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ON MULTIPLE SOLUTIONS FOR STRONGLY RESONANT PROBLEMS WITH THE p-LAPLACIAN

EVGENIA H. PAPAGEORGIOU AND NIKOLAOS S. PAPAGEORGIOU

Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece.

ABSTRACT. We study a nonlinear elliptic problem driven by the p-Laplacian and with a nonsmooth potential function (hemivariational inequality). On the nonsmooth potential we impose conditions of strong resonance. Following a variational approach based on the nonsmooth critical point theory and the second deformation theorem, we establish the existence of at least two nontrivial smooth solutions.

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1. PRELIMINARIES

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. We consider the following nonlinear elliptic problem with a nonmooth potential (hemivariational inequality):

$$\left\{ \begin{array}{l} -div(\|Dx(z)\|^{p-2}Dx(z)) \\ \qquad\qquad -\lambda_1|x(z)|^{p-2}x(z) \in \partial j(z, x(z)) \ \text{ a.e. on } \ Z, \\ x|_{\partial Z} = 0, \quad 1 < p < \infty. \end{array} \right\} \tag{1.1}$$

Here $\lambda_1 > 0$ is the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$ and $j(z, x)$ is a measurable potential function which is locally Lipschitz and in general nonsmooth in the $x \in \mathbb{R}$ variable. By $\partial j(z, x)$ we denote the generalized subdifferential of the locally Lipschitz function $x \to j(z, x)$ (see Section 2). Our goal is to prove a multiplicity result for problem (1.1) under conditions of strong resonance.

To better motivate the work of this paper, let us momentarily restrict ourselves to the semilinear, smooth framework of the seminal work of Landesman-Lazer [15]. So let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and consider the following semilinear elliptic problem

$$\left\{ \begin{array}{l} -\Delta x(z) = f(x(z)) \ \text{ a.e. on } \ Z, \\ x|_{\partial Z} = 0. \end{array} \right\} \tag{1.2}$$

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It is well-known that in order to produce existence and multiplicity results for problem (1.2), we need to know the asymptotic behavior of the right hand side nonlinearity f. Suppose that f is asymptotically linear at infinity and interacts with the principal eigenvalue $\lambda_1 > 0$ of $(-\Delta_p, H_0^1(Z))$. So we have

$$f(x) = \lambda_1 x + g(x)$$

with $\lim\limits_{|x|\to\infty} \frac{g(x)}{x} = 0$. Then such a problem is called "resonant" (at the first eigenvalue $\lambda_1 > 0$). These problems were first studied by Landesman-Lazer [15], who introduced a sufficient condition for their solvability. Resonant problems arise frequently in mechanics. Landesman-Lazer [15] also introduced a classification of resonant problems, according to the rate of growth of the nonlinearity g as $|x| \to \infty$. So they considered the following three distinct cases:

(a): $\lim\limits_{x\to\pm\infty} g(x) = g_\pm \in \mathbb{R}$ and $(g_+, g_-) \neq (0,0)$.
(b): $\lim\limits_{x\to\pm\infty} g(x) = 0$ and $\lim\limits_{x\to\pm\infty} G(x) = \pm\infty$, where $G(x) = \int_0^x g(s)ds$.
(c): $\lim\limits_{x\to\pm\infty} g(x) = 0$ and $\lim\limits_{x\to\pm\infty} G(x) \in \mathbb{R}$.

Case (c) is the strongly resonant at infinity case. The term "strongly resonant" was coined by Bartolo-Benci-Fortunato [5] and of the three cases, this is the most difficult, because as we will also see in the sequel, in the strongly resonant case, the Euler functional exhibits a partial lack of compactness.

Resonant problems, were studied in the past primarily within the framework of semilinear (i.e. $p = 2$) problems with a smooth potential (i.e. $x \to F(x) = \int_0^x f(s)ds$ belongs in $C^1(\mathbb{R})$). We mention the works of Ahmad-Lazer-Paul [1] (they treat case (b)) and Ambrosetti-Mancini [2], Hess [14], Rabinowitz [18] (who consider cases (a) and (b)). The strongly resonant case was investigated by Thews [21], Bartolo-Benci-Fortunato [5], Ward [23], Solimini [19], Lupo-Solimini [16] and more recently by Costa-Silva [9], Arcoya-Costa [3] and Goncalves-Miyagaki [13], who prove multiplicity results. The study of the corresponding problem for the p-Laplacian differential operator, is lagging behind. The work of Arcoya-Orsina [4] employs classical Landesman-Lazer conditions, which preclude strong resonance as this was described earlier. Similarly the recent work of Bouchala-Drabek [6], extends to a nonlinear setting a generalized Landesman-Lazer condition first introduced by Tang [20], which does not incorporate the strongly resonant situation. In addition, both works deal with the problem of existence of solutions and do not address the question of multiplicity of solutions. Finally, there is also the very recent work of Filippakis-Gasinski-Papageorgiou [10], who treat the strongly resonant case using a different set of hypotheses and a different solution method based on an extended version of the Ekeland variational principle due to Zhong [24]. In contrast here, we use a nonsmooth version of the second deformation theorem due to Corvellec [8].

We should mention that hemivariational inequalities provide the right framework to study several problems in mechanics and engineering. For several such applications, we refer to the book of Naniewicz-Panagiotopoulos [17].

2. MATHEMATICAL BACKGROUND

Our approach is variational, based on the nonsmooth critical point theory which uses the subdifferential theory of locally Lipschitz functions. For easy reference, we recall some basic definitions and facts from these theories, which we will need in the sequel. Details can be found in Clarke [7] and Gasinski-Papageorgiou [11].

So let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, the generalized directional derivative $\varphi^0(x; h)$ of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda},$$

whereas the generalized subdifferential $\partial\varphi(x)$ of φ at $x \in X$, is defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \leq \varphi^0(x; h) \quad \text{for all} \;\; h \in X\}.$$

If $\varphi \in C^1(X)$, then φ is locally Lipshitz and $\partial\varphi(x) = \{\varphi'(x)\}$. Also, if $\varphi : X \to \mathbb{R}$ is continuous, convex, then φ is locally Lipschitz and the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis, defined by

$$\partial_c\varphi(x) = \{x^* \in X^* :< x^*, h >\leq \varphi(x + h) - \varphi(x) \;\; \text{for all} \;\; h \in X\}.$$

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we say that $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. We say that φ satisfies the nonsmooth Palais-Smale condition at level $c \in \mathbb{R}$ (the nonsmooth PS$_c$-condition for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ and $m(x_n) = \inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence.

Definition 2.1. If Y is a subset of the Banach space X, a "deformation of Y" is a continuous map $h : [0, 1]\times Y \to Y$ such that $h(0, \cdot) = id_Y$. If $V \subseteq Y$, then we say that V is a "weak deformation retract of Y", if there exists a deformation $h : [0, 1]\times Y \to Y$ such that $h(1, Y) \subseteq V$ and $h(t, V) \subseteq V$ for all $t \in [0, 1]$.

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$ and $c \in \mathbb{R}$, we set

$$\varphi^{0^c} = \{x \in X : \varphi(x) < c\}$$
$$\text{and} \;\; K_c = \{x \in X : 0 \in \partial\varphi(x), \;\; \varphi(x) = c\}.$$

The next theorem is a partial extension to a nonsmooth setting of the so-called "second deformation theorem" (see for example Gasinski-Papageorgiou [12], p. 628) and it is due to Corvellec [8]. In fact the result of Corvelle is formulated in the more general context of metric spaces, for continuous functions, using the so-called weak slope. For our purposes, it suffices the particular form of the result stated next.

Theorem 2.2. *If X is a Banach space, $a, b \in \mathbb{R}$, $a < b$, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the nonsmooth PS_c-condition for every $c \in (a, b)$, φ has no critical points in $\varphi^{-1}(a, b)$ and the set K_a is discrete or empty then there exists a deformation $h : [0, 1] \times \varphi^{0^b} \to \varphi^{0^b}$ such that*

- **(a):** $h(t, \cdot)|_{K_a} = id|_{K_a}$ *for all* $t \in [0, 1]$;
- **(b):** $h(1, \varphi^{0^b}) \subseteq \varphi^{0^a} \cup K_a$;
- **(c):** $\varphi(h(t, x)) \leq \varphi(x)$ *for all* $t \in [0, 1]$ *and all* $x \in \varphi^{0^b}$.

Finally we recall some basic facts about the spectrum of the negative p-Laplacian with Dirichlet boundary condition. So we consider the following nonlinear eigenvalue problem:

$$(2.1) \qquad \left\{ \begin{array}{l} -div(\|Dx(z)\|^{p-2}Dx(z)) = \lambda|x(z)|^{p-2}x(z) \text{ a.e on } Z, \\ x|_{\partial Z} = 0, \quad \lambda \in \mathbb{R}, \quad 1 < p < \infty. \end{array} \right\}$$

The least real number λ, denoted by λ_1, for which problem (2.1) has a nontrivial solution in $W_0^{1,p}(Z)$, is called the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$. We know that λ_1 is positive, isolated and simple (i.e. the corresponding eigenspace in one-dimensional). Moreover, it has a variational characterization using the Rayleigh quotient, namely

$$(2.2) \qquad \lambda_1 = \inf\{\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z), x \neq 0\}.$$

The infimum is actually realized at the normalized eigenfunction u_1. From nonlinear regularity theory (see Gasinski-Papageorgiou [12], p. 737–738), we have that $u_1 \in C_0^{1,\beta}(\bar{Z})$ with $0 < \beta < 1$ and moreover, u_1 does not change sign, hence we may assume that $u_1(z) \geq 0$ for all $z \in \bar{Z}$. In fact the nonlinear strict maximum principle of Vazquez [22] implies that $u_1(z) > 0$ for all $z \in Z$. Note that if $u \in C_0^1(\bar{Z})$ is an eigenfunction for any eigenvalue $\lambda \neq \lambda_1$, then u much change sign.

If V is a topological complement of $\mathbb{R}u_1$ (i.e. $W_0^{1,p}(Z) = \mathbb{R}u_1 \oplus V$, note that $\mathbb{R}u_1$ is the eigenspace corresponding to $\lambda_1 > 0$), then because $\lambda_1 > 0$ is isolated

$$(2.3) \qquad \lambda_V = \inf[\|Dv\|_p^p : v \in V, \|v\|_p = 1] > \lambda_1$$

In our case $V = \{v \in W_0^{1,p}(Z) : \int_Z vu_1^{p-1}dz = 0\}$. When $p = 2$, then $\lambda_V = \lambda_2$ =the second eigenvalue of $(-\Delta, H_0^1(Z))$

3. AUXILIARY RESULTS

The hypotheses on the nonsmooth potential $j(z,x)$ are the following:

$\underline{H(j)}$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z,0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(iii) for every $r > 0$, there exists $a_r \in L^\infty(Z)_+$ such that for almost all $z \in Z$, all $|x| \leq r$ and all $u \in \partial j(z,x)$, we have

$$|u| \leq a_r(z);$$

(iv) there exist functions $j_\pm \in L^1(Z)$ such that $\int_Z j_\pm(z)dz \leq 0$ and

$$\lim_{x \to \pm\infty} j(z,x) = j_\pm(z) \text{ uniformly for a.a. } z \in Z$$

while $\lim\limits_{|x|\to\infty} \frac{u}{|x|^{p-2}x} = 0$ uniformly for a.a. $z \in Z$ and all $u \in \partial j(z,x)$;

(v) there exists $t^* > 0$ such that

$$\int_Z j(z, \pm t^* u_1(z))dz > 0;$$

(vi) for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$j(z,x) \leq \frac{1}{p}(\lambda_V - \lambda_1)|x|^p.$$

Remark 3.1. Because of hypothesis $H(j)(iv)$, problem (1.1) is classified as a strongly resonant problem. The following nonsmooth locally Lipschitz function satisfies hypotheses $H(j)$ but not those of Filippakis-Gasinski-Papageorgiou [10] (for simplicity we drop the z-dependence):

$$j(x) = \begin{cases} \frac{1}{p}(\lambda_V - \lambda_1)|x|^p & \text{if } |x| \leq 1 \\ \frac{c}{p}\frac{1}{|x|} - \frac{c}{p} + \frac{1}{p}(\lambda_V - \lambda_1) & \text{if } |x| > 1, \end{cases}$$

with $c \geq \lambda_V - \lambda_1 > 0$. Another possibility is the function

$$j(x) = \begin{cases} \frac{1}{p}(\lambda_V - \lambda_1)|x|^p & \text{if } |x| \leq 1 \\ (\lambda_V - \lambda_1)\frac{1}{p|x|} & \text{if } |x| > 1. \end{cases}$$

The Euler functional $\varphi : W_0^{1,p}(Z) \to \mathbb{R}$ for problem (1.1), is defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z j(z,x(z))dz \text{ for all } x \in W_0^{1,p}(Z).$$

We know that φ is locally Lipschitz on bounded sets, hence locally Lipschitz. Moreover, if by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(W^{-1,p'}(Z) = W_0^{1,p}(Z)^*$, $W_0^{1,p}(Z))$ $(\frac{1}{p} + \frac{1}{p'} = 1)$, $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ is the nonlinear operator defined by

$$\langle A(x), y \rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N} dz \text{ for all } x, y \in W_0^{1,p}(Z),$$

$K : L^p(Z) \to L^{p'}(Z)$ is defined by

$$K(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot) \ \text{ for all } \ x \in L^p(Z)$$

and $N : L^p(Z) \to 2^{L^{p'}(Z)} \setminus \{\emptyset\}$ is the multifunction defined by

$$N(x) = \{u \in L^{p'}(Z) : u(z) \in \partial j(z, x(z)) \ \text{ a.e. on } \ Z\}$$

then we have

$$\partial\varphi(x) = A(x) - \lambda_1 K(x) - N(x) \ \text{ for all } \ x \in W_0^{1,p}(Z) \tag{3.1}$$

(see Clarke [7], p. 83). Recall that by the Sobolev embedding theorem, the Sobolev space $W_0^{1,p}(Z)$ is embedded compactly in $L^p(Z)$.

The next proposition illustrates the partial lack of compactness characterizing strongly resonant problems, namely the nonsmooth Palais-Smale condition is satisfied only for a certain range of levels.

Proposition 3.1. *If hypotheses $H(j)$ hold, then φ satisfies the PS_c-condition for every $c < \min\{-\int_Z j_+ dz, -\int_Z j_- dz\}$.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that for $c < \min\{-\int_Z j_+ dz, -\int_Z j_- dz\}$, we have

$$\varphi(x_n) \to c \ \text{ and } \ m(x_n) \to c.$$

Exploiting the fact that the norm functional in a Banach space is weakly lower semicontinuous and the set $\partial\varphi(x_n) \subseteq W^{-1,p'}(Z)$ is weakly compact, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi(x_n)$ such that

$$m(x_n) = \inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\} = \|x_n^*\|, \ n \geq 1.$$

From (3.1), we have that

$$x_n^* = A(x_n) - \lambda_1 K(x_n) - u_n$$

with $u_n \in N(x_n)$, $n \geq 1$.

We claim that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. We argue indirectly. So suppose that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is unbounded. We may assume that $\|x_n\| \to \infty$. We set $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. By passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \ \text{ in } \ W_0^{1,p}(Z), \ y_n \to y \ \text{ in } \ L^p(Z), \ y_n(z) \to y(z) \ \text{ a.e. on } \ Z$$

and $|y_n(z)| \leq k(z)$ for a.a $z \in Z$, all $n \geq 1$, with $k \in L^p(Z)_+$.

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$, we have

$$| < x_n^*, x_n > | \leq \varepsilon_n \|x_n\| \ \text{ with } \ \varepsilon_n \downarrow 0,$$

$$\Rightarrow \left| \|Dx_n\|_p^p - \lambda_1 \|x_n\|_p^p - \int_Z u_n x_n dz \right| \leq \varepsilon_n \|x_n\|.$$

We divide with $\|x_n\|^p$ and obtain

$$|\|Dy_n\|_p^p - \lambda_1\|y_n\|_p^p - \frac{1}{\|x_n\|^p}\int_Z u_n x_n dz| \leq \frac{\varepsilon_n}{\|x_n\|^{p-1}} \tag{3.2}$$

By virtue of hypothesis $H(j)(iv)$, we have

$$\frac{1}{\|x_n\|^p}\int_Z u_n x_n dz \to 0 \text{ as } n \to \infty.$$

Therefore, if we pass to the limit as $n \to \infty$ in (3.2), we obtain

$$\begin{aligned} &\|Dy\|_p^p = \lambda_1\|y\|_p^p \qquad \text{(see also (2.2))} \\ &\Rightarrow y = 0 \qquad \text{or } \; y = \pm u_1. \end{aligned}$$

If $y = 0$, then we have $\|Dy_n\|_p \to 0$ as $n \to \infty$ and this by virtue of Poincare's inequality implies that $y_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$.

So $y = \pm u_1$. To fix things we assume $y = u_1$ (the reasoning is similar if $y = -u_1$). Then we have

$$\begin{aligned} &x_n(z) \to +\infty \quad \text{for a.a } z \in Z, \\ \Rightarrow\; &j(z, x_n(z)) \to j_+(z) \quad \text{for a.a } z \in Z \quad \text{(see hypothesis } H(j)(iv)). \end{aligned}$$

Note that by virtue of hypotheses $H(j)(iii), (iv)$ and the mean value theorem for locally Lipschitz functions (see Clarke [7], p. 41), we have that

$$|j(z,x)| \leq h(z) \text{ for a.a } z \in Z \text{ and all } x \in \mathbb{R}, \text{ with } h \in L^1(Z)_+.$$

Therefore an application of the Lebesgue dominated convergence theorem implies

$$\int_Z j(z, x_n(z))dz \to \int_Z j_+(z)dz \text{ as } n \to \infty. \tag{3.3}$$

By hypothesis $\varphi(x_n) \to c$ as $n \to \infty$. So given $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon) \geq 1$ such that

$$\begin{aligned} &\varphi(x_n) \leq c + \varepsilon \quad \text{for all } n \geq n_0, \\ \Rightarrow &\frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p - \int_Z j(z, x_n(z))dz \leq c + \varepsilon \quad \text{for all } n \geq n_0, \\ \Rightarrow &-\int_Z j(z, x_n(z))dz \leq c + \varepsilon \quad \text{for all } n \geq n_0 \quad \text{(see (2.2))}, \\ \Rightarrow &-\int_Z j_+(z)dz \leq c + \varepsilon \quad \text{(see (3.3))}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$ to obtain

$$-\int_Z j_+(z)dz \leq c,$$

a contradiction to the choice of the level c. This means that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. Therefore we may assume that

$$x_n \xrightarrow{w} x \text{ in } W_0^{1,p}(Z) \text{ and } x_n \to x \text{ in } L^p(Z).$$

We have

$$|\langle A(x_n), x_n - x\rangle - \lambda_1 \int_Z |x_n|^{p-2} x_n (x_n - x) dz - \int_Z u_n (x_n - x) dz| \leq \varepsilon_n \|x_n - x\|. \tag{3.4}$$

Evidently

$$\int_Z |x_n|^{p-2} x_n (x_n - x) dz \to 0 \text{ and } \int_Z u_n (x_n - x) dz \to 0$$

(see hypotheses $H(j)(iii), (iv)$).

So from (3.4), we have

$$\lim_{n\to\infty} < A(x_n), x_n - x >= 0. \tag{3.5}$$

But it is easy to see that A is bounded, continuous, monotone, hence it is maximal monotone. A maximal monotone operator is generalized pseudomonotone (see Gasinski-Papageorgiou [12], p. 330). Therefore from (3.5) it follows that

$$\|Dx_n\|_p^p =< A(x_n), x_n >\to \langle A(x), x\rangle = \|Dx\|_p^p.$$

We already know that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$. Since $L^p(Z, \mathbb{R}^N)$ is uniformly convex, from the Kadec-Klee property, we deduce that $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$. This convergence by Poincare's inequality implies $x_n \to x$ in $W_0^{1,p}(Z)$. □

Now we are ready to produce the first nontrivial solution for problem (1.1).

Proposition 3.2. *If hypotheses $H(j)$ hold, then there exists $x_0 \in C_0^1(\bar{Z})$, $x_0 \neq 0$, solution of problem (1.1).*

Proof. Recall that there exists $h \in L^1(Z)_+$ such that

$$|j(z, x)| \leq h(z) \text{ for all a.a } z \in Z \text{ and all } x \in \mathbb{R}.$$

This combined with (2.2), implies that

$$-\|h\|_1 \leq \varphi(x) \text{ for all } x \in W_0^{1,p}(Z).$$

Therefore φ is bounded below. Hence

$$-\infty < \hat{m} = \inf[\varphi(x) : x \in W_0^{1,p}(Z)].$$

Because of hypothesis $H(j)(v)$ and since $\|Du_1\|_p^p = \lambda_1 \|u_1\|_p^p$, we have

$$\hat{m} \leq \varphi(t^* u_1) = -\int_Z j(z, t^* u_1(z)) dz < 0 = \varphi(0),$$

$$\Rightarrow \hat{m} < 0 \leq \min\{-\int_Z j_+ dz, -\int_Z j_- dz\} \text{ (see hypothesis } H(j)(iv)).$$

By Proposition 3.1, φ satisfies the $\mathrm{PS}_{\hat{m}}$-condition.

Invoking the Ekeland variational principle (see for example Gasinski-Papageorgiou [12], p. 582), we can find a minimizing sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ for φ, i.e. $\varphi(x_n) \downarrow \hat{m}$, such that

$$\varphi(x_n) \leq \varphi(x) + \frac{1}{n}\|x - x_n\| \text{ for all } n \geq 1 \text{ and all } x \in W_0^{1,p}(Z).$$

Let $x = x_n + \lambda h$, with $\lambda > 0$ and $h \in W_0^{1,p}(Z)$ arbitrary. We obtain

$$\begin{aligned} -\frac{1}{n}\|h\| &\leq \frac{\varphi(x_n + \lambda h) - \varphi(x_n)}{\lambda}, \\ \Rightarrow -\frac{1}{n}\|h\| &\leq \varphi^0(x_n; h) \text{ for all } n \geq 1 \text{ and all } h \in W_0^{1,p}(Z). \end{aligned}$$

Because $\varphi^0(x_n; \cdot)$ is sublinear continuous, we can apply Lemma 1.3.2, p. 66, of Gasinski-Papageorgiou [11] and obtain $x_n^* \in X^*$ with $\|x_n^*\| \leq 1$ such that

$$\frac{1}{n} < x_n^*, h > \leq \varphi^0(x_n; h) \text{ for all } h \in W_0^{1,p}(Z).$$

This then from the definition of the generalized subdifferential of φ at x_n (see Section 2) implies that

$$\frac{1}{n}x_n^* \in \partial\varphi(x_n) \text{ for all } n \geq 1.$$

Hence we have

$$m(x_n) = \inf[\|x^*\| : x^* \in \partial\varphi(x_n)] \leq \frac{1}{n}\|x^*\| \leq \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

This means that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is a $\mathrm{PS}_{\hat{m}}$-sequence and so by virtue of Proposition 3.1, we can say (at least for a subsequence), that $x_n \to x_0$ in $W_0^{1,p}(Z)$. Then

$$\begin{aligned} &\varphi(x_n) \to \varphi(x_0) = \hat{m} < 0 = \varphi(0), \\ \Rightarrow &x_0 \neq 0 \text{ and } 0 \in \partial\varphi(x_0). \end{aligned}$$

From the last inclusion, it follows that

$$A(x_0) - \lambda_1 K(x_0) = u_0 \text{ with } u_0 \in N(x_0),$$

$$\Rightarrow \left\{ \begin{array}{l} -div(\|Dx_0(z)\|^{p-2}Dx_0(z)) - \lambda_1|x_0(z)|^{p-2}x_0(z) = u_0(z) \text{ a.e on } Z, \\ x_0|_{\partial Z} = 0. \end{array} \right\}$$

So $x_0 \in W_0^{1,p}(Z)$ is a nontrivial solution of problem (1.1) and in addition from nonlinear regularity theory (see for example Gasinski-Papageorgiou [12], p. 738), we have $x_0 \in C_0^1(\bar{Z})$. $\square$

Let $\mathbb{R}u_1$ be the eigenspace corresponding to the principal eigenvalue $\lambda_1 > 0$ of $(-\Delta_p, W_0^{1,p}(Z))$ (recall that $\lambda_1 > 0$ is simple). Let

$$V = \{v \in W_0^{1,p}(Z) : \int_Z v u_1^{p-1} dz = 0\}.$$

This is a topological complement of $\mathbb{R}u_1$, i.e. we have

$$W_0^{1,p}(Z) = \mathbb{R}u_1 \oplus V.$$

Proposition 3.3. *If hypotheses $H(j)$ hold, then $\varphi|_V \geq 0$.*

Proof. Because of hypothesis $H(j)(vi)$, for all $v \in V$, we have

$$\begin{aligned}\varphi(v) &= \frac{1}{p}\|Dv\|_p^p - \frac{\lambda_1}{p}\|v\|_p^p - \int_Z j(z, v(z))dz \\ &\geq \frac{1}{p}\|Dv\|_p^p - \frac{\lambda_1}{p}\|v\|_p^p - \frac{\lambda_V - \lambda_1}{p}\|v\|_p^p \\ &= \frac{1}{p}\|Dv\|_p^p - \frac{\lambda_V}{p}\|v\|_p^p \geq 0 \ \text{ (see (2.3))}.\end{aligned}$$

□

4. MULTIPLICITY THEOREM

In this section using the auxiliary results of the previous section and Theorem 2.1, we prove a multiplicity theorem for problem (1.1).

Theorem 4.1. *If hypotheses $H(j)$ hold, then problem (1.1) has at least two nontrivial solutions $x_0, y_0 \in C_0^1(\bar{Z})$.*

Proof. From Proposition 3.2, we already have one nontrivial solution $x_0 \in C_0^1(\bar{Z})$.

Suppose that $\{x_0\}$ is the only critical point of φ. Then we can apply Theorem 2.1 and obtain a deformation $\eta : [0,1] \times \varphi^{0^0} \to \varphi^{0^0}$ such that

$$\eta(t,\cdot)|_{K_{\hat{m}}} \ \text{is the identity for all} \ t \in [0,1] \tag{4.1}$$

$$\varphi(\eta(t,x)) \leq \varphi(x) \ \text{ for all } \ t \in [0,1] \ \text{ and all } \ x \in \varphi^{0^0} \tag{4.2}$$

$$\text{and } \ \eta(1, \varphi^{0^0}) \subseteq \varphi^{0^{\hat{m}}} \cup K_{\hat{m}}. \tag{4.3}$$

Since we have assumed that $\{x_0\}$ is the only critical point of φ and $\hat{m} < 0 = \varphi(0)$, it follows that $K_{\hat{m}} = \{x_0\}$. Hence

$$\eta(1,y) = x_0 \ \text{ for all } \ y \in \varphi^{0^0} = \{y \in W_0^{1,p}(Z) : \varphi(y) < 0\} \ \text{ (see (4.3))}.$$

For $r > 0$,we set $B_r = \{x \in W_0^{1,p}(Z) : \|x\| < r\}$ and $\partial B_r = \{x \in W_0^{1,p}(Z) : \|x\| = r\}$. If we take $r = t^*$ with $t^* \in \mathbb{R}_+ \setminus \{0\}$ as in hypothesis $H(j)(v)$, we have

$$\mu = \sup[\varphi(x) : x \in \partial B_r \cap \mathbb{R}u_1] < 0. \tag{4.4}$$

We consider the set

$$\Gamma = \{\gamma \in C(\bar{B}_r \cap \mathbb{R}u_1, W_0^{1,p}(Z)) : \gamma\,|_{\partial B_r \cap \mathbb{R}u_1} = \text{identity}\}.$$

Then we introduce $\gamma_0 : \bar{B}_r \cap \mathbb{R}u_1 \to W_0^{1,p}(Z)$, defined by

$$\gamma_0(x) = \begin{cases} x_0 & \text{if } \|x\| < \frac{r}{2} \\ \eta(\frac{2(r-\|x\|)}{r}, \frac{rx}{\|x\|}) & \text{if } \|x\| \geq \frac{r}{2} \end{cases}. \tag{4.5}$$

If $x \in \bar{B}_r \cap \mathbb{R}u_1$ and $\|x\| = \frac{r}{2}$, then

$$\gamma_0(x) = \eta(1, 2x).$$

Since $\|2x\| = r$, from (4.4) and (4.3) we have

$$\gamma_0(x) = \eta(1, 2x) = x_0.$$

Therefore γ_0 is continuous, i.e. $\gamma_0 \in C(\bar{B}_r \cap \mathbb{R}u_1, W_0^{1,p}(Z))$. Moreover, since η is a deformation, we have

$$\begin{aligned} &\eta(0, \cdot) = \text{identity}, \\ \Rightarrow &\gamma_0|_{\partial B_r \cap \mathbb{R}u_1} = \text{identity (see(4.5))}, \\ \Rightarrow &\gamma_0 \in \Gamma. \end{aligned}$$

Then from (4.2), we have

$$\begin{aligned} &\varphi(\eta(t, x)) \leq \varphi(x) \text{ for all } t \in [0, 1] \text{ and all } x \in \bar{B}_r \cap \mathbb{R}u_1, \\ \Rightarrow &\varphi(\gamma_0(x)) < 0 \text{ and all } x \in \bar{B}_r \cap \mathbb{R}u_1 \text{ (see (4.4) and (4.5)).} \end{aligned} \tag{4.6}$$

The pair of sets $\{\partial B_r \cap \mathbb{R}u_1, \bar{B}_r \cap \mathbb{R}u_1\}$ is linking with V in $W_0^{1,p}(Z)$ (see for example Gasinski-Papageorgiou [12], p. 642). Therefore

$$\begin{aligned} &\gamma_0(\bar{B}_r \cap \mathbb{R}u_1) \cap V \neq \emptyset, \\ \Rightarrow &\sup[\varphi(\gamma_0(x)) : x \in \bar{B}_r \cap \mathbb{R}u_1] \geq 0 \text{ (see Proposition 3.3).} \end{aligned} \tag{4.7}$$

Comparing (4.6) and (4.7), we reach a contradiction. This means that φ must have a third critical point $y_0 \in W_0^{1,p}(Z)$, distinct from $\{x_0, 0\}$. Then

$$\begin{aligned} &0 \in \partial\varphi(y_0) \\ \Rightarrow &A(y_0) - \lambda_1 K(y_0) = \hat{u}_0 \text{ with } \hat{u}_0 \in N(y_0), \\ \Rightarrow &\left\{ \begin{array}{l} -div(\|Dy_0(z)\|^{p-2} Dy_0(z)) - \lambda_1 |y_0(z)|^{p-2} y_0(z) = \hat{u}_0(z) \text{ a.e. on } Z, \\ y_0|_{\partial Z} = 0. \end{array} \right\} \end{aligned}$$

So $y_0 \in W_0^{1,p}(Z)$ is a nontrivial solution of (1.1) and as before from nonlinear regularity theory we have $y_0 \in C_0^1(\bar{Z})$. □

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