Dynamic Systems and Applications 16 (2007) 155-174

NONLINEAR IMPULSIVE EVOLUTION INCLUSIONS OF SECOND ORDER

STANISŁAW MIGÓRSKI AND ANNA OCHAL

Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Computer Science, Nawojki 11
30072 Krakow, Poland

ABSTRACT. In this paper we consider strongly nonlinear second order impulsive evolution inclusions. We provide the existence results for the Cauchy problems with convex and nonconvex valued right hand sides. The compactness of the solution set in the convex case is proved. Applications to a distributed parameter control system with a priori feedback and to a hyperbolic hemivariational inequality with impulses are provided.

2000 Mathematics Subject Classification: 34A37, 34G20, 35A05, 35L70, 47H04, 93C25.

1. INTRODUCTION

In this paper we study second order evolution inclusions with impulses considered in the framework of an infinite dimensional evolution triple of spaces. The inclusions are governed by pseudomonotone coercive damping operator and linear continuous operator depending on solution. Our purpose is to provide results on the existence of solutions to the Cauchy problems for such inclusions with convex and nonconvex valued right hand sides. We also demonstrate the compactness of the solution set to the convex problem. As an application we present two examples which lead to impulsive evolution inclusions of second order. In the first example we deal with a distributed parameter system governed by a nonlinear hyperbolic equation with a priori feedback. The second example concerns a dynamical hemivariational inequality arising from a contact problem in viscoelasticity. The main feature of the hemivariational inequality is a nonmonotone multivalued term which is expressed by the Clarke subdifferential of a nonsmooth and nonconvex superpotential. For more information on models described by hemivariational inequalities and on mathematical results, we refer to Panagiotopoulos [20], Naniewicz and Panagiotopoulos [18], Migórski and Ochal [16], [17] and Ochal [19].

Recently the impulsive differential equations and inclusions have been studied by several authors. This is due to the fact that many real phenomena and processes in

Received August 30, 2006

mechanics, biology, physics, chemistry, biotechnology, etc. are characterized by the situation that at certain instants in time, the system parameters (e.g. displacement, velocities) undergo rapid changes. The duration of these changes is often neglected and it is assumed that the changes are represented by parameter jumps. One of natural tools for mathematical modeling and simulation of such phenomena is the theory of impulsive differential equations. This theory has started in the 1990s (cf. [11]) and today it covers various kind of problems which are motivated by numerous applications.

The impulsive evolution equations and inclusions for the first order problems have been studied recently by Liu [12] by using the semigroup approach, by Ahmed in a series of papers [1, 2, 3] and by Sattayatham [22] in the framework of evolution triple with applications to optimal control. The results for the second order impulsive systems can be found e.g. in Hernandez [10] who used the cosine function theory via semigroup method, in Benchohra et al. [4] for inclusions in finite dimensional spaces, in Benchohra and Ouahab [5] and Yong-Kui and Wang-Tong [23] for functional differential inclusions by a fixed point approach. In all papers mentioned above the jumps sizes were single-valued. Multivalued jump operators for problems described by functional differential inclusions of first order were considered by Benedetti [6]. For related results on second order inclusions without impulses we refer, among other papers, to Denkowski et al. [9], Papageorgiou and Yannakakis [21] and Migórski [13, 14].

To our knowledge the impulsive second order evolution inclusions with multivalued jump operators treated in the present paper have not been considered in the literature. We mention that the model with multivalued jump sizes may arise in a control problem where we want to control the jump sizes in order to achieve given objectives. Further properties of the solution set of second order impulsive inclusion and the corresponding optimal control problems will be studied elsewhere.

The paper is organized as follows. In Section 2 we recall definitions which will be used later. In Section 3 we deliver the existence results for impulsive evolution inclusions of second order and, under an additional assumption on the closedness of the graphs of jump operators, we establish the compactness of the solution set. Finally, in Section 4 we present two examples where our results can be applied.

2. **PRELIMINARIES**

In this section we introduce the notation and recall some definitions needed in the sequel.

Let H be a separable Hilbert space and let V be a dense subspace of H carrying the structure of a separable reflexive Banach space with continuous embedding $V \subset$

H. Identifying H with its dual, the triple of spaces (V, H, V^*) is called an evolution triple (cf. [9]). Moreover, we assume that the embedding $V \subset H$ is compact (hence also $H \subset V^*$ compactly).

Given $I = [0, T]$, $0 < T < +\infty$, $0 \leq \tau_1 < \tau_2 \leq T$ and an evolution triple (V, H, V^*), we define

$$W(\tau_1, \tau_2) = \{v \in L^2(\tau_1, \tau_2; V) \, : \, \dot{v} \in L^2(\tau_1, \tau_2; V^*)\},$$

where the time derivative is understood in the sense of vector-valued distributions. Endowed with the norm $\|v\|_{W(\tau_1,\tau_2)} = \|v\|_{L^2(\tau_1,\tau_2;V)} + \|\dot{v}\|_{L^2(\tau_1,\tau_2;V^*)}$, the space $W(\tau_1, \tau_2)$ becomes a separable, reflexive Banach space. It is well known (cf. [9]) that the space $W(\tau_1, \tau_2)$ is embedded continuously in $C(\tau_1, \tau_2; H)$ (the space of continuous functions on $[\tau_1, \tau_2]$ with values in H), i.e. every element of $W(\tau_1, \tau_2)$, after a possible modification on a set of measure zero, has a unique continuous representative in $C(\tau_1, \tau_2; H)$. Moreover, since V is embedded compactly in H, then so does $W(\tau_1, \tau_2)$ into $L^2(\tau_1, \tau_2; H)$ (cf. [9]). We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_{V^*}$ the norms in V, H and V^*, respectively. The duality brackets for the pair (V, V^*) is denoted by $\langle \cdot, \cdot \rangle$.

Let $D = \{t_1, \ldots, t_m\}$ be a finite set of points such that

$$0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T.$$

The elements of D are called impulsive points. In what follows, we need the following space of piecewise continuous functions

$$PC(I; V) = \{v \colon I \to V \text{ such that } v \text{ is continuous at } t \in I \setminus D, v \text{ is left}$$
$$\text{continuous at } t \in D, \text{the right limits } v(t_i^+) \text{ exist for } i = 1, \ldots, m\}.$$

Evidently, $PC(I; V)$ is a Banach space with norm $\|v\|_{PC(I;V)} = \sup_{t \in I} \|v(t)\|$. Analogously we define the space $PC(I; H)$ furnished with the supremum norm.

Let $\sigma_i = (t_i, t_{i+1})$ for $i = 0, 1, \ldots, m$. We define

$$PW(I) = \{v \colon I \to V \text{ such that } v|_{\sigma_i} \in W(\sigma_i) \text{ for } i = 1, \ldots, m\}$$

which becomes a Banach space with norm $\|v\|_{PW(I)} = \displaystyle\sum_{i=0}^{m} \|v|_{\sigma_i}\|_{W(\sigma_i)}$.

Let (Ω, Σ) be a measure space, X be a separable Banach space and let 2^X be a family of all subsets of X. A multifunction $F \colon \Omega \to 2^X$ is called graph measurable if $Gr\, F = \{(\omega, x) \in \Omega \times Y \, : \, x \in F(\omega)\} \in \Sigma \times \mathcal{B}(X)$ with $\mathcal{B}(X)$ being the Borel σ-field of X. It is said to be measurable if for each closed set $C \subset X$, the set $F^-(C) = \{\omega \in \Omega \, ; \, F(\omega) \cap C \neq \emptyset\} \in \Sigma$ (cf. Section 4.2 of [8]).

Let X and Y be Banach spaces. A multifunction $F \colon X \to 2^Y \setminus \{\emptyset\}$ is lsc (usc, respectively) if for $C \subset Y$ closed, the set $F^+(C) = \{x \in X \, : \, F(x) \subset C\}$ $(F^-(C) =$

$\{x \in X : F(x) \cap C \neq \emptyset\}$, respectively) is closed in X. F is bounded on bounded sets if $F(B) = \cup_{x \in B} F(x)$ is a bounded subset of Y for all bounded sets B in X.

Let Y be a reflexive Banach space. An operator $T\colon Y \to Y^*$ is pseudomonotone if $y_n \to y_0$ weakly in Y and $\limsup \langle Ty_n, y_n - y_0 \rangle \leq 0$ imply that $\langle Ty_0, y_0 - y \rangle \leq \liminf \langle Ty_n, y_n - y \rangle$ for all $y \in Y$. It is said to be demicontinuous if $y_n \to y_0$ in Y implies $Ty_n \to Ty_0$ weakly in Y^*.

Given a Banach space $(X, \| \cdot \|_X)$, the symbol w-X is always used to denote the space X endowed with the weak topology. By $\mathcal{L}(X, X^*)$ we denote the class of linear and bounded operators from X to X^*. If $U \subset X$, then we write $\|U\|_X = \sup\{\|x\|_X : x \in U\}$.

Finally, we recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $h\colon X \to \mathbb{R}$, where X is a Banach space (see [7]). The generalized directional derivative of h at $x \in X$ in the direction $v \in X$, denoted by $h^0(x; v)$, is defined by

$$h^0(x; v) = \limsup_{y \to x,\ t \downarrow 0} \frac{h(y + tv) - h(y)}{t}.$$

The generalized gradient of h at x, denoted by $\partial h(x)$, is a subset of a dual space X^* given by $\partial h(x) = \{\zeta \in X^* : h^0(x; v) \geq \langle \zeta, v \rangle_{X^* \times X}$ for all $v \in X\}$.

3. IMPULSIVE INCLUSIONS

In this section we provide main results of the paper on existence of solutions to the Cauchy problems for evolution inclusions considered in a framework of evolution triple (V, H, V^*).

Let us consider the following impulsive second order evolution inclusion

$$\begin{cases} \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) \in F(t, u(t), \dot{u}(t)) & \text{for } t \in I \setminus D \\ u(0) = u_0, \quad \dot{u}(0) = v_0 \\ u(t_i^+) \in u(t_i^-) + G_i(u(t_i^-), \dot{u}(t_i^-)) \\ \dot{u}(t_i^+) \in \dot{u}(t_i^-) + H_i(u(t_i^-), \dot{u}(t_i^-)) \text{ for } i = 1, \ldots, m. \end{cases} \tag{3.1}$$

Here $A\colon I \times V \to V^*$ is a nonlinear operator, $B\colon V \to V^*$ is linear and continuous, $F\colon I \times H \times H \to 2^H \setminus \{\emptyset\}$ is a multivalued function, $u_0 \in V$, $v_0 \in H$, $G_i\colon V \times H \to 2^V \setminus \{\emptyset\}$, $H_i\colon V \times H \to 2^H \setminus \{\emptyset\}$, $i = 1, \ldots, m$, are multivalued maps and $u(t_i^+)$, $u(t_i^-)$ (and $\dot{u}(t_i^+)$, $\dot{u}(t_i^-)$, respectively) denote the right and left limits of $u(t)$ (and of $\dot{u}(t)$, respectively) at $t = t_i$. The difference $u(t_i^+) - u(t_i^-)$ (and $\dot{u}(t_i^+) - \dot{u}(t_i^-)$, respectively) represents the jump in the state u (and its derivative $\dot{u}$, respectively) at time $t = t_i$ with G_i (and H_i, respectively) determining the size of the jump at time t_i.

Definition 3.1. A function $u \in PC(I;V)$ is called a solution to (3.1) if $\dot{u} \in PW(I) \cap PC(I;H)$ and there exists $f \in L^2(I;H)$ such that

$$\left.\begin{array}{l} \ddot{u}(t) + A(t,\dot{u}(t)) + Bu(t) = f(t) \\ f(t) \in F(t,u(t),\dot{u}(t)) \end{array}\right\} \quad \text{for a.e. } t \in \sigma_i,\ i = 0,1,\ldots,m,$$

$u(0) = u_0$, $\dot{u}(0) = v_0$ and $u(t_i^+) = u(t_i^-) + \zeta_i$, $\dot{u}(t_i^+) \in \dot{u}(t_i^-) + \eta_i$ with $\zeta_i \in G_i(u(t_i^-),\dot{u}(t_i^-))$ and $\eta_i \in H_i(u(t_i^-),\dot{u}(t_i^-))$ for all $i = 1,\ldots,m$.

Remark 3.2. If u is a solution to problem (3.1), then $(u,\dot{u}) \in PC(I;V) \times PC(I;H)$ and we write $u(t^-) = u(t)$ and $\dot{u}(t^-) = \dot{u}(t)$ at every point $t = t_i$, $i = 1,\ldots,m$.

We need the following hypotheses on the data.

$\underline{H(A)}$: $A\colon I \times V \to V^*$ is an operator such that
- (i) $A(\cdot,v)$ is measurable on I, for every $v \in V$;
- (ii) $A(t,\cdot)$ is pseudomonotone and demicontinuous, for a.e. $t \in I$;
- (iii) $\|A(t,v)\|_{V^*} \le a(t) + c\|v\|$ for a.e. $t \in I$, for all $v \in V$ with $a \in L^2_+(I)$ and $c > 0$;
- (iv) $\langle A(t,v),v\rangle \ge c_1\|v\|^2 - a_1(t)$ for a.e. $t \in I$, for all $v \in V$ with $c_1 > 0$ and $a_1 \in L^1(I)$.

$\underline{H(B)}$: $B \in \mathcal{L}(V,V^*)$ is nonnegative and symmetric operator.

$\underline{H(F)}$: $F\colon I \times H \times H \to 2^H \setminus \{\emptyset\}$ is a multifunction with convex and closed values such that
- (i) $F(\cdot,u,v)$ is measurable on I for all $u, v \in H$;
- (ii) $Gr\, F(t,\cdot,\cdot)$ is sequentially closed in $H \times H \times (w\text{-}H)$ topology for a.e. $t \in I$;
- (iii) $|F(t,u,v)| \le a_2(t) + c_2(|u| + |v|)$ for a.e. $t \in I$, for all $u, v \in H$ with $a_2 \in L^2_+(I)$ and $c_2 > 0$.

$\underline{H(G,H)}$: the multifunctions $G_i\colon V \times H \to 2^V \setminus \{\emptyset\}$ and $H_i\colon V \times H \to 2^H \setminus \{\emptyset\}$ are bounded on bounded sets, for $i = 1,\ldots,m$.

$\underline{(H_0)}$: $u_0 \in V$, $v_0 \in H$.

The following result is a consequence of Theorem 1 of [21] and it will be used in proving the main result of the paper.

Theorem 3.3. *Let $0 \le \tau_1 < \tau_2 \le T$. Under hypotheses $H(A)$, $H(B)$, $H(F)$ and (H_0), the solution set of the problem*

$$\begin{cases} \ddot{u}(t) + A(t,\dot{u}(t)) + Bu(t) \in F(t,u(t),\dot{u}(t)) \ \text{a.e. } t \in (\tau_1,\tau_2) \\ u(\tau_1) = u_0, \quad \dot{u}(\tau_1) = v_0 \end{cases} \tag{3.2}$$

is nonempty, weakly compact in $H^1(\tau_1,\tau_2;V)$ and compact in $C^1(\tau_1,\tau_2;H)$.

Recall that a function $u \in L^2(\tau_1, \tau_2; V)$ is a solution to (3.2), if $\dot{u} \in W(\tau_1, \tau_2)$ and there exists $f \in L^2(\tau_1, \tau_2; H)$ such that

$$\begin{cases} \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) = f(t) \quad \text{a.e. } t \in (\tau_1, \tau_2) \\ f(t) \in F(t, u(t), \dot{u}(t)) \quad \text{a.e. } t \in (\tau_1, \tau_2) \\ u(0) = u_0, \quad \dot{u}(0) = v_0. \end{cases}$$

We also remark that the statement: $u \in L^2(\tau_1, \tau_2; V)$ such that $\dot{u} \in W(\tau_1, \tau_2)$ is equivalent to: $u \in C(\tau_1, \tau_2; V)$ such that $\dot{u} \in W(\tau_1, \tau_2)$.

Theorem 3.4. *If hypotheses $H(A)$, $H(B)$, $H(F)$, $H(G, H)$ and (H_0) hold, then problem (3.1) has a solution.*

Proof. We divide the construction of the solution to (3.1) into steps. We solve the problem in the interval $\sigma_0 = (0, t_1)$, then in the interval $\sigma_1 = (t_1, t_2)$ and so on until the final interval $\sigma_m = (t_m, T)$. More precisely, we proceed in the following way.

1) Consider the following problem without impulses

$$\begin{cases} \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e. } t \in \sigma_0 \\ u(0) = u_0, \quad \dot{u}(0) = v_0. \end{cases} \tag{3.3}$$

¿From Theorem 3.3, applied to the above problem, it follows that (3.3) admits a solution $u^{(0)} \in L^2(\sigma_0; V)$ such that $\dot{u}^{(0)} \in W(\sigma_0)$. Hence $u^{(0)} \in C(\sigma_0; V)$ and $\dot{u}^{(0)} \in C(\sigma_0; H)$. Thus, the left limits $u^{(0)}(t_1^-)$ and $\dot{u}^{(0)}(t_1^-)$ exist in V and H, respectively and we define $u^{(0)}(t_1) = u^{(0)}(t_1^-) \in V$ and $\dot{u}^{(0)}(t_1) = \dot{u}^{(0)}(t_1^-) \in H$. By assumption $H(G, H)$, $u^{(0)}(t_1^+)$ and $\dot{u}^{(0)}(t_1^+)$ are well defined and they are given by

$$u_1 := u^{(0)}(t_1^+) = u^{(0)}(t_1) + \zeta_1, \quad u_1 \in V,$$

$$v_1 := \dot{u}^{(0)}(t_1^+) = \dot{u}^{(0)}(t_1) + \eta_1, \quad v_1 \in H,$$

where $\zeta_1 \in G_1(u^{(0)}(t_1), \dot{u}^{(0)}(t_1))$ and $\eta_1 \in H_1(u^{(0)}(t_1), \dot{u}^{(0)}(t_1))$.

2) Consider the following problem without impulses

$$\begin{cases} \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e. } t \in \sigma_1 \\ u(t_1) = u_1, \quad \dot{u}(t_1) = v_1. \end{cases} \tag{3.4}$$

Using Theorem 3.3 we obtain a solution $u^{(1)}$ to (3.4) such that $u^{(1)} \in L^2(\sigma_1; V)$ and $\dot{u}^{(1)} \in W(\sigma_1)$. Therefore $u^{(1)} \in C(\sigma_1; V)$ with $\dot{u}^{(1)} \in C(\sigma_1; H)$. Analogously as in Step 1, we set

$$u_2 := u^{(1)}(t_2^+) = u^{(1)}(t_2) + \zeta_2, \quad u_2 \in V,$$

$$v_2 := \dot{u}^{(1)}(t_2^+) = \dot{u}^{(1)}(t_2) + \eta_2, \quad v_2 \in H,$$

where $\zeta_2 \in G_2(u^{(1)}(t_2), \dot{u}^{(1)}(t_2))$ and $\eta_2 \in H_2(u^{(1)}(t_2), \dot{u}^{(1)}(t_2))$.

3) Further we continue the process and for $k = 0, 1, \ldots, m$, we obtain $u^{(k)} \in L^2(\sigma_k; V)$ such that $\dot{u}^{(k)} \in W(\sigma_k)$ and it is a solution to the problem

$$\begin{cases} \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e. } t \in \sigma_k \\ u(t_k) = u_k, \quad \dot{u}(t_k) = v_k, \end{cases}$$

where

$$u_k = u^{(k-1)}(t_k) + \zeta_k, \quad u_k \in V,$$

$$v_k = \dot{u}^{(k-1)}(t_k) + \eta_k, \quad v_k \in H,$$

with $\zeta_k \in G_k(u^{(k-1)}(t_k), \dot{u}^{(k-1)}(t_k))$ and $\eta_k \in H_k(u^{(k-1)}(t_k), \dot{u}^{(k-1)}(t_k))$ for $k = 1, \ldots, m$. Now we define the function $u \colon I \to V$ by

$$u(t) = \begin{cases} u^{(0)}, & t \in [0, t_1] \\ u^{(1)}, & t \in (t_1, t_2] \\ \cdots & \cdots \\ u^{(m)}, & t \in (t_m, T]. \end{cases}$$

It is easy to see that $u \in PC(I; V)$ with $\dot{u} \in PW(I) \cap PC(I; H)$ is a solution to (3.1). The proof is complete. $\square$

In what follows we establish the compactness of the solution map to (3.1). We need to strengthen the conditions on the multifunctions defining the jumps sizes.

<u>$H(G,H)_1$</u> : the multifunctions $G_i \colon V \times H \to 2^V \setminus \{\emptyset\}$, $H_i \colon V \times H \to 2^H \setminus \{\emptyset\}$ satisfy $H(G,H)$, $Gr\, G_i$ is closed in $H \times H \times (w\text{-}V)$ topology and $Gr\, H_i$ is closed in $H \times H \times (w\text{-}H)$ topology, for $i = 1, \ldots, m$.

Let $S \colon V \times H \to 2^{PC^1(I;H)}$ be a solution map to (3.1), i.e. the multifunction defined by

$$S(u_0, v_0) = \{u : u \text{ is a solution to (3.1)}\},$$

where $PC^1(I; H) = \{v \in PC(I; H) : \dot{v} \in PC(I; H)\}$.

Proposition 3.5. Under hypotheses $H(A)$, $H(B)$, $H(F)$, $H(G,H)_1$ and (H_0), the solution set $S(u_0, v_0)$ is a nonempty compact subset of $PC^1(I; H)$.

Proof. Let $(u_0, v_0) \in V \times H$ and $\{u_n\} \subset S(u_0, v_0)$. The nonemptiness follows from Theorem 4. In the following steps we find a subsequence of $\{u_n\}$ and constract its limit.

1) Define $u_n^{(0)} = u_n|_{\sigma_0}$. Then $u_n^{(0)} \in C(\sigma_0; V)$ with $\dot{u}_n^{(0)} \in W(\sigma_0)$ solves the problem without impulses

$$\begin{cases} \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e. } t \in \sigma_0 \\ u(0) = u_0, \quad \dot{u}(0) = v_0. \end{cases} \tag{3.5}$$

¿From Theorem 3.3, applied to (3.5), there exist a subsequence $\{u_{n_k}^{(0)}\}$ of $\{u_n^{(0)}\}$ and $u^{(0)} \in C(\sigma_0; V)$ with $\dot{u}^{(0)} \in W(\sigma_0) \subset C(\sigma_0; H)$ such that

$$u_{n_k}^{(0)} \to u^{(0)} \ \text{ in } \ C^1(\sigma_0; H), \ \text{ as } k \to \infty$$

and $u^{(0)}$ is a solution to (3.5). For simplicity of notation, we denote the subsequence $\{u_{n_k}\}$ of $\{u_n\} \subset S(u_0, v_0)$ by the same symbol $\{u_n\}$. Subsequently, we define $\{u_{n1}\} \subset V$ and $\{v_{n1}\} \subset H$ by

$$u_{n1} = u_n^{(0)}(t_1) + \zeta_{n1}, \quad v_{n1} = \dot{u}_n^{(0)}(t_1) + \eta_{n1},$$

where

$$\zeta_{n1} \in G_1(u_n^{(0)}(t_1), \dot{u}_n^{(0)}(t_1)), \quad \eta_{n1} \in H_1(u_n^{(0)}(t_1), \dot{u}_n^{(0)}(t_1)). \tag{3.6}$$

Since G_1 and H_1 are bounded-valued multifunctions, $\{\zeta_{n1}\}$ and $\{\eta_{n1}\}$ are bounded in V and H, respectively, uniformly with respect to n. Passing to a subsequence, if necessary, we may assume

$$\zeta_{n1} \to \zeta_1 \ \text{ weakly in } V,$$
$$\eta_{n1} \to \eta_1 \ \text{ weakly in } H$$

with $\zeta_1 \in V$ and $\eta_1 \in H$. Using the convergences

$$u_n^{(0)}(t_1) \to u^{(0)}(t_1), \quad \dot{u}_n^{(0)}(t_1) \to \dot{u}^{(0)}(t_1) \ \text{ both in } H$$

and the closedness of the graphs of G_1 and H_1 in suitable topologies (cf. $H(G,H)_1$), from (3.6) we deduce

$$\zeta_1 \in G_1(u^{(0)}(t_1), \dot{u}^{(0)}(t_1)), \quad \eta_1 \in H_1(u^{(0)}(t_1), \dot{u}^{(0)}(t_1)).$$

Hence, putting

$$u_1 = u^{(0)}(t_1) + \zeta_1, \quad v_1 = \dot{u}^{(0)}(t_1) + \eta_1,$$

we have

$$u_{n1} \to u_1, \quad v_{n1} \to v_1 \ \text{ both in } H, \ \text{ as } n \to \infty.$$

2) We define $u_n^{(1)} = u_n|_{\sigma_1}$. Then $u_n^{(1)} \in C(\sigma_1; V)$ is such that $\dot{u}_n^{(1)} \in W(\sigma_1)$ and it is a solution of the problem without impulses

$$\begin{cases} \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e. } t \in \sigma_1 \\ u(t_1) = u_1, \quad \dot{u}(t_1) = v_1. \end{cases} \tag{3.7}$$

Again from Theorem 3.3, there exist a subsequence $\{u^{(1)}_{n_k}\}$ of $\{u^{(1)}_n\}$ and $u^{(1)} \in C(\sigma_1; V)$ with $\dot{u}^{(1)} \in W(\sigma_1) \subset C(\sigma_1; H)$ such that

$$u^{(1)}_{n_k} \to u^{(1)} \quad \text{in} \quad C^1(\sigma_1; H), \text{ as } k \to \infty \tag{3.8}$$

and $u^{(1)}$ is a solution to problem (3.7). We denote the subsequence $\{u_{n_k}\}$ of $\{u_n\} \subset S(u_0, v_0)$ again by $\{u_n\}$. Next, we define $\{u_{n2}\} \subset V$ and $\{v_{n2}\} \subset H$ by

$$u_{n2} = u^{(1)}_n(t_2) + \zeta_{n2}, \quad v_{n2} = \dot{u}^{(1)}_n(t_2) + \eta_{n2}$$

with

$$\zeta_{n2} \in G_2(u^{(1)}_n(t_2), \dot{u}^{(1)}_n(t_2)), \quad \eta_{n2} \in H_2(u^{(1)}_n(t_2), \dot{u}^{(1)}_n(t_2)).$$

By the boundedness of values of G_2 and H_2 we may suppose $\zeta_{n2} \to \zeta_2$ weakly in V and $\eta_{n2} \to \eta_2$ weakly in H, where $\zeta_2 \in V$ and $\eta_2 \in H$. Exploiting (3.8) and the closedness of the graphs of G_2 and H_2, we obtain

$$u_{n2} \to u_2, \quad v_{n2} \to v_2 \quad \text{both in } H,$$

where $u_2 = u^{(1)}(t_2) + \zeta_2$, $v_2 = \dot{u}^{(1)}(t_2) + \eta_2$ with $\zeta_2 \in G_2(u^{(1)}(t_2), \dot{u}^{(1)}(t_2))$ and $\eta_2 \in H_2(u^{(1)}(t_2), \dot{u}^{(1)}(t_2))$.

3) Step by step, we obtain a family of functions $\{u^{(i)}\}_{i=0}^m$, where $u^{(i)} \in C(\sigma_i; V)$, $\dot{u}^{(i)} \in W(\sigma_i)$ such that $u^{(i)}$ is a solution of the following problem

$$\begin{cases} \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e. } t \in \sigma_i \\ u(t_i) = u_i, \quad \dot{u}(t_i) = v_i, \end{cases}$$

for $i = 0, 1, \ldots, m$. Finally, we define the function $u^*\colon I \to V$ by

$$u^*(t) = \begin{cases} u^{(0)}, & t \in [0, t_1] \\ u^{(1)}, & t \in (t_1, t_2] \\ \cdots & \cdots \\ u^{(m)}, & t \in (t_m, T]. \end{cases}$$

Then $u^* \in PC(I; V)$, $\dot{u}^* \in PW(I) \cap PC(I; H)$ is a solution to (3.1), i.e. $u^* \in S(u_0, v_0)$. It is easy to see that the subsequence of $\{u_n\}$ successively chosen in m steps converges to u^* in $PC^1(I; H)$-norm. This proves the proposition. $\square$

Remark 3.6. Analogously as in Theorem 3.4 and Proposition 3.5 we can prove the nonemptiness and compactness of the solution set of the following impulsive second order evolution inclusion

$$\begin{cases} \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) \in F(t, u(t), \dot{u}(t)) \text{ for } t \in I \setminus D \\ u(0) = u_0, \quad \dot{u}(0) = v_0 \\ u(s_i^+) \in u(s_i^-) + G_i(u(s_i^-), \dot{u}(s_i^-)) \text{ for } s_i \in D_1 \\ \dot{u}(t_i^+) \in \dot{u}(t_i^-) + H_i(u(t_i^-), \dot{u}(t_i^-)) \text{ for } t_i \in D_2, \ i = 1, \ldots, m, \end{cases}$$

where $D = D_1 \cup D_2$, $D_1 = \{s_1, s_2, \ldots, s_m\}$ and $D_2 = \{t_1, t_2, \ldots, t_m\}$.

¿From the existence result for nonconvex second order inclusions without impulses (cf. Theorem 2 in [21]), we establish a new existence result for problem (3.1) with a nonconvex set valued map F. The hypothesis on the nonconvex term is as follows.

$\underline{H(F)_1}$: $F \colon I \times H \times H \to 2^H \setminus \{\emptyset\}$ is a multifunction with closed values such that

(i) $F(\cdot, \cdot, \cdot)$ is graph measurable;

(ii) $F(t, \cdot, \cdot)$ is lsc for a.e. $t \in I$;

(iii) $|F(t, u, v)| \leq a_2(t) + c_2(|u| + |v|)$ for a.e. $t \in I$, for all u, $v \in H$ with $a_2 \in L^2_+(I)$ and $c_2 > 0$.

The proof of the next theorem is analogous to that of Theorem 3.4 and it is omitted.

Theorem 3.7. *Under hypotheses $H(A)$, $H(B)$, $H(F)_1$, $H(G, H)$ and (H_0), the problem (3.1) admits a solution.*

We conclude this section by providing an example of the multifunction G_i which satisfies the hypothesis $H(G, H)_1$. An analogous example can also be given for the multifunction H_i.

Let Y be a separable reflexive Banach space, let $C \colon V \times H \to \mathcal{L}(Y, V)$ and $U \colon V \times H \to 2^Y \setminus \{\emptyset\}$. Define the multifunction $G \colon V \times H \to 2^V \setminus \{\emptyset\}$ by

$$G(u, v) = \{C(u, v)y \, : \, y \in U(u, v)\} = \bigcup_{y \in U(u,v)} C(u, v)y \tag{3.9}$$

for $u \in V$, $v \in H$. We introduce the following hypotheses.

$\underline{H(C)}$: $C \colon V \times H \to \mathcal{L}(Y, V)$ is such that

(i) $(u, v) \to C(u, v)$ is continuous from $H \times H$ into $\mathcal{L}(Y, V)$;

(ii) $\|C(u, v)\|_{\mathcal{L}(Y,V)} \leq c_0(1 + \|u\| + |v|)$ for all $(u, v) \in V \times H$ with $c_0 > 0$.

$\underline{H(U)}$: $U \colon V \times H \to 2^Y \setminus \{\emptyset\}$ is such that

(i) $Gr\, U \subset V \times H \times Y$ is closed in $H \times H \times (w\text{-}Y)$;

(ii) $\|U(u, v)\|_Y \leq M$ for all $(u, v) \in V \times H$ with $M > 0$.

Lemma 3.8. *Under hypotheses $H(C)$ and $H(U)$, the multifunction $G\colon V \times H \to 2^V \setminus \{\emptyset\}$ defined by (3.9) is bounded on bounded sets and its graph is closed in $H \times H \times (w\text{-}V)$.*

Proof. It is easy to observe that from $H(C)(ii)$ and $H(U)(ii)$ the multifunction G is bounded on bounded sets. To show the closedness of the graph of G, let $\{(u_n, v_n, w_n)\} \subset Gr\, G$ be such that $u_n \to u$, $v_n \to v$ both in H and $w_n \to w$ weakly in V. ¿From the definition of G, we have $w_n = C(u_n, v_n)y_n$ with $y_n \in U(u_n, v_n)$. By $H(U)(ii)$, we may suppose that $y_n \to y$ weakly in Y with $y \in U(u, v)$ (cf. $H(U)(i)$). Let $h \in V^*$. We have

$$\langle C(u_n, v_n)y_n - C(u, v)y, h\rangle =$$

$$= \langle C(u_n, v_n)y_n - C(u_n, v_n)y, h\rangle + \langle C(u_n, v_n)y - C(u, v)y, h\rangle =$$

$$= \langle C(u_n, v_n)(y_n - y), h\rangle + \langle (C(u_n, v_n) - C(u, v))y, h\rangle \leq$$

$$\leq \langle y_n - y, C(u_n, v_n)^* h\rangle + \|C(u_n, v_n) - C(u, v)\|_{\mathcal{L}(Y,V)} \|y\|_Y \|h\|_{V^*},$$

where $C(u_n, v_n)^* \in \mathcal{L}(V^*, Y^*)$ denotes the adjoint operator to $C(u_n, v_n)$. Because

$$\|C(u_n, v_n)^* h - C(u, v)^* h\|_{Y^*} \leq \|C(u_n, v_n)^* - C(u, v)^*\|_{\mathcal{L}(V^*,Y^*)} \|h\|_{V^*} =$$

$$= \|C(u_n, v_n) - C(u, v)\|_{\mathcal{L}(Y,V)} \|h\|_{V^*} \to 0$$

(by $H(C)(i)$), we obtain $\langle y_n - y, C(u_n, v_n)^* h\rangle \to 0$. Thus $C(u_n, v_n)y_n \to C(u, v)y$ weakly in V and $w = C(u, v)y$. Therefore $w \in G(u, v)$ which means that $Gr\, G$ is sequentially closed in $H \times H \times (w\text{-}V)$ topology. □

4. APPLICATIONS

In this section we present two examples of hyperbolic partial differential equations with multivalued terms leading to impulsive evolution inclusions of second order to which our results apply.

Example 1. We consider a distributed parameter system governed by nonlinear hyperbolic equation with a feedback control.

Let $I = [0, T]$, $0 < T < \infty$, let the set $D = \{t_1, \ldots, t_m\}$ be a partition of I such that $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T$ and let Ω be an open bounded subset of $\mathbb{R}^d$ with boundary $\Gamma = \partial\Omega$. Let $Q = \Omega \times (I \setminus D)$ and let $B_{\mathbb{R}}$ be a ball in $\mathbb{R}$. Consider the following system

$$
(4.1)\qquad
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2}(x,t) - \displaystyle\sum_{k=1}^{d} \dfrac{\partial}{\partial x_k} a_k\Big(x,t,\dfrac{\partial u}{\partial t}(x,t), \nabla \dfrac{\partial u}{\partial t}(x,t)\Big) + \\
\qquad + a_0\Big(x,t,\dfrac{\partial u}{\partial t}(x,t)\Big) - \displaystyle\sum_{i,j=1}^{d} \dfrac{\partial}{\partial x_i}\Big(a_{ij}(x)\dfrac{\partial u}{\partial x_j}(x,t)\Big) = \\
\qquad = f\Big(x,t,u(x,t),\tfrac{\partial u}{\partial t}(x,t)\Big)\bar{u}(x,t) \ \text{ in } Q \\
u(x,0) = u_0(x), \dfrac{\partial u}{\partial t}(x,0) = v_0(x) \text{ in } \Omega \\
u = 0 \text{ on } \Gamma \times (0,T) \\
|\bar{u}(x,t)| \le \varrho(t, \|u(t,\cdot)\|_{L^2(\Omega)}) \quad \text{a.e. in } Q \\
u(x,t_i^+) \in u(x,t_i^-) + B_{\mathbb{R}}\Big(0, g_i\Big(|u(x,t_i^-)|, |\dfrac{\partial u}{\partial t}(x,t_i^-)|\Big)\Big) \\
\qquad\qquad \text{for } i = 1,\dots,m \\
\dfrac{\partial u}{\partial t}(x,t_i^+) \in \dfrac{\partial u}{\partial t}(x,t_i^-) + B_{\mathbb{R}}\Big(0, h_i\Big(|u(x,t_i^-)|, |\dfrac{\partial u}{\partial t}(x,t_i^-)|\Big)\Big) \\
\qquad\qquad \text{for } i = 1,\dots,m
\end{cases}
$$

The conditions on the data involved are the following.

$\underline{H(a)}$: $a_k \colon \Omega \times I \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $k = 1,\dots,d$ are functions such that

(i) $(x,t) \to a_k(x,t,r,\xi)$ is measurable for all $r \in \mathbb{R}$, $\xi \in \mathbb{R}^d$;
(ii) $(r,\xi) \to a_k(x,t,r,\xi)$ is continuous for a.e. $(x,t) \in \Omega \times I$;
(iii) $|a_k(x,t,r,\xi)| \le a(x,t) + c(x,t)(|r| + \|\xi\|_{\mathbb{R}^d})$ for a.e. $(x,t) \in \Omega \times I$, for all $(r,\xi) \in \mathbb{R} \times \mathbb{R}^d$ with $a \in L^2_+(\Omega \times I)$, $c \in L^\infty_+(\Omega \times I)$;
(iv) $\sum_{k=1}^d (a_k(x,t,r,\xi) - a_k(x,t,r,\eta))(\xi_k - \eta_k) \ge 0$ for a.e. $(x,t) \in \Omega \times I$, all $r \in \mathbb{R}$ and all ξ, $\eta \in \mathbb{R}^d$;
(v) $\sum_{k=1}^d a_k(x,t,r,\xi)\xi_k \ge c_1\|\xi\|^2_{\mathbb{R}^d} - a_1(x,t)$ for a.e. $(x,t) \in \Omega \times I$, all $r \in \mathbb{R}$ and all $\xi \in \mathbb{R}^d$ with $a_1 \in L^1(\Omega \times I)$, $c_1 > 0$.

$\underline{H(a_0)}$: $a_0 \colon \Omega \times I \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) $(x,t) \to a_0(x,t,r)$ is measurable for all $r \in \mathbb{R}$;
(ii) $r \to a_k(x,t,r)$ is continuous for a.e. $(x,t) \in \Omega \times I$;
(iii) $|a_0(x,t,r)| \le a_2(x,t) + c|r|$ for a.e. $(x,t) \in \Omega \times I$, for all $r \in \mathbb{R}$ with $a_2 \in L^1_+(\Omega \times I)$, $c > 0$.

$\underline{H(a_1)}$: $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$, $\sum_{k=1}^d a_{ij}(x)\xi_i\xi_j \ge 0$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$.

$\underline{H(f)}$: $f \colon \Omega \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) $(x,t) \to f(x,t,r,s)$ is measurable for all r, $s \in \mathbb{R}$;
(ii) $(r,s) \to f(x,t,r,s)$ is continuous for a.e. $(x,t) \in \Omega \times I$;
(iii) $|f(x,t,r,s)| \le a_3(x,t) + c(x,t)(|r| + |s|)$ for a.e. $(x,t) \in \Omega \times I$, for all r, $s \in \mathbb{R}$ with $a_3 \in L^2_+(\Omega \times I)$, $c \in L^\infty_+(\Omega \times I)$.

$\underline{H(\varrho)}$: $\varrho\colon I\times\mathbb{R}_+\to\mathbb{R}_+$ is measurable in t, upper semicontinuous in $s\in\mathbb{R}_+$ and $\varrho(t,s)\le\beta(t)$ a.e. on I with $\beta\in L^\infty_+(I)$.

$\underline{H(g,h)}$: $g_i,h_i\colon\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ are bounded and are upper semicontinuous with respect to both variables, for $i=1,\dots,m$.

In the problem (4.1) the evolution triple consists of the Sobolev spaces $V=H^1_0(\Omega)$, $H=L^2(\Omega)$ and $V^*=H^{-1}(\Omega)$. By the Sobolev embedding theorem, we know that the embeddings $V\subset H\subset V^*$ are compact. We define $A\colon I\times V\to V^*$ and $B\colon V\to V^*$ by

$$\langle A(t,u),v\rangle=\int_\Omega\sum_{k=1}^d a_k(x,t,u,\nabla v)\frac{\partial v}{\partial x_k}\,dx+\int_\Omega a_0(x,t,u)v\,dx,$$

$$\langle Bu,v\rangle=\int_\Omega\sum_{i,j=1}^d a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx$$

for $t\in I$ and u, $v\in V$. It can be seen (cf. [21], Section 6) that these operators satisfy $H(A)$ and $H(B)$, respectively. Next, let $\widehat{f}\colon I\times H\times H\to H$ be the Nemitsky (superposition) operator corresponding to f, i.e. $\widehat{f}(t,u,v)(\cdot)=f(\cdot,t,u(\cdot),v(\cdot))$ for $t\in I$, u, $v\in H$. Define $U\colon I\times H\to 2^{L^\infty(\Omega)}$ by

$$U(t,v)=\{u\in L^\infty(\Omega)\,:\,\|u\|_{L^\infty(\Omega)}\le\varrho(t,|v|)\}$$

for $t\in I$ and $v\in H$. Let $F\colon I\times H\times H\to 2^H$ be defined by

$$F(t,u,v)=\widehat{f}(t,u,v)U(t,u)=\bigcup_{y\in U(t,u)}\widehat{f}(t,u,v)y$$

for $t\in I$, u, $v\in H$. Analogously as in [21] we can show that the multifunction F satisfies $H(F)$.

Finally, let the multifunctions $G_i\colon V\times H\to 2^V\setminus\{\emptyset\}$ and $H_i\colon V\times H\to 2^H\setminus\{\emptyset\}$ be defined by

$$G_i(u,v)(\cdot)=B_V\left(0,g_i\left(|u(\cdot)|,|v(\cdot)|\right)\right),$$

$$H_i(u,v)(\cdot)=B_H\left(0,h_i\left(|u(\cdot)|,|v(\cdot)|\right)\right)$$

for $u\in V$, $v\in H$, where B_V and B_H represent balls in V and H, respectively. We observe that $Gr\,G_i$ is closed in $H\times H\times(w\text{-}V)$ topology for all $i=1,\dots,m$. Indeed, let $(u_n,v_n,z_n)\in Gr\,G_i$, $u_n\to u$, $v_n\to v$ in H and $z_n\to z$ weakly in V. So $z_n\in G_i(u_n,v_n)$ and $\|z_n\|\le g_i(|u_n(x)|,|v_n(x)|)$ for a.e. $x\in\Omega$. By passing to a subsequence, we have $u_n(x)\to u(x)$, $v_n(x)\to v(x)$ a.e. $x\in\Omega$. Exploiting the weak lower semicontinuity of the norm and $H(g,h)$, we have

$$\|z\|\le\liminf_n\|z_n\|\le\limsup_n g_i(|u_n(x)|,|v_n(x)|)\le g_i(|u(x)|,|v(x)|)$$

for a.e. $x \in \Omega$. Hence $z \in G_i(u, v)$ which implies the closedness of G_i. Analogously, we prove that H_i is closed in $H \times H \times (w\text{-}H)$ topology, for $i = 1, \ldots, m$. It is straighforward to see that $H(G, H)_1$ holds.

Applying the results of previous section, we obtain the following result for weak solutions to the system under consideration.

Theorem 4.1. *If $H(a)$, $H(a_0)$, $H(a_1)$, $H(f)$, $H(\varrho)$, $H(g, h)$ hold and $u_0 \in H_0^1(\Omega)$, $v_0 \in L^2(\Omega)$, then problem (4.1) has a solution $u \in PC(I; H_0^1(\Omega))$ with $\frac{\partial u}{\partial t} \in PW(I) \cap PC(I; L^2(\Omega))$ and the set of all solutions is compact in $PC^1(I; L^2(\Omega))$.*

Example 2. In this example we consider a mechanical contact problem involving a nonmonotone multivalued term. First, we formulate this problem as a hyperbolic hemivariational inequality with impulses. Then, we associate with the hemivariational inequality an impulsive evolution inclusion. The existence of solutions to the latter and, in consequence, the existence of weak solutions to the contact problem will follow from Theorem 3.4.

Let $I = [0, T]$, $0 < T < \infty$ and let $D = \{t_1, \ldots, t_m\}$ be such that $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T$. Let Ω be an open bounded subset of $\mathbb{R}^d$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$. Put $Q = \Omega \times (I \setminus D)$.

We consider the following problem.

$$\frac{\partial^2 u}{\partial t^2}(t) - \operatorname{div} \sigma(t) = \varphi(t) \text{ in } Q \tag{4.2}$$

$$\sigma(t) = \mathcal{C}\varepsilon\Big(\frac{\partial u}{\partial t}(t)\Big) + \mathcal{G}\varepsilon(u(t)) \text{ in } Q \tag{4.3}$$

$$\varphi(x, t) = \varphi_1(x, t) + \varphi_2(x, t) \text{ in } Q \tag{4.4}$$

$$u = 0 \text{ on } \Gamma_D \times (I \setminus D) \tag{4.5}$$

$$\sigma(t)n = \psi(t) \text{ on } \Gamma_N \times (I \setminus D) \tag{4.6}$$

$$-\varphi_1(x, t) \in \partial j(x, t, u(x, t)) \text{ in } Q \tag{4.7}$$

$$u(0) = u_0, \quad \frac{\partial u}{\partial t}(0) = v_0 \text{ in } \Omega \tag{4.8}$$

$$u(x, t_i^+) \in u(x, t_i^-) + B_{\mathbb{R}}\Big(0, g_i\Big(|u(x, t_i^-)|, |\frac{\partial u}{\partial t}(x, t_i^-)|\Big)\Big) \quad \text{for } \ i = 1, \ldots, m \tag{4.9}$$

$$\frac{\partial u}{\partial t}(x, t_i^+) \in \frac{\partial u}{\partial t}(x, t_i^-) + B_{\mathbb{R}}\Big(0, h_i\Big(|u(x, t_i^-)|, |\frac{\partial u}{\partial t}(x, t_i^-)|\Big)\Big), \quad \text{for } \ i = 1, \ldots, m. \tag{4.10}$$

The system (4.2)–(4.10) serves as a mathematical model for a contact problem in viscoelasticity with nonmonotone and nonconvex superpotential laws. More precisely, let us consider a viscoelastic body which occupies the reference configuration $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. We suppose that the boundary Γ is divided into two disjoint measurable parts Γ_D and Γ_N such that $meas(\Gamma_D) > 0$. The body is clamped along Γ_D, so the displacement field vanishes there. We denote by ψ the density of surface tractions on Γ_N. We suppose that the nonmonotone skin effects (e.g. skin friction, adhesive forces etc.) appear in Ω (cf. [20, 17]). In order to describe such effects we assume that the volume forces φ consist of two parts: φ_2 is given and φ_1 is the reaction of the constraint introducing the skin effects. So we may write $\varphi = \varphi_1 + \varphi_2$ (cf. (4.4)), where φ_2 is the prescribed external loading and φ_1 is a possibly multivalued function of the displacement satisfying (4.7), where ∂j is the Clarke generalized gradient of a given function j.

We denote by $u = (u_1, \ldots, u_d)$ the displacement vector, by $\sigma = (\sigma_{ij})$ the stress tensor and by $\varepsilon(u) = (\varepsilon_{ij}(u))$, $\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, the linearized (small) strain tensor, where $i, j = 1, \ldots, d$. The relation (4.2) is a dynamic equation of motion. We suppose the viscoelastic constitutive relationship of Kelvin-Voigt type (4.3), where $\mathcal{C}$ and $\mathcal{G}$ are given viscosity and elasticity functions, respectively. Conditions (4.5) and (4.6) represent the displacement and traction boundary conditions, respectively while u_0 and u_1 in (4.8) denote the initial displacement and the initial velocity, respectively. Conditions (4.9) and (4.10) are the impulse constraints relations. The classical problem with impulses is to find a displacement field $u\colon Q \to \mathbb{R}^d$ such that (4.2)–(4.10) hold.

In order to set the above problem in a variational form, we consider $\mathcal{S}_d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$ with the inner product and the corresponding norm $\sigma : \tau = \sum_{ij} \sigma_{ij}\tau_{ij}$, $\|\tau\|^2_{\mathcal{S}_d} = \tau : \tau$, respectively. Let $H = L^2(\Omega; \mathbb{R}^d)$ and $\mathcal{H} = L^2(\Omega; \mathcal{S}_d)$ be Hilbert spaces equipped with the inner products

$$\langle u, v\rangle_H = \int_\Omega u \cdot v \, dx, \qquad \langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma : \tau \, dx.$$

We denote by V the closed subspace of $H^1(\Omega; \mathbb{R}^d)$ defined by $V = \{v \in H^1(\Omega; \mathbb{R}^d) : v = 0 \text{ on } \Gamma_D\}$ and on V we consider the inner product and the corresponding norm given by

$$\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \quad \|v\| = \|\varepsilon(v)\|_{\mathcal{H}} \text{ for } u, v \in V.$$

Then the spaces (V, H, V^*) form an evolution triple of spaces.

In the study of the problem (4.2)-(4.10) we use the following assumptions.

$\underline{H(\mathcal{C})}$: the viscosity operator $\mathcal{C}\colon Q \times \mathcal{S}_d \to \mathcal{S}_d$ satisfies the Carathéodory condition (i.e. $\mathcal{C}(\cdot, \cdot, \varepsilon)$ is measurable on Q for all $\varepsilon \in \mathcal{S}_d$ and $\mathcal{C}(x, t, \cdot)$ is continuous on $\mathcal{S}_d$ for a.e. $(x, t) \in Q$) and

(i) $\|\mathcal{C}(x,t,\varepsilon)\|_{\mathcal{S}_d} \le c_1\,(b(x,t) + \|\varepsilon\|_{\mathcal{S}_d})$ for $\varepsilon \in \mathcal{S}_d$, a.e. $(x,t) \in Q$ with $b \in L^2_+(Q)$, $c_1 > 0$;

(ii) $(\mathcal{C}(x,t,\varepsilon_1) - \mathcal{C}(x,t,\varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) \ge 0$ for all ε_1, $\varepsilon_2 \in \mathcal{S}_d$ and a.e. $(x,t) \in Q$;

(iii) $\mathcal{C}(x,t,\varepsilon) : \varepsilon \ge c_2 \|\varepsilon\|^2_{\mathcal{S}_d}$ for all $\varepsilon \in \mathcal{S}_d$ and a.e. $(x,t) \in Q$ with $c_2 > 0$.

$\underline{H(\mathcal{G})}$: the elasticity operator $\mathcal{G}\colon \Omega \times \mathcal{S}_d \to \mathcal{S}_d$ is of the form $\mathcal{G}(x,\varepsilon) = \mathbb{E}(x)\varepsilon$ (Hooke's law) with a symmetric and positive elasticity tensor $\mathbb{E} \in L^\infty(\Omega)$, i.e. $\mathbb{E} = (g_{ijkl})$, $i,j,k,l = 1,\dots,d$ with $g_{ijkl} = g_{jikl} = g_{lkij}$ and $g_{ijkl}(x)\chi_{ij}\chi_{kl} \ge 0$ for a.e. $x \in \Omega$ and for all symmetric tensors $\chi = \{\chi_{ij}\}$.

$\underline{H(j)}$: $j\colon Q \times \mathbb{R}^d \to \mathbb{R}$ is a function such that

(i) $j(\cdot,\cdot,\xi)$ is measurable on Q for all $\xi \in \mathbb{R}^d$;

(ii) $j(x,t,\cdot)$ is locally Lipschitz for all $(x,t) \in Q$;

(iii) $\|\partial j(x,t,\xi)\|_{\mathbb{R}^d} \le c\,(1 + \|\xi\|_{\mathbb{R}^d})$ for all $(x,t) \in Q$, $\xi \in \mathbb{R}^d$ with $c > 0$.

In the hypotheses $H(j)$ the symbol ∂j denotes the Clarke subdifferential of j with respect to the last variable.

$\underline{H(g,h)}$: $g_i, h_i\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are bounded and upper semicontinuous with respect to both variables, for $i = 1,\dots,m$.

The external loading, boundary tractions and the initial data have the following regularity.

$\underline{(H_0)}$: $\psi \in L^2(I; H^{1/2}(\Gamma_N;\mathbb{R}^d))$, $\varphi_2 \in L^2(I;H)$, $u_0 \in V$ and $u_1 \in H$.

The variational formulation of the problem (4.2)–(4.10) reads as follows: find a displacement field $u\colon I \setminus D \to V$ such that $u \in PC(I;V)$, $\dot u \in PW(I) \cap PC(I;H)$ and

$$
\begin{cases}
\langle \ddot u(t), v\rangle + \langle \mathcal{C}\varepsilon(\dot u(t)) + \mathcal{G}\varepsilon(u(t)), \varepsilon(v)\rangle_{\mathcal{H}} + \\
\qquad\qquad + \displaystyle\int_\Omega j^0(x,t,u(x,t); v(x))\,dx \ge \langle \widetilde{\psi}(t), v\rangle_{H^*\times H} \\
\qquad\qquad\qquad\qquad\qquad \text{for all } \ v \in V \text{ and a.e. } t \in I \setminus D \\
u(0) = u_0, \quad \dot u(0) = v_0 \\
u(t_i^+) \in u(t_i^-) + B_{\mathbb{R}}\left(0, g_i\left(|u(t_i^-)|, |\dot u(t_i^-)|\right)\right) \text{ for } i = 1,\dots,m \\
\dot u(t_i^+) \in \dot u(t_i^-) + B_{\mathbb{R}}\left(0, h_i\left(|u(t_i^-)|, |\dot u(t_i^-)|\right)\right) \text{ for } i = 1,\dots,m,
\end{cases}
\tag{4.11}
$$

where $\langle \widetilde{\psi}, v\rangle_{H^*\times H} = \langle \varphi_2(t), v\rangle_H + \langle \psi(t), v\rangle_\Gamma$.

We associate with the hemivariational inequality (4.11) an evolution impulsive inclusion. To this end, let $A\colon I \times V \to V^*$, $B\colon V \to V^*$ and $J\colon I \times H \to \mathbb{R}$ be defined by

$$
\langle A(t,u), v\rangle = \langle \mathcal{C}(x,t,\varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}} \quad \text{for } u, v \in V \text{ and } t \in I,
\tag{4.12}
$$

$$\langle Bu, v\rangle = \langle \mathcal{G}(x, \varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}} \quad \text{for } u, v \in V, \tag{4.13}$$

$$J(t, v) = \int_\Omega j(x, t, v(x))\, dx \quad \text{for } v \in H \text{ and } t \in I. \tag{4.14}$$

Lemma 4.2. *Under the hypothesis $H(\mathcal{C})$ the operator $A\colon I \times V \to V^*$ defined by (4.12) satisfies $H(A)$. Under the assumption $H(\mathcal{G})$ the operator $B\colon V \to V^*$ defined by (4.13) satisfies $H(B)$. Under the assumption $H(j)$ the functional J defined by (4.14) satisfies*

$\underline{H(J)}$: $J\colon I \times H \to \mathbb{R}$ *is a functional such that*

(i) $J(\cdot, v)$ is measurable on I for all $v \in H$;

(ii) $J(t,\cdot)$ is well defined and Lipschitz continuous on bounded subsets of H, for a.e. $t \in I$;

(iii) $|\partial J(t, u)| \le \widetilde{c}\,(1 + |u|)$ for a.e. $t \in I$ and all $u \in H$ with $\widetilde{c} > 0$;

(iv) for a.e. $t \in I$ and all u, $v \in H$, we have

$$J^0(t, u; v) \le \int_\Omega j^0(x, t, u(x); v(x))\, dx,$$

where $\partial J(t, u)$ denotes the Clarke subdifferential of $J(t,\cdot)$ at a point $u \in H$ and $J^0(t, u; v)$ stands for the directional derivative of $J(t,\cdot)$ at a point $u \in H$ in the direction $v \in H$.

We define the multifunctions $G_i\colon V \times H \to 2^V$ and $H_i\colon V \times H \to 2^H$ by $G_i(u, v)(\cdot) = B_V(0, g_i(|u(\cdot)|, |v(\cdot)|))$, $H_i(u, v)(\cdot) = B_H(0, h_i(|u(\cdot)|, |v(\cdot)|))$ for $u \in V$, $v \in H$. From Example 1, we know that under the condition $H(g, h)$ the hypothesis $H(G, H)_1$ holds. We also observe that if (H_0) is satisfied, then $\widetilde{\psi} \in \mathcal{H}^*$, $u_0 \in V$ and $v_0 \in H$.

Consider the following evolution inclusion with impulses.

$$\begin{cases} \ddot{u}(t) + A(t, \dot{u}(t)) + Bu(t) + \partial J(t, u(t)) \ni \widetilde{\psi}(t) \text{ for } t \in I \setminus D \\ u(0) = u_0, \quad \dot{u}(0) = v_0 \\ u(t_i^+) \in u(t_i^-) + G_i(u(t_i^-), \dot{u}(t_i^-)) \\ \dot{u}(t_i^+) \in \dot{u}(t_i^-) + H_i(u(t_i^-), \dot{u}(t_i^-)) \text{ for } i = 1, \dots, m. \end{cases} \tag{4.15}$$

The reason to introduce problem (4.15) is stated in the following lemma. For details we refer to [15] (cf. also [19]).

Lemma 4.3. *Under condition $H(J)$, every solution to problem (4.15) is a solution to the hemivariational inequality (4.11).*

Now, it is clear that defining multifunction $F\colon I \times H \times H \to 2^H$ by

$$F(t, u, v) = -\partial J(t, u) + \widetilde{\psi}(t) \text{ for } t \in I,\ u, v \in H \tag{4.16}$$

and using the above notation, we formulate the inclusion (4.15) in the form of (3.1).

Lemma 4.4. *If $H(J)$ holds and $\widetilde{\psi} \in \mathcal{H}^*$, then F given by (4.16) satisfies $H(F)$.*

Proof. The condition $H(F)(i)$ follows from $H(J)(i)$. Since the subdifferential has nonempty, weakly compact and convex values, the multifunction F is closed and convex valued. It is also known (cf. Proposition 5.6.10 of [9]) that $\partial J(t, \cdot)$ is usc from H into w-H. Hence (cf. Proposition 4.1.14 of [9]) F satisfies $H(F)(ii)$. The condition $H(F)(iii)$ is a consequence of $H(J)(iii)$. □

Summing up, from the result of Section 3, Lemmata 4.2, 4.3, 4.4, we obtain the following.

Theorem 4.5. *If $H(\mathcal{C})$, $H(\mathcal{G})$, $H(j)$, (H_0), $H(g,h)$ hold and $u_0 \in H_0^1(\Omega)$, $v_0 \in L^2(\Omega)$, then problem (4.2)–(4.10) has a solution $u \in PC(I; H_0^1(\Omega))$ with $\frac{\partial u}{\partial t} \in PW(I) \cap PC(I; L^2(\Omega))$ and the set of all solutions is compact in $PC^1(I; L^2(\Omega))$.*

5. ACKNOWLEDGEMENTS

Research supported in part by the State Committee for Scientific Research of the Republic of Poland (KBN) under Grants no. 2 P03A 003 25 and 4 T07A 027 26.

REFERENCES

[1] N.U. Ahmed, Measure solution for impulsive systems in Banach spaces and their control, *Dyn. Cont. Discrete Impulsive Systems*, 6:519–535, 1999.

[2] N.U. Ahmed, Systems governed by impulsive differential inclusions on Hilbert spaces, *Nonlinear Analysis*, 45:693–706, 2001.

[3] N.U. Ahmed, K.L. Teo and S.H. Hou, Nonlinear impulsive systems on infinite dimensional spaces, *Nonlinear Analysis*, 54:907–925, 2003.

[4] M. Benchohra, A. Boucherif and A. Ouahabi, On nonresonance impulsive functional differential inclusions with nonconvex valued right-hand side, *J. Math. Anal. Appl.*, 282:85–94, 2003.

[5] M. Benchohra and A. Ouahab, Initial and boundary value problems for second order impulsive functional differential inclusions, *Electron. J. Qual. Theory Differ. Equ.*, 3:1–10, 2003.

[6] I. Benedetti, An existence result for impulsive functional differential inclusions in Banach spaces, *Discussiones Math.*, 24:13–30, 2004.

[7] F.H. Clarke, *Optimization and Nonsmooth Analysis,* Wiley, Interscience, New York, 1983.

[8] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.

[9] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.

[10] E. Hernández, A second order impulsive Cauchy problem, *Cadernos de Matematica*, 02:229–238, 2001.

[11] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[12] J.H. Liu, Nonlinear impulsive evolution equations, *Dynam. Cont. Discrete Impuls. Systems*, 6:77–85, 1999.

[13] S. Migórski, Existence and relaxation results for nonlinear second order evolution inclusions, *Discussiones Math.*, 15:129–148, 1995.

[14] S. Migórski, Control problems for systems described by nonlinear second order evolution inclusions, *Nonlinear Analysis*, 30:419–428, 1997.

[15] S. Migórski and A. Ochal, Hemivariational inequality for viscoelastic contact problem with slip dependent friction, *Nonlinear Analysis*, 61:135–161, 2005.

[16] S. Migórski and A. Ochal, Existence of solutions for second order evolution inclusions and their applications to mechanical contact problems, *Optimization*, 55:101–120, 2006.

[17] S. Migórski and A. Ochal, Vanishing viscosity for hemivariational inequality modeling dynamic problems in elasticity, *Nonlinear Analysis*, 2006, in press.

[18] Z. Naniewicz and P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications,* Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.

[19] A. Ochal, Existence results for evolution hemivariational inequalities of second order, *Nonlinear Analysis*, 60:1369–1391, 2005.

[20] P.D. Panagiotopoulos, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, 1993.

[21] N.S. Papageorgiou and N. Yannakakis, Second order nonlinear evolution inclusions I: existence and relaxation results, *Acta Mat. Sinica, English Series*, 21:977–996, 2005.

[22] P. Sattayatham, Strongly nonlinear impulsive evolution equations and optimal control, *Nonlinear Analysis*, 57:1005–1020, 2004.

[23] C. Yong-Kui and L. Wang-Tong, Existence results for second order impulsive functional differential inclusions, *J. Math. Anal. Appl.*, 301:477–490, 2005.