbb{R}^m \to Cl(\mathbb{R}^m)$ satisfy conditions (A) then $BSDI(F, H)$ possesses a strong solution on $\mathcal{P}_{\mathbb{F}}$.*

Proof. By virtue Theorem 3.3 there is a sequence $(x^n)_{n=1}^\infty$ of $\mathcal{S}^2(\mathbb{F}, \mathbb{R}^m)$ such that $x_T^n = \xi$, $x_s^n \in E[x_t^n + \int_s^t F(\tau, x_\tau^{n-1} d\tau |\mathcal{F}_t]$ a.s. for $0 \le s \le t \le T$ and

$$E \sup_{t\le u\le T} |x_u^{n+1} - x_u^n|^2 \le 4E\left(\int_t^T k(\tau) \sup_{\tau\le s\le T} |x_s^n - x_s^{n-1}| d\tau\right)^2,$$

where $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^m)$ is such that $\xi \in H(\xi)$ a.s. Hence it follows

$$E \sup_{t \leq u \leq T} |x_u^{n+1} - x_u^n|^2 \leq 4T \int_t^T k^2(\tau) E \sup_{\tau \leq u \leq T} |x_u^n - x_u^{n-1}|^2 d\tau$$

for $n = 1, 2, \dots$. and $0 \leq t \leq T$. By the properties of F and H one has $E \sup_{t \leq u \leq T} |x_u^1 - x_u^0|^2 \leq \mathbb{L}$, where $\mathbb{L} = 4(E|\xi|^2 + \int_0^T m^2(\tau)d\tau) + 2E \sup_{0 \leq t \leq T} |x_t^0|^2$. Therefore,

$$E \sup_{t \leq u \leq T} |x_u^2 - x_u^1|^2 \leq 4T\mathbb{L} \int_t^T k^2(\tau)d\tau.$$

Hence it follows

$$E \sup_{t \leq u \leq T} |x_u^3 - x_u^2|^2 \leq (4T)^2\mathbb{L} \int_t^T \left(k^2(\tau) \int_\tau^T k^2(s)ds \right) d\tau$$

$$= \frac{(4T)^2\mathbb{L}}{2} \left(\int_t^T k^2(\tau)d\tau \right)^2.$$

By the induction procedure for every $n = 1, 2, \dots$ and $0 \leq t \leq T$ we get

$$E \sup_{t \leq u \leq T} |x_u^{n+1} - x_u^n|^2 \leq \frac{(4T)^n\mathbb{L}^{n-1}}{n!} \left(\int_t^T k^2(\tau)d\tau \right)^n.$$

Then $E \sup_{0 \leq t \leq T} |x_t^n - x_t^m|^2 \to 0$ as $n, m \to \infty$. Therefore, there is a process $(x_t)_{0 \leq t \leq T} \in \mathcal{S}^2(\mathbb{F}, \mathbb{R}^m)$ such that $E \sup_{0 \leq t \leq T} |x_t^n - x_t|^2 \to 0$ as $n \to \infty$. Hence and Proposition 4.1 it follows

$$E\text{dist}\left(x_s, E\left[x_t + \int_s^t F(\tau, x_\tau)d\tau | \mathcal{F}_s \right] \right)$$

$$\leq E|x_s - x_s^n| + E\text{dist}\left(x_s^n, E\left[x_t^n + \textstyle\int_s^t F(\tau, x_\tau^{n-1})d\tau | \mathcal{F}_s \right] \right) +$$

$$+Eh\left(E\left[x_t^n + \int_s^t F(\tau, x_\tau^{n-1})d\tau | \mathcal{F}_s \right], E\left[x_t + \int_s^t F(\tau, x_\tau)d\tau | \mathcal{F}_s \right] \right)$$

$$\leq E|x_s^n - x_s| + E|x_t^n - x_t| + \int_s^t k(\tau)E|x_\tau^{n-1} - x_\tau|d\tau$$

$$\leq 2\|x^n - x\|_{\mathcal{S}^2} + \left(\int_0^T k^2(\tau)d\tau \right)^{\frac{1}{2}} \|x^{n-1} - x\|_{\mathcal{S}^2}$$

for every $0 \leq s \leq t \leq T$ and $n = 1, 2, \dots$ Therefore, $\text{dist}(x_s, E[x_t + \int_s^t F(\tau, x_\tau)d\tau | \mathcal{F}_s]) = 0$ a.s. for every $0 \leq s \leq t \leq T$. Then $x_s \in E\left[x_t + \int_s^t F(\tau, x_\tau)d\tau | \mathcal{F}_s \right]$ a.s. for every $0 \leq s \leq t \leq T$. By the definition of $(x_t^n)_{0 \leq t \leq T}$ we have $x_T^n = \xi \in H(\xi)$ a.s. for every $n = 1, 2, \dots$. Therefore, we also have $x_T = \xi$ a.s. Then $x_T \in H(x_T)$ a.s. $\square$

5. EXISTENCE OF WEAK SOLUTIONS

Asume that $F : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : \mathbb{R}^m \to Cl(\mathbb{R}^m)$ satisfy the following conditions (B).

(i) *F is measurable and uniformly square integrably bouned by a function $m \in L^2([0,T], \mathbb{R}_+)$,*

ii) *H takes on convex values is measurable and bounded by a number $L > 0$,*

(iii) *$F(t,\cdot)$ and H are lower semicontinuous for a.e. fixed $0 \le t \le T$.*

We shall prove that for F and H satisfying conditions (B) there exists a continuous weak solution to BSDI(F,H), i.e. there exists a pair $(\mathcal{P}_{\mathbb{F}}, x)$, with x having a.a. continuous trajectories and satisfying BSDI(F,H). The result is obtained by the construction of the Tonelli's type approximations for a backward stochastic differential equation defined by some special selectors of F and H on a filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, P, \mathbb{F}^B)$ with $(\Omega, \mathcal{F}, P)$ supporting an d-dimensional Browninan motion B and $\mathbb{F}^B = (\mathcal{F}_t^B)_{0 \le t \le T}$ beeing a natural augmented filtration of B. The tightness of such approximation sequence will follow from the following extension of the classical tightness criterion (see [1], Th. 2.12.3).

Theorem 5.1 ([9], Th. 3)**.** *A sequence $(x^n)_{n=1}^\infty$ of continuous m-dimensional stochastic processes $x^n = (x^n(t))_{0 \le t \le T}$ on a probability space $(\Omega, \mathcal{F}, P)$ is tight if for every $\varepsilon > 0$ there is a number $a > 0$ such that $P(\{|x^n(0)| > a\}) \le \varepsilon$ for $n \ge 1$ and there are $\gamma \ge 0$, an integer $\alpha > 1$ and a continuous nondecreasing bounded stochastic process $(\Gamma(t))_{0 \le t \le T}$ on $(\Omega, \mathcal{F}, P)$ such that*

$$P\left(\{|x^n(t) - x^n(s)| \ge \eta\}\right) \le \frac{1}{\eta^\gamma} E\left|\Gamma(t) - \Gamma(s)\right|^\alpha$$

for every $n \ge 1$, $\eta > 0$ and $s, t \in [0,T]$.

We shall also need the following results.

Proposition 5.2. *Let $F : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : \mathbb{R}^m \to Cl(\mathbb{R}^m)$ be measurable and uniformly square integrably bounded and bounded, respectively. A pair $(\mathcal{P}_{\mathbb{F}}, x)$ is a continuous weak solution to BSDI(F,H) if and only if there exist $\xi \in S(H \circ x_T)$ and $f \in S_{\mathbb{F}}(coF \circ x)$ such that $x_t = E[\xi + \int_t^T f_\tau d\tau | \mathcal{F}_t]$ a.s. for $0 \le t \le T$ and such that a martingale $M = (M_t)_{0 \le t \le T}$ defined by $M_t = E[\xi + \int_0^T f_\tau d\tau | \mathcal{F}_t] - E[\xi + \int_0^T f_\tau d\tau | \mathcal{F}_0]$ is continuous.*

Proof. By virtue of Theorem 3.1 a pair $(\mathcal{P}_{\mathbb{F}}, x)$ is a weak solution to BSDI(F,H) if and only if there are $\xi \in S(H \circ x_T)$ and $f \in S_{\mathbb{F}}(coF \circ x)$ such that $x_t = E[\xi + \int_t^T f_\tau d\tau | \mathcal{F}_t]$ a.s. for $0 \le t \le T$. By Corollary 3.2 we have $x_t = x_0 + M_t - \int_0^t f_\tau d\tau$ a.s. for $0 \le t \le T$. Hence it follows that x is continuous if and only if M is continuous. $\square$

Proposition 5.3. *Let $F : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : \mathbb{R}^m \to Cl(\mathbb{R}^m)$ be measurable and uniformly square integrably bounded and bounded, respectively. Assume $F(t,\cdot)$ and H are continuos and let $(x^n)_{n=1}^{\infty}$ be a sequence of continuous solutions to BSDI(F,H) on a filtered probability space $\mathcal{P}_{\mathbb{F}}$. Then $(x^n)_{n=1}^{\infty}$ is tight.*

Proof. By virtue of Corollary 3.2 we have $x_t^n = x_0^n + M_t^n - \int_0^t f_\tau^n d\tau$ a.s. for $0 \leq t \leq T$, where $f^n \in S_{\mathbb{F}}(coF \circ x^n)$ and M^n is an $\mathbb{F}$-martingale defined above for $n = 1, 2, \dots$. By properties of F and Proposition 5.2, M^n is for every $n = 1, 2, \dots$ a square integrable continuous martingale such that $M_0^n = 0$ for $n \geq 1$. Furthermore $|M_t^n| \leq 2\lambda$ a.s. for $0 \leq t \leq T$, where $\lambda = L + \int_0^T m(t)dt$. Denote by N_i^n for $n \geq 1$ and $i = 1, 2, \dots, m$ a real-valued $\mathbb{F}$-martingale such that $M_t^n = (N_1^n(t), \dots, N_m^n(t))$ for $0 \leq t \leq T$. For every $i = 1, \dots, m$ and every partition $\Delta = \{0 = t_0 < t_1 < \cdots < t_r = T\}$ of $[0,T]$ one gets

$$\langle N_i^n \rangle_t^{\Delta} \doteq \sum_{k=0}^{r-1} (N_i^n(t \wedge t_{k+1}) - N_i^n(t \wedge t_k))^2 \leq$$

$$\leq \sum_{j=1}^{r-1} |N_i^n(t \wedge t_{j+1}) - N_i^n(t \wedge t_j)| \max_k |N_i^n(t \wedge t_{k+1}) - N_i^n(t \wedge t_k)|$$

$$\leq |N_i^n(t)| \max_k |N_i^n(t \wedge t_{k+1}) - N_i^n(t \wedge t_k)| \leq 8\lambda^2$$

a.s. for $0 \leq t \leq T$. Moreover, by ([10], Th. 2.2.2) we have $\sup_{0 \leq t \leq T} E|\langle N_i^n \rangle_t^{\Delta_r} - \langle N_i^n \rangle_t|^2 \to 0$ as $|\Delta_r| \to 0$, where $|\Delta_r| = \max_{0 \leq k \leq r-1}(t_{k+1} - t_k)$. Then there is a subsequence $(\Delta_{r_j})_{j=1}^{\infty}$ of $(\Delta_r)_{r=1}^{\infty}$ such that $\sup_{0 \leq t \leq T} |\langle N_i^n \rangle_t^{\Delta_{r_j}} - \langle N_i^n \rangle_t| \to 0$ a.s. as $j \to 0$. Hence it follows

$$\sup_{0 \leq t \leq T} |\langle N_i^n \rangle| \leq \sup_{0 \leq t \leq T} \left| \langle N_i^n \rangle_t - \langle N_i^n \rangle_t^{\Delta_{r_j}} \right| + \left| \langle N_i^n \rangle_t^{\Delta_{r_j}} \right|$$

$$\leq \sup_{0 \leq t \leq T} \left| \langle N_i^n \rangle_t - \langle N_i^n \rangle_t^{\Delta_{r_j}} \right| + 8\lambda^2$$

a.s. for every $n \geq 1$ and $i = 1, \dots, m$. Then $\sup_{0 \leq t \leq T} |\langle N_i^n \rangle_t| \leq 8\lambda^2$ a.s. for every $n \geq 1$ and $i = 1, \dots, m$. Let us observe that quadratic variational process $(\langle N_i^n \rangle_t)_{0 \leq t \leq T}$ is increasing in t a.s. for every $n \geq 1$ and $i = 1, \dots, m$. Then for every $n \geq 1$, $i = 1, \dots, m$ and P-a.e. $\omega \in \Omega$ it generates a measure $\mu_i^n(\omega)$ on $\beta_T = \beta([0,T)$ such that $\mu_i^n(\omega)((s,t]) = \langle N_i^n \rangle_t - \langle N_i^n \rangle_s$ and $\mu_i^n(\omega)(\{0\}) = 0$. Let $\mu^n(\omega)(A) = \max_{0 \leq i \leq m} \mu_i^n(\omega)(A)$ and $\mu(\omega)(A) = \sup_{n \geq 1} \mu^n(\omega)(A)$ for $A \in \beta_T$ and P-a.e. $\omega \in \Omega$. Similarly as in the proof of ([5], Prop. 8.5.17) it can be verified that $\mu(\omega)$ is a measure on β_T for P-a.e. $\omega \in \Omega$. It can be also verified that for every $A \in \beta_T$ a mapping $\Omega \ni \to \mu(\omega)(A) \in \mathbb{R}^+$ is a random variable such that $\mu(\omega)((0,T]) \leq 8\lambda^2$ for P-a.e $\omega \in \Omega$. By Itô's formula and Doob's inequality one obtains

$$E\left(N_i^n(t) - N_i^n(s)\right)^{2k} =$$

$$E\left(\int_s^t dN_i^n(u)\right)^{2k} = k(k-1)E\int_s^t\left(\int_s^\tau dN_i^n(u)\right)^{2(k-1)} d\left\langle N_i^n\right\rangle_\tau$$

$$\leq k(k-1)\left[E\sup_{s\leq\tau\leq t}\left(\int_s^\tau dN_i^n(u)\right)^{4(k-1)}\right]^{\frac{1}{2}}\left[E\left(\int_s^t d\left\langle N_i^n\right\rangle_\tau\right)^2\right]^{\frac{1}{2}} \leq$$

$$\leq C_k\left[E\left(\int_s^t dN_i^n(u)\right)^{4(k-1)}\right]^{\frac{1}{2}}\left[E\left(\int_s^t d\left\langle N_i^n\right\rangle_\tau\right)^2\right]^{\frac{1}{2}},$$

where $C_k = k(k-1)\left(\frac{4(k-1)}{4k-3}\right)^{4(k-1)}$ for $0 \leq s < t \leq T$, $n \geq 1$ and $i = 1, \ldots, m$. Hence, in particular for $k = 2$ it follows

$$E\left(\mathbf{1}_{A_i^n(s,t)}\left(N_i^n(t) - N_i^n(s)\right)^4\right) =$$

$$\leq C_2\left[E\left(\mathbf{1}_{A_i^n(s,t)}\left(N_i^n(t) - N_i^n(s)\right)^4\right)\right]^{\frac{1}{2}}\left[E\left(\int_s^t d\mu\right)^2\right]^{\frac{1}{2}},$$

where $A_i^n(s,t) = \{\omega \in \Omega : N_i^n(t) - N_i^n(s) > 0\}$ for fixed $0 \leq s < t \leq T$, $n \geq 1$ and $i = 1, \ldots, m$. Then

$$\left[E\mathbf{1}_{A_i^n(s,t)}\left(N_i^n(t) - N_i^n(s)\right)^4\right]^{\frac{1}{2}} \leq C_2\left[E\left(\int_s^t d\mu\right)^2\right]^{\frac{1}{2}},$$

which implies that

$$E\left(N_i^n(t) - N_i^n(s)\right)^4 \leq C_2^2 E\left(\int_s^t d\mu\right)^2$$

for $0 \leq s < t \leq T$, $n \geq 1$ and $i = 1, \ldots, m$. Hence it follows

$$E\left|M_t^n - M_s^n\right|^4 = E\left(\sum_{i=1}^m \left|N_i^n(t) - N_i^n(s)\right|\right)^4 \leq$$

$$C_m\sum_{i=1}^m E\left(N_i^n(t) - N_i^n(s)\right)^4 \leq C_m m C_2^2 E\left(\int_s^t d\mu\right)^2$$

for $0 \leq s < t \leq T$ and $n \geq 1$, where C_m is a positive number depending on m. Finally, there is a positive number $C = \int_0^T m^2(t)dt$ such that

$$E\left|x_t^n - x_s^n\right|^4 \leq 4E\left|M_t^n - M_s^n\right|^4 + 4E\left|\int_s^t f_\tau^n d\tau\right|^4$$

$$\leq 4C_m m C_2^2 E\left(\int_s^t d\mu\right)^2 + 4C^2(t-s)^2 \leq E\left[2C_2\sqrt{mC_m}\int_s^t d\mu + 2C(t-s)\right]^2$$

$$= E\left|\Gamma(t) - \Gamma(s)\right|^2,$$

for $0 \leq s < t \leq T$ and $n \geq 1$, where $\Gamma(t) = 2C_2\sqrt{mC_m}\int_0^t d\mu + 2Ct$. Hence, by Doob's inequality it follows

$$P\left(\{|x_t^n - x_s^n| \geq \eta\}\right) \leq \frac{1}{\eta^4}E|x_t^n - x_s^n|^4 \leq \frac{1}{\eta^4}E\left|\Gamma(t) - \Gamma(s)\right|^2$$

for $\eta > 0$, $0 \le s < t \le T$ and $n \ge 1$. Finally, let us recall that $|x_t^n| \le \lambda$ a.s. for $0 \le t \le T$ and $n \ge 1$ with $\lambda = L + \int_0^T m(t)dt$. Therefore, for every $N \ge 1$ one has $P(\{|x_0^n| > N\}) \le \lambda/N$ for $n \ge 1$. Then $\sup_{n\ge1} P(\{|x_0^n| > N\}) \to 0$ as $N \to \infty$. Therefore, for every $\varepsilon > 0$ there is $a > 0$ such that $P(\{|x_0^n| > a\}) \le \varepsilon$ for $n \ge 1$. Then by virtue of Theorem 5.1 a sequence $(x^n)_{n=1}^\infty$ is tight. □

We can prove now the existence of continuous weak solutions to BSDI(F,H).

Theorem 5.4. *Let $F : [0,T] \times \mathbb{R}^m \to Cl(\mathbb{R}^m)$ and $H : \mathbb{R}^m \to Cl(\mathbb{R}^m)$ satisfy conditions (B). Then BSDI(F,H) possesses a continuous weak solution.*

Proof. By Michael's and Rybinski's continuous selection theorems (see [8], Th. II.4.1 and [15], Th. 2) there exist a continuous selector h of H and a Carathéodory type selector $f : [0,T]\times\mathbb{R}^m \to \mathbb{R}^m$ of coF. Let $\mathcal{P} = (\Omega, \mathcal{F}, P)$ be a probability space such that there is a d-dimensional Brownian motion B defined on this space. Let $\mathbb{F} = (\mathcal{F}_t)_{t\ge0}$ be a natural augmented filtration of B. Let $\xi \in \mathcal{B}_\lambda$ be a fixed point of h, where $\mathcal{B}_\lambda$ is a closed ball of $\mathbb{R}^m$ centered at the origin with the radius $\lambda = L + \int_0^T m(t)dt$. Define on $\mathcal{P}_\mathbb{F} = (\Omega, \mathcal{F}, P, \mathbb{F})$ a sequence $(x^n)_{n=1}^\infty$ of stochastic processes $x^n = (x_t^n)_{0\le t\le 2T}$ such that $x_t^n = \xi$ a.s. for $t \in [T, 2T]$ and $x_t^n = E[h(\xi) + \int_t^T f(\tau, x^n_{\tau+T/n}d\tau|\mathcal{F}_t]$ a.s. for $0 \le t \le T$ and $n = 1, 2 \ldots$. Let us observe that for every $n \ge 1$ a process x^n is defined step by step begining with the interval $[T - T/n, T]$. For example, for $t \in [T - T/n, T]$ we have $x_t^n = E[h(\xi) + \int_t^T f(\tau, \xi)d\tau|\mathcal{F}_t]$ a.s. For $t \in [T - 2T/n, T - T/n]$ we have $x_t^n = E[h(\xi) + \int_t^T f(\tau, \tilde{x}^n_{\tau+T/n}d\tau|\mathcal{F}_t]$ a.s. with $\tilde{x}^n_{\tau+T/n} = E[h(\xi) + \int_{\tau+T/n}^T f(u, \xi)du|\mathcal{F}_{\tau+T/n}]$ because $\tau + T/n \in [T - T/n, T]$. Let us observe that x^n is for every $n \ge 1$ a continuous process because of ([14], Corollary IV.1) a process $M^n = (M_t^n)_{0\le t\le T}$ defined by

$$M_t^n = E\left[h(\xi) + \int_0^T f(\tau, x^n_{\tau+T/n}d\tau|\mathcal{F}_t\right] - E\left[h(\xi) + \int_0^T f(\tau, x^n_{\tau+T/n}d\tau|\mathcal{F}_0\right]$$

is a continuous $\mathbb{F}$-martingale for every $n \ge 1$. Similarly as in the proof of Proposition 5.3 we can verify that the sequence $(x^n)_{n=1}^\infty$ is tight. Then by ([6], Th. I.2.7) there are a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, a sequence $(\tilde{x}^{n_k})_{k=1}^\infty$ of continuous m-dimensional stochastic processe $(\tilde{x}_t^{n_k})_{0\le t\le 2T}$ and a continuous stochastic process $\tilde{x} = (\tilde{x}_t)_{0\le t\le 2T}$ such that $P(x^{n_k})^{-1} = P(\tilde{x}^{n_k})^{-1}$ for $k \ge 1$ and $\sup_{0\le t\le 2T} |\tilde{x}_t^{n_k} - \tilde{x}_t| \to 0$ a.s. as $k \to \infty$. Let $\tilde{\mathcal{F}}_t = \bigcap_{\varepsilon>0} \sigma[\tilde{x}_u : u \le t + \varepsilon]$ and let $\phi : C_T \to \mathbb{R}$ be a continuous and bounded function such that ϕ is $\beta_s(C_T)$-measurable, where $\beta_s(C_T) = \rho_s^{-1}(\beta(C_T))$ with $\rho_s(x) = x(s \wedge u)$ for $x \in C_T$ and $u \in [0,T]$. Similarly as in the proof of Proposition 5.3 we can verify that $|x_t^n| \le \lambda$ a.s. for $0 \le t \le T$ and $n \ge 1$, where $\lambda = L + \int_0^T m(t)dt$. Hence by the properties of $\tilde{x}^n$ we also have $|\tilde{x}_t^n| \le \lambda$ a.s. for $0 \le t \le T$ and $n \ge 1$. Let $\mathcal{B}_\lambda$ be a closed ball of $\mathbb{R}^m$ such as above and let $g : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous extension of a mapping $I : \mathcal{B}_\lambda \to \mathbb{R}^m$ defined by $I(x) = x$ for $x \in \mathcal{B}_\lambda$. We have

$|g(x)| \leq \lambda$ for $x \in \mathbb{R}^m$, $g(x_t^n) = x_t^n$ and $g(\tilde{x}_t^n) = \tilde{x}_t^n$ a.s. for $0 \leq t \leq T$ and $n \geq 1$. Therefore,

$$\begin{aligned}
0 &= E\left\{\phi(x^n)\left(x_s^n - E[x_t^n + \int_s^t f(\tau, x_{\tau+T/n}^n)d\tau|\mathcal{F}_s]\right)\right\} \\
&= E\left\{E[\phi(x^n)\left(x_s^n - x_t^n - \int_s^t f(\tau, x_{\tau+T/n}^n)d\tau|\mathcal{F}_s]\right)\right\} \\
&= E\left\{\phi(x^n)\left(g(x_s^n) - g(x_t^n) - \int_s^t f(\tau, x_{\tau+T/n}^n)d\tau\right)\right\} \\
&= \tilde{E}\left\{\phi(\tilde{x}^n)\left(g(\tilde{x}_s^n) - g(\tilde{x}_t^n) - \int_s^t f(\tau, \tilde{x}_{\tau+T/n}^n)d\tau\right)\right\} \\
&= \tilde{E}\left\{\phi(\tilde{x}^n)\left(\tilde{x}_s^n - \tilde{x}_t^n - \int_s^t f(\tau, \tilde{x}_{\tau+T/n}^n)d\tau\right)\right\}
\end{aligned}$$

for $0 \leq s \leq t \leq T$ and $n \geq 1$. Hence, it follows

$$\begin{aligned}
&\tilde{E}\left\{\phi(\tilde{x})\left(\tilde{x}_s - \tilde{x}_t - \int_s^t f(\tau, \tilde{x}_\tau)d\tau\right)\right\} \\
&= \lim_{n\to\infty} \tilde{E}\left\{\phi(\tilde{x})\int_s^t [f(\tau, \tilde{x}_\tau) - f(\tau, \tilde{x}_{\tau+T/n})]d\tau\right\} \\
&+ \lim_{n\to\infty} \tilde{E}\left\{\phi(\tilde{x})\left[(\tilde{x}_t - \tilde{x}_t^n) - ((\tilde{x}_s - \tilde{x}_s^n)\right]\right\} + \\
&\lim_{n\to\infty} \tilde{E}\left\{\phi(\tilde{x})\left(\int_s^t [f(\tau, \tilde{x}_{\tau+T/n}) - f(\tau, \tilde{x}_{\tau+T/n}^n)]d\tau\right)\right\} \\
&+ \lim_{n\to\infty} \tilde{E}\left\{(\phi(\tilde{x}) - \phi(\tilde{x}^n))\left(\tilde{x}_s^n - \tilde{x}_t^n - \int_s^t f(\tau, \tilde{x}_{\tau+T/n}^n)d\tau\right)\right\} = 0
\end{aligned}$$

for $0 \leq s \leq t \leq T$, because $\sup_{0\leq t\leq T}|\tilde{x}_t^n - \tilde{x}_t| \to 0$ a.s. as $n \to \infty$, ϕ and a mapping $C_T \ni x \to \int_s^t f(\tau, x_\tau)d\tau \in \mathbb{R}^m$ are continuous on C_T and $|\tilde{x}_s^n - \tilde{x}_t^n - \int_s^t f(\tau, \tilde{x}_{\tau+T/n}^n)d\tau| \leq 2\lambda + \int_0^T m(t)dt < \infty$ a.s. By the Monotone Class Theorem ([14], Th. I.1.8)it follows that

$$\tilde{E}\left\{\psi(\tilde{x})\left(\tilde{E}[\tilde{x}_s - \tilde{x}_t - \int_s^t f(\tau, \tilde{x}_\tau)d\tau|\tilde{\mathcal{F}}_s]\right)\right\} = 0$$

for $0 \leq s \leq t \leq T$ and every $\mathcal{F}_s$-measurable bounded function ψ on C_T, which implies that $\tilde{E}[\tilde{x}_s - \tilde{x}_t - \int_s^t f(\tau, \tilde{x}_\tau)d\tau|\tilde{\mathcal{F}}_s] = 0$ a.s. for every $0 \leq s \leq t \leq T$. Hence by the properties of f, it follows that $\tilde{x}_s \in \tilde{E}[\tilde{x}_t + \int_s^t F(\tau, \tilde{x}_\tau)d\tau|\tilde{\mathcal{F}}_s]$ a.s. for $0 \leq s \leq t \leq T$. Finally, let us observe that $\mathrm{dist}(g(x_T^n), H(x_T^n)) = 0$ a.s. for $n \geq 1$ and a function $\mathbb{R}^m \ni x \to \mathrm{dist}(g(x), H(x)) \in \mathbb{R}$ is continuous. Therefore, we also have $\mathrm{dist}(g(\tilde{x}_T^n), H(\tilde{x}_T^n)) = 0$ a.s. for $n \geq 1$, which implies that $\tilde{x}_T \in H(\tilde{x}_T)$ a.s. $\square$

6. WEAK COMPACTNESS OF SOLUTIONS SET

We shall consider here measurable set-valued mappings F and H satisfying conditions (B) and such that H and $F(t,\cdot)$ are continuous for a.e. fixed $t \in [0,T]$. In what follows we shall say that such F and H satisfy conditions (C). Denote by $\mathcal{X}(F,H)$ the set of all continuous weak solutions to BSDI(F,H). We shall show that if F and H satisfy conditions (C) then the set $\mathcal{X}(F,H)$ is weakly compact with respect to the Prohorov's topology. We begin with the following result.

Proposition 6.1. *If F and H satisfy conditions (C) and $\{(\mathcal{P}^n_{\mathbb{F}^n}, x^n)\}_{n=1}^{\infty}$ is a sequence of $\mathcal{X}(F,H)$ then there are a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and a sequence $(\tilde{x}^n)_{n=1}^{\infty}$ of m-dimensional continuous stochastic processes $\tilde{x}^n = (\tilde{x}^n_t)_{0\le t\le T}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that $P^n(x^n)^{-1} = \tilde{P}(\tilde{x}^n)^{-1}$ and such that*

$$\begin{cases} \tilde{x}^n_s \in \tilde{E}\left[\tilde{x}^n_t + \int_s^t F(\tau, \tilde{x}^n_\tau)d\tau | \tilde{\mathcal{F}}^n_s\right] \\ \tilde{x}^n_T \in H(\tilde{x}^n_T) \end{cases}$$

for $n \ge 1$, where $\tilde{\mathcal{F}}^n_t = \bigcap_{\varepsilon>0} \sigma[\tilde{x}^n_u : u \le t+\varepsilon]$ for $n \ge 1$ and $t \in [0,T]$.

Proof. Similarly as in the proof of ([6], Th. I.2.7) we define $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ by taking $\tilde{\Omega} = [0,1)$, $\tilde{\mathcal{F}} = \beta([0,1))$ and $\tilde{P} = \mu$, where μ is a Lebesgue'a measure on $\tilde{\mathcal{F}}$. Similarly, we define a sequence $(\tilde{x}^n)_{n=1}^{\infty}$ of random variables $\tilde{x}^n : \tilde{\Omega} \to C_T$ such that $P^n(x^n)^{-1} = \tilde{P}(\tilde{x}^n)^{-1}$ fo r $n \ge 1$. By virtue of Theorem 3.1 for every $n \ge 1$ there are $f^n \in S_{\mathbb{F}^n}(coF \circ x^n)$ and $\xi^n \in S(H \circ x^n_T)$ such that $x^n_t = E^n[\xi^n + \int_t^T f^n_\tau d\tau | \mathcal{F}^n_t]$ a.s. for $0 \le t \le T$. Hence it follows that $|x^n_t| \le \lambda$ a.s. for $n \ge 1$ and $0 \le t \le T$, where λ is such as above. Let g and ϕ be such as in the proof of Theorem 5.4. Hence and the properties of Aumann's integral (see [8],Th. II.3.20) we obtain

$$E^n\sigma(p, \phi(x^n)g(x^n_s)) \le E^n\left\{\sigma\left(p, E^n[\phi(x^n)(g(x^n_t) + \int_s^t F(\tau, x^n_\tau)d\tau | \mathcal{F}^n_s])\right)\right\}$$
$$= E^n\left\{\sigma\left(p, \phi(x^n)(g(x^n_t) + \int_s^t F(\tau, x^n_\tau)d\tau)\right)\right\}$$

for $p \in \mathbb{R}^m$, $n \ge 1$ and $0 \le s \le t \le T$, where $\sigma(p,\cdot)$ is a support function on $\mathbb{R}^m$. But x^n and $\tilde{x}^n$ have the same distributions and a function defined by superpositions of $\sigma(p,\cdot)$, ϕ, g and $F(t,\cdot)$ is continuous and bounded on C_T. Then the last inequality implies

$$\tilde{E}\sigma(p, \phi(\tilde{x}^n)g(\tilde{x}^n_s)) \le \tilde{E}\left\{\sigma\left(p, \phi(\tilde{x}^n)(g(\tilde{x}^n_t) + \int_s^t F(\tau, \tilde{x}^n_\tau)d\tau)\right)\right\}$$

for $p \in \mathbb{R}^m$, $n \ge 1$ and $0 \le s \le t \le T$. Therefore, for $n \ge 1$ and $0 \le s \le t \le T$ one has $\tilde{E}[\Phi(\tilde{x}^n)\tilde{x}^n_s] \in \tilde{E}\left\{\Phi(\tilde{x}^n)\left(\tilde{E}[\tilde{x}^n_t + \int_s^t F(\tau, \tilde{x}^n_\tau)d\tau\right)\right\}$. Let $\tilde{f}^n \in S_{\tilde{\mathbb{F}}^n}(coF \circ \tilde{x}^n)$ be such that

$$\tilde{E}\left\{\phi(\tilde{x}^n)\left(\tilde{x}^n_s - \tilde{x}^n_t - \int_s^t \tilde{f}^n_\tau)d\tau\right)\right\} = 0$$

for $n \geq 1$ and $0 \leq s \leq t \leq T$. Hence, similarly as in the proof of Theorem 5.4, it follows that $\tilde{x}^n_s = \tilde{E}[\tilde{x}^n_t + \int_s^t \tilde{f}^n_\tau d\tau|\tilde{\mathcal{F}}^n_s]$ for $n \geq 1$ and $0 \leq s \leq t \leq T$. Therefore $\tilde{x}^n$ satisfies the first condition of (9). Similarly as in the proof of Theorem 5.4 we also get $\tilde{x}^n_T \in H(\tilde{x}^n_T)$ a.s. for $n \geq 1$. $\square$

We can prove now the main result of the section.

Theorem 6.2. *If F and H satisfy conditions (C) then $\mathcal{X}(F,H)$ is nonempty weakly compact with respect to the convergence in distributions.*

Proof. By virtue of Theorem 5.4 we have $\mathcal{X}(F,H) \neq \emptyset$. By virtue of Proposition 6.1 for every sequence $\{(\mathcal{P}^n_{\mathbb{F}^n}, x^n)\}_{n=1}^\infty$ of $\mathcal{X}(F,H)$ there are a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and a sequence $(\tilde{x}^n)_{n=1}^\infty$ of m-dimensional continuous stochastic processes $\tilde{x}^n = (\tilde{x}^n_t)_{0\leq t\leq T}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that $P^n(x^n)^{-1} = \tilde{P}(\tilde{x}^n)^{-1}$ and such that conditions (9) are satisfied. By virtue of Proposition 5.3 a sequence $(\tilde{x}^n)_{n=1}^\infty$ is tight, which implies the tightness of a given sequence $(x^n)_{n=1}^\infty$. Then there is a subsequence $(x^{n_k})_{k=1}^\infty$ converging in distributions to a probability measure $\mathcal{P}$ on $\beta(C_T)$ as $k \to \infty$. By virtue of ([6], Th. I.2.7) there are a complete probability space, denoted again by $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and a sequence $(\tilde{x}^{n_k})_{k=1}^\infty$ of m-dimensional continuous stochastic processes $\tilde{x}^{n_k} = (\tilde{x}^{n_k}_t)_{0\leq t\leq T}$ and a continuous process $\tilde{x} = (\tilde{x}_t)_{0\leq t\leq T}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that $P(x^{n_k})^{-1} = P(\tilde{x}^{n_k})^{-1}$ for $k \geq 1$, $P(\tilde{x})^{-1} = \mathcal{P}$ and $\sup_{0\leq t\leq T} |\tilde{x}^{n_k}_t - \tilde{x}_t| \to 0$ a.s. as $k \to \infty$. Let $\tilde{\mathcal{F}}_t = \bigcap_{\varepsilon>0} \sigma[\tilde{x}_u : u \leq t+\varepsilon]$ and let $\phi : C_T \to \mathbb{R}$ be a continuous and bounded function such that ϕ is $\beta_s(C_T)$-measurable, where $\beta_s(C_T)$ is such as above. Similarly as in the proofs of Theorem 5.4 and Proposition 6.1 we get

$$\tilde{E}(\phi(\tilde{x})\tilde{x}_s) \in \tilde{E}\left(\phi(\tilde{x})\tilde{E}[\tilde{x}_t + \int_s^t F(\tau, \tilde{x}_\tau)d\tau|\tilde{\mathcal{F}}_s]\right)$$

for $0 \leq s \leq t \leq T$. Hence by virtue of Theorem 3.1 there is $\tilde{f} \in S_{\tilde{\mathbb{F}}}(coF \circ \tilde{x})$ such that

$$\tilde{E}\left(\tilde{E}\left[\phi(\tilde{x})(\tilde{x}_s - \tilde{x}_t - \int_s^t \tilde{f}_\tau d\tau)|\tilde{\mathcal{F}}_s\right]\right) = 0$$

for $0 \leq s \leq t \leq T$, which implies that

$$\tilde{E}\left(\phi(\tilde{x})(\tilde{x}_s - \tilde{x}_t - \int_s^t \tilde{f}_\tau d\tau)\right) = 0$$

for $0 \leq s \leq t \leq T$. Hence, similarly as in the proof of Theorem 5.4 it follows that $\tilde{x}_s = \tilde{E}[\tilde{x}_t + \int_s^t \tilde{f}_\tau d\tau)|\tilde{\mathcal{F}}_s]$ a.s. for every $0 \leq s \leq t \leq T$. Then $\tilde{x}_s \in \tilde{E}[\tilde{x}_t + \int_s^t F(\tau, \tilde{x}_\tau)d\tau)|\tilde{\mathcal{F}}_s]$ a.s. for $0 \leq s \leq t \leq T$. Similarily as in the proof of Proposition 5.3 we also get $\tilde{x}_T \in H(\tilde{x}_T)$ a.s. Then there is a subsequence $(x^{n_k})_{k=1}^\infty$ of a sequence $(x^n)_{n=1}^\infty$ converging in distributions to a solution $\tilde{x}$ to BSDI(F,H) on a complete filtered probability space $\mathcal{P}_{\tilde{\mathbb{F}}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathbb{F}})$ with a filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0\leq t\leq T}$. Thus $\mathcal{X}(F,H)$ is weakly compact with respect to the convergence in distributions. $\square$

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