Dynamic Systems and Applications 16 (2007) 233-250

MULTIPLICITY RESULTS OF POSITIVE SOLUTIONS FOR SINGULAR BOUNDARY VALUE PROBLEMS WITH TIME DEPENDENT NONLINEARITY

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ABSTRACT. We investigate bifurcation phenomena of positive solutions for problems of the form;

$$u''(t) + \lambda f(t, u(t)) = 0, \quad t \in (0, 1),$$
$$u(0) = 0 = u(1),$$

when f satisfies that there exists $r \in C((0,1),(0,\infty))$ with $\int_0^1 s(1-s)r(s)ds < \infty$ such that $0 < \lim_{u\to 0+} \frac{f(t,u)}{r(t)u} < \infty$ uniformly in $t \in (0,1)$. Here λ is a positive real parameter and $f \in C((0,1) \times [0,\infty), [0,\infty))$ may be singular at $t = 0$ and/or $t = 1$.

AMS (MOS) Subject Classification. 34A37, 34B15

1. INTRODUCTION

In this paper, we study the existence of multiple positive solutions for singular boundary value problems of the form

$$(P_\lambda) \qquad u''(t) + \lambda f(t, u(t)) = 0, \quad t \in (0, 1),$$
$$u(0) = 0 = u(1),$$

where λ is a positive real parameter, $f \in C((0,1) \times \mathbf{R}_+, \mathbf{R}_+)$ may be singular at $t = 0$ and/or $t = 1$ and $\mathbf{R}_+ = [0,\infty)$. Existence, nonexistence and multiplicity of positive solutions for (P_λ) have been widely studied by several authors. For the separable case i.e. f is of the form $f(t, u) = q(t)\tilde{f}(u)$, one may refer to Choi ([6]), Dalmasso ([7]), Ha and Lee ([9]), Lee ([12], [13]), Liu and Li ([14]), Wong ([18]), Xu and Ma ([19]), Zhang ([21]) and Zhao ([22]). Studies for general cases were initiated by Agarwal, Wang and Lian ([2]) and have been discussed recently in Cheng and Zhang ([5]), Erbe and Mathsen ([8]), Orpel ([15]), Stanczy ([17]) and Yang ([20]).

In particular, among other results, Cheng and Zhang proved under assumptions

(Z_1) There exist $q \in \mathcal{A}$ and $\Phi \in C(\mathbf{R}_+, \mathbf{R}_+)$ such that

$$f(t, u) \le q(t)\Phi(u), \quad \text{for all } (t, u) \in (0, 1) \times \mathbf{R}_+$$

Received May 1, 2006 1056-2176 $15.00 ©Dynamic Publishers, Inc.

This work was supported by grant R01-2003-000-11731-0 from the Basic Research Program of the Korea Science and Engineering Foundation.

(Z_2) There exist $a < b$ in $(0,1)$ such that one of the following conditions is satisfies;

$$\lim_{u\to+\infty}\frac{\Phi(u)}{u}=0 \quad \text{and} \quad \lim_{u\to 0+}\min_{t\in[a,b]}\frac{f(t,u)}{u}=\infty \quad \text{(sublinear)},$$

$$\lim_{u\to 0+}\frac{\Phi(u)}{u}=0 \quad \text{and} \quad \lim_{u\to+\infty}\min_{t\in[a,b]}\frac{f(t,u)}{u}=\infty \quad \text{(superlinear)}$$

that problem (P_λ) has at least one positive solution for all $\lambda > 0$, here we denote $\mathcal{A} = \{q \in C((0,1),(0,\infty)) | \int_0^1 s(1-s)q(s)ds < \infty\}$.
Stanczy generalized the superlinear case as follows; assume

(C_0) There exists $r \in \mathcal{A}$ such that

$$\lim_{u\to 0^+}\frac{f(t,u)}{r(t)u}=0 \quad \text{uniformly in } t\in(0,1).$$

(C_1) For any $M > 0$, there exists $h_M \in \mathcal{A}$ such that

$$f(t,u) \le h_M(t) \quad \text{for all } (t,u)\in(0,1)\times[0,M].$$

$(S\ )$ There exists a set $A \subset (0,1)$ of positive measure such that

$$\lim_{u\to\infty}\frac{f(t,u)}{u}=\infty \quad \text{uniformly in } t\in A.$$

Then (P_λ) has at least one positive solution for all $\lambda > 0$.

It is interesting to consider that the limit in (C_0) is neither 0 nor ∞. For simplicity, we assume that the limit equals 1 and we consider the following hypothesis;

(C_2) There exists $r \in \mathcal{A}$ such that

$$\lim_{u\to 0^+}\frac{f(t,u)}{u}=r(t) \quad \text{uniformly in } t\in(0,1).$$

In this case, we may expect certain bifurcation phenomena for solutions with respect to parameter λ and investigating this phenomena is our main goal for this paper. Recently Agarwal, Lü and O'Regan ([1]) considered this case for a p-Laplacian problem with superlinear and sublinear growth at ∞, respectively. As an example for superlinear case, they proved under the following main assumption,

$(F\ )$ There exist $\tilde{f} \in C(\mathbf{R}_+, \mathbf{R}_+)$ and $\alpha, \beta \in C((0,1), \mathbf{R}_+)$ with $\beta \in \mathcal{A}$ such that

$$\alpha(t)\tilde{f}(u) \le f(t,u) \le \beta(t)\tilde{f}(u), \quad \text{for all } (t,u)\in(0,1)\times\mathbf{R}_+$$

that if $0 < \lim_{u\to 0^+}\frac{\tilde{f}(u)}{u} = l < \infty$ and $\lim_{u\to\infty}\frac{\tilde{f}(u)}{u} = \infty$, then problem (P_λ) has at least one positive solution for $\lambda \in (0, \frac{1}{lA_1})$ where $A_1 = \min\{\int_0^{\frac{1}{2}} s\beta(s)ds, \int_{\frac{1}{2}}^1 (1-s)\beta(s)ds\}$.
Our main theorem for superlinear case comes out as follows;

Theorem *Assume* $(C_1), (C_2)$ *and* (S). *Also assume*

(C_3) $f(t,u) > 0$, *for all* $(t,u) \in (0,1)\times(0,\infty)$.

Then there exist $0 \leq \lambda_* \leq \lambda^*$ *such that problem* (P_λ) *has at least one positive solution for* $0 < \lambda < \lambda_*$ *and no positive solution for* $\lambda > \lambda^*$.

We notice that combination of conditions (C_1) and (C_2) generalizes condition (F) in Agarwal, Lü and O'Regan. For example, let

$$f(t,u) = \left[(t(1-t))^{-2+\frac{1}{u+1}}\right] u + u^2.$$

Then f satisfies conditions (C_1) and (C_2). But if we suppose that f satisfies condition (F), then $\beta \notin \mathcal{A}$.

Most results mentioned above are based on topological methods which are associated with several fixed point theorems in cones, upper and lower solutions method and Leray-Schauder degree theory. Because, on the other hand, we are interested in the bifurcation phenomena, we employ the global bifurcation theorem of Rabinowitz ([16]). A version of this theorem for singular boundary value problems was recently proved in Im et al. (Theorem 3.1 in [10]) and applied in Im and Lee ([11]). But Theorem 3.1 in [10] is not directly applicable in our situation mainly due to condition (C_2). Therefore we modify the global bifurcation theorem for condition (C_2) and use it to obtain several existence, nonexistence and multiplicity results of positive solutions for problem (P_λ) including Theorem mentioned above.

This paper is organized as follows. In Section 2, we induce the existence of bifurcation branches of solutions for problem (P_λ). In Section 3, we prove the existence of unbounded continuum of positive solutions using the global bifurcation theorem of Rabinowitz. In Section 4, figuring the shape of the unbounded continuum in Section 3 we get several results about existence, nonexistence and multiplicity of positive solutions and then apply them to a problem with generalized linear operator of the form $Lu = (pu')'$. In Section 5, we apply our results to show the existence and multiplicity of positive radial solutions for semilinear elliptic problems defined on exterior domains.

2. PRELIMINARY

We first introduce the global bifurcation theorem due to Rabinowitz ([16]). Consider

$$u = \lambda L u + H(\lambda, u), \tag{2.1}$$

where $L : E \to E$ is a bounded linear operator, $H : \mathbf{R} \times E \to E$ continuous and E a real Banach space with norm $\|\cdot\|$. Let $r(L)$ denote the set of real characteristic values of L and $\mathcal{S}$ the closure of set of nontrivial solutions of (2.1). Assume L and H are compact on E and $\mathbf{R} \times E$ respectively. Furthermore, assume $H(\lambda, 0) = 0$ for all $\lambda \in \mathbf{R}$ and $H(\lambda, u) = o(\|u\|)$ as $\|u\| \to 0$. Then we have a global bifurcation theorem from the trivial branch as follows.

Theorem 2.1. ([16]) *If $\mu \in r(L)$ is of odd multiplicity, then there exists a subcontinuum C in $\mathcal{S}$ bifurcating from $\{(\lambda, 0) \mid \lambda \in \mathbf{R}\}$ at $(\mu, 0)$and either*

(i) *C is unbounded in $\mathbf{R} \times E$ or*
(ii) *C contains $(\hat{\mu}, 0)$, where $\mu \neq \hat{\mu} \in r(L)$.*

Now let us consider the following problem

$$(H_\lambda) \qquad \begin{aligned} &u''(t) + \lambda h(t, u(t)) = 0, \quad t \in (0,1),\\ &u(0) = 0 = u(1), \end{aligned}$$

where $h \in C((0,1) \times \mathbf{R}, \mathbf{R})$. Denote $\mathcal{A} = \{q \in C((0,1),\, (0,\infty)) | \int_0^1 s(1-s)q(s)ds < \infty\}$ and consider the following assumptions.

(H_1) For any $M > 0$, there exists $p_M \in \mathcal{A}$ such that

$$|h(t,u)| \leq p_M(t) \ \text{ for all } |u| \leq M \text{ and } t \in (0,1).$$

(H_2) There exists $\gamma(t) \in \mathcal{A}$ such that $\lim_{u \to 0} \frac{h(t,u)}{u} = \gamma(t)$ uniformly in $t \in (0,1)$.

Problem (H_λ) can be equivalently written as the following integral equation

$$u(t) = \lambda \int_0^1 G(t,s) h(s, u(s)) ds, \tag{2.2}$$

where $G(t,s)$ is Green's function explicitly written as

$$G(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t, \\ t(1-s), & t \leq s \leq 1. \end{cases}$$

Define $L : C_0[0,1] \to C_0[0,1]$ and $H : \mathbf{R} \times C_0[0,1] \to C_0[0,1]$ by taking

$$\begin{aligned} Lu(t) &= \int_0^1 G(t,s)\gamma(s)u(s)ds, \\ H(\lambda, u)(t) &= \lambda \int_0^1 G(t,s)[h(s,u(s)) - \gamma(s)u(s)]ds \end{aligned}$$

respectively. Then (2.2) can be equivalently written as $u = \lambda Lu + H(\lambda, u)$ and by (H_1) and (H_2), it is not hard to check that L is a bounded linear operator and H continuous satisfying $H(\lambda, u) = o(\|u\|_\infty)$ as $\|u\|_\infty \to 0$. It is also known that L and H are completely continuous in $C_0[0,1]$ and $\mathbf{R} \times C_0[0,1]$ respectively. Recently, the existence of characteristic values of L and its properties are studied by Asakawa ([3]) that there exists a sequence of simple characteristic values $0 < \mu_1 < \cdots < \mu_n < \mu_{n+1} < \cdots \to \infty$ and corresponding characteristic function u_n to μ_n has $n-1$ interior zeros. We notice that the characteristic value of L is identical with the eigenvalue of

$$\begin{aligned} &u''(t) + \lambda\gamma(t)u(t) = 0, \quad t \in (0,1),\\ &u(0) = 0 = u(1). \end{aligned}$$

Let us assume (H_1) and (H_2), then by Theorem 2.1, we conclude that there exists a subcontinuum C_k of solutions of (H_λ) bifurcating from $(\mu_k, 0)$ and either it is unbounded in $\mathbf{R} \times C_0[0,1]$ or it meets $(\mu_j, 0)$, for some $j \neq k$. In the following section, we will prove that the first alternative is the only possibility. We end up with this section giving some notations and useful lemmas for later use. Let N_k^+ denote the set of $u \in C_0[0,1]$ such that u has exactly $k-1$ simple interior zeros, $u > 0$ near 0 and all zeros of u in [0,1] are simple. Let ${N_k}^- = -N_k$ and $N_k = {N_k}^+ \cup {N_k}^-$. We notice that $N_k \cap N_j = \emptyset$ if $k \neq j$. Also notice that ${N_k}^\pm$ and N_k are neither open nor closed in $C_0[0,1]$.

We give well-known Gronwall-Bellman inequality.

Lemma 2.2. ([4]) *Let $\epsilon > 0$ and let $m \in L^1(0,T)$ be such that $m \geq 0$ a.e. in $(0,T)$. Suppose $u \in C[0,T]$ and*

$$u(t) \leq \epsilon + \int_0^t m(s)u(s)ds,$$

for all $t \in [0,T]$. Then $u(t) \leq \epsilon e^{\int_0^t m(s)ds}$ for all $t \in [0,T]$.

We also give a modification of Gronwall-Bellman inequality for readers convenience. Proof can be done by obvious modification of Lemma 2.2.

Lemma 2.3. *Let $\epsilon > 0$ and let $m \in L^1(0,T)$ be such that $m \geq 0$ a.e. in $(0,T)$. Suppose $u \in C[0,T]$ and*

$$u(t) \leq \epsilon + \int_t^T m(s)u(s)ds,$$

for all $t \in [0,T]$. Then $u(t) \leq \epsilon e^{\int_t^T m(s)ds}$ for all $t \in [0,T]$.

3. UNBOUNDED BRANCHES

In this section, we prove that subcontinuum C_k of solutions of (H_λ) known to exist in Section 2 is unbounded. Throughout this section, we assume the hypotheses (H_1), (H_2) and

(H_3) $uh(t,u) > 0$ for all $u \in \mathbf{R} \setminus \{0\}$ and all $t \in (0,1)$¿

Lemma 3.1. *For given $M > 0$, there exists $q_M \in \mathcal{A}$ such that*

$$|h(t,u)| \leq q_M(t)|u| \quad \textit{for all } |u| \leq M \textit{ and } t \in (0,1).$$

Proof. Let $\epsilon > 0$ be given, then by (H_2), we may take $\delta > 0$ such that

$$\left|\frac{h(t,u)}{u}\right| - \gamma(t) \leq \left|\frac{h(t,u)}{u} - \gamma(t)\right| \leq \epsilon,$$

for all $|u| \leq \delta$ and $t \in (0,1)$. Thus $|h(t,u)| \leq (\gamma(t)+\epsilon)|u|$, for all $|u| \leq \delta$ and $t \in (0,1)$. By (H_1), there exists $p_M \in \mathcal{A}$ such that if $\delta \leq |u| \leq M$, then $|h(t,u)| \leq p_M(t) \leq \frac{p_M(t)}{\delta}|u|$. Thus

$$|h(t,u)| \leq \max\{\gamma(t)+\epsilon, \frac{p_M(t)}{\delta}\}|u|,$$

for all $|u| \leq M$ and $t \in (0,1)$. If we take $q_M(t) = \max\{\gamma(t)+\epsilon, \frac{p_M(t)}{\delta}\}$, then $q_M \in \mathcal{A}$ and the proof is done. $\square$

Lemma 3.2. *If u is a solution of (H_λ) and u has a double zero, then $u \equiv 0$.*

Proof. Let u be a solution of (H_λ) and $t^* \in [0,1]$ be a double zero of u. *i.e.* $u(t^*) = u'(t^*) = 0$. Let us consider the case $t^* \in (0,1)$. Then from (H_λ), u satisfies

$$u(t) = -\lambda \int_{t^*}^{t}\int_{t^*}^{s} h(\tau,u(\tau))d\tau ds = -\lambda \int_{t^*}^{t}(t-s)h(s,u(s))ds. \tag{3.1}$$

By Lemma 3.1, there exists $q_M \in \mathcal{A}$ such that

$$|h(t,v)| \leq q_M(t)|v|, \tag{3.2}$$

for all $|v| \leq M = \|u\|_\infty + 1$ and $t \in (0,1)$. First, we consider $t \in (t^*,1)$. From (3.1) and (3.2), we get

$$|u(t)| \leq |\lambda| \textstyle\int_{t^*}^{t}(t-s)|h(s,u(s))|ds \leq \int_{t^*}^{t} |\lambda|(1-s)q_M(s)|u(s)|ds.$$

Applying Lemma 2.2 on the interval $(t^*,1)$ with $m(s) = |\lambda|(1-s)q_M(s)$ and $\epsilon = 0$, we get $u \equiv 0$ on $[t^*,1]$. Next, consider $t \in (0,t^*)$. By similar computation, we get

$$|u(t)| \leq \int_{t}^{t^*} |\lambda| s q_M(s)|u(s)|ds.$$

Applying Lemma 2.3 on $(0,t^*)$ with $m(s) = |\lambda| s q_M(s)$ and $\epsilon = 0$, we also get $u \equiv 0$ on $[0,t^*]$. Now, let us consider the case $t^* = 0$, i.e. $u(0) = u'(0) = 0$. The proof for $t^* = 1$ is similar. By similar computation, we get

$$|u(t)| \leq \int_{0}^{t} |\lambda|(t-s)q_M(s)|u(s)|ds.$$

Let $z(t) = \frac{u(t)}{t}$. Then

$$\begin{aligned} |z(t)| &\leq \tfrac{|\lambda|}{t}\textstyle\int_0^t (t-s)q_M(s)|sz(s)|ds. \\ &\leq |\lambda| \textstyle\int_0^t s(1-\frac{s}{t})q_M(s)|z(s)|ds. \\ &\leq |\lambda| \textstyle\int_0^t s(1-s)q_M(s)|z(s)|ds. \end{aligned}$$

Applying Lemma 2.1 on $(0,1)$ with $m(s) = |\lambda| s(1-s)q_M(s)$ and $\epsilon = 0$, we also get $z \equiv 0$ on $[0,1]$. Thus $u \equiv 0$ on $[0,1]$ and the proof is complete. $\square$

Let μ_k denote the k-th characteristic value of linear operator L given in Section 2. The proof of the following lemma is similar to the proof of Lemma 3.2 in [10].

Lemma 3.3. *Let C_k be a subcontinuum of solutions of (H_λ) bifurcating from $(\mu_k, 0)$. Then $C_k \cap \mathbf{R} \times \{0\} \subset \cup_{j=1}^{\infty}\{(\mu_j, 0)\}$.*

Lemma 3.4. *Let u_n and u be nontrivial solutions of (H_{λ_n}) and (H_λ) respectively. If $u_n \to u$ and $\lambda_n \to \lambda \neq 0$, then there exist $\delta_1, \delta_2 > 0$ such that*

$$\cup_{n=1}^{\infty}\{t \in (0,1) \mid u_n(t) = 0\} \subset [\delta_1, \delta_2] \subset (0,1).$$

Proof. Let u_n and u be nontrivial solutions of (H_{λ_n}) and (H_λ) respectively. Let t_n be the first interior zero of u_n. Then we show that there exists $\delta_1 > 0$ such that $t_n > \delta_1$ for all n. We can prove the result for the sequence of last interior zeros of u_n by similar fashion. Suppose on the contrary, $t_n \to 0$. We know that u_n is a solution of the problem

$$\begin{aligned} &u''(t) + \lambda_n h(t, u(t)) = 0, \quad t \in (0, t_n), \\ &u(0) = 0 = u(t_n). \end{aligned}$$

u_n satisfies

$$u_n(t) = \frac{\lambda}{t_n}(t_n - t)\int_0^t sh(s, u_n(s))ds + \frac{\lambda}{t_n}t\int_t^{t_n}(t_n - s)h(s, u_n(s))ds,$$

and

$$u_n'(t) = \frac{-\lambda}{t_n}\int_0^t sh(s, u_n(s))ds + \frac{\lambda}{t_n}\int_t^{t_n}(t_n - s)h(s, u_n(s))ds. \tag{3.3}$$

Let $|u_n(\tilde{t}_n)| = \max_{t\in[0,t_n]}|u_n(t)|$. Then $u_n'(\tilde{t}_n) = 0$ and from (3.3), we get

$$\int_0^{\tilde{t}_n} sh(s, u_n(s))ds = \int_{\tilde{t}_n}^{t_n}(t_n - s)h(s, u_n(s))ds.$$

Since $u_n \to u$, we may assume $|u_n(t)| < \|u\|_\infty + 1$ for all $t \in [0,1]$ and large n. Then by Lemma 3.1, there exists $q_M \in \mathcal{A}$ such that $|h(s, u_n(s))| \leq q_M(s)|u_n(s)|$, where $M = \|u\|_\infty + 1$ and thus

$$|u_n(\tilde{t}_n)| = \lambda_n \int_0^{\tilde{t}_n} s|h(s, u_n(s))|ds \leq |\lambda_n||u_n(\tilde{t}_n)|\int_0^{\tilde{t}_n} sq_M(s)ds.$$

This implies

$$1 \leq |\lambda_n|\int_0^{\tilde{t}_n} sq_M(s)ds.$$

Since $|\lambda_n| \to |\lambda| \neq 0$, $tq_M(t) \in L^1(0,\delta]$ and $\tilde{t}_n \to 0$, the above inequality is not possible and this completes the proof. $\square$

Lemma 3.5. *Every nontrivial solution of (H_λ) has at most finite interior zeros.*

Proof. Let u be a nontrivial solution of (H_λ). Suppose that u has a sequence of distinct interior zeros (t_n). Considering a subsequence if necessary, let $t_n \to t^*$. By Lemma 3.4, we know that $t^* \neq 0$ and $t^* \neq 1$, i.e. $t^* \in (0,1)$. Since u is a solution of

$$\begin{aligned} &u''(t) + \lambda h(t, u(t)) = 0, \quad t \in (t_n, t_{n+1}), \\ &u(t_n) = 0 = u(t_{n+1}), \end{aligned}$$

Following the same computation as in the proof of Lemma 3.4, we get,

$$|u(\tilde{t_n})| \leq \lambda |u(\tilde{t_n})| \int_{t_n}^{\tilde{t_n}} (s - t_n) q_M(s) ds.$$

This implies

$$1 \leq \lambda \int_{t_n}^{\tilde{t_n}} (s - t_n) q_M(s) ds.$$

Since $t_n \to t^*$ and $q_M \in C[t^* - \delta, t^* + \delta]$ for some $\delta > 0$, the integral should converge to 0 as $n \to \infty$ and this contradicts to the inequality. $\square$

By using Lemma 3.5, we may obtain the following lemmas, the proofs are the same as those of Lemmas 3.5 and 3.6 in [10].

Lemma 3.6. *Let u_n and u be nontrivial solutions of (H_{λ_n}) and (H_λ) respectively. Assume that there exists $k > 0$ such that $u_n \in N_k$ for all n, $u_n \to u$ and $\lambda_n \to \lambda \neq 0$. Then $u \in N_k$.*

Lemma 3.7. *For each $j > 0$, there exists a neighborhood $\mathcal{O}_j$ of $(\mu_j, 0)$ such that $(\lambda, u) \in \mathcal{O}_j \cap \mathcal{S}$ and $u \not\equiv 0$ implies $u \in N_j$.*

Now we give the main theorem in this section. The proof is also the same as that of Theorem 3.1 in [10], but we give the proof for readers convenience.

Theorem 3.8. *The subcontinuum C_k known to exist in Section 2 is unbounded.*

Proof. If we show $C_k \subset (\mathbf{R} \times N_k) \cup \{(\mu_k, 0)\}$, then C_k is unbounded by Lemma 3.7, Theorem 2.1 and by the fact $N_j \cap N_k = \emptyset$ for $j \neq k$. Suppose $C_k \not\subset (\mathbf{R} \times N_k) \cup \{(\mu_k, 0)\}$. Then there exists $(\lambda, u) \in C_k \cap (\mathbf{R} \times \partial N_k)$ such that $(\lambda, u) \neq (\mu_k, 0), u \notin N_k$ and $(\lambda_n, u_n) \to (\lambda, u)$ with $(\lambda_n, u_n) \in C_k \cap (\mathbf{R} \times N_k)$. By Lemma 3.3 and Lemma 3,7, $u \not\equiv 0$. Since $\partial N_k \subset \cup_{i=1}^k N_i \cup \{u \in C_0[0,1] |\ u \text{ has a double zero}\}$, $u \notin N_k$ and u does not have a double zero by Lemma 3.2, there exists j with $0 < j < k$ such that $u \in N_j$. Consequently $u_n \in N_k$ and $u \in N_j$ $(j \neq k)$ are nontrivial solutions of (G_{λ_n}) and (G_λ) respectively and $u_n \to u$, $\lambda_n \to \lambda \neq 0$. Thus by Lemma 3.6, we get a contradiction. $\square$

4. MAIN RESULTS

In this section, we prove various existence results of positive solutions for the following problem

$$(P_\lambda) \qquad \begin{aligned} &u''(t) + \lambda f(t, u(t)) = 0, \quad t \in (0,1),\\ &u(0) = 0 = u(1), \end{aligned}$$

where λ is a positive real parameter and $f \in C((0,1) \times \mathbf{R}_+, \mathbf{R}_+)$. The assumptions we are interested in this section are as follows.

(C_1) For any $M > 0$, there exists $h_M \in \mathcal{A}$ such that

$$f(t,u) \le h_M(t) \ \text{ for all } (t,u) \in (0,1) \times [0, M].$$

(C_2) There exists $r \in \mathcal{A}$ such that

$$\lim_{u \to 0^+} \frac{f(t,u)}{u} = r(t) \quad \text{uniformly in } t \in (0,1).$$

(C_3) $f(t,u) > 0$, for all $(t,u) \in (0,1) \times (0, \infty)$.

Define $h : (0,1) \times \mathbf{R} \to \mathbf{R}$ by

$$h(t,u) = \begin{cases} f(t,u), \ u \ge 0, \\ -f(t,-u), \ u < 0, \end{cases}$$

and consider the following problem

$$(H_\lambda) \qquad \begin{aligned} &u''(t) + \lambda h(t, u(t)) = 0, \quad t \in (0,\ 1),\\ &u(0) = 0 = u(1). \end{aligned}$$

We know that a positive solution of problem (H_λ) is a positive solution of problem (P_λ). Assume $(C_1), (C_2)$ and (C_3), then we can easily check that problem (H_λ) satisfies conditions (H_1), (H_2) and (H_3). Thus by theorem 3.8, (H_λ) has an unbounded subcontinuum C_k bifurcating from $(\mu_k, 0)$, where μ_k is the k-th eigenvalue of problem

$$(4.1) \qquad \begin{aligned} &u''(t) + \lambda r(t) u(t) = 0, \quad t \in (0,\ 1),\\ &u(0) = 0 = u(1). \end{aligned}$$

Since we are interested in positive solutions of (H_λ), we focus on the shape of branch C_1. First, we consider the superlinear case so that f satisfies the following condition.

$(S\ )$ There is a set $A \subset (0,1)$ of positive measure such that

$$\lim_{u \to \infty} \frac{f(t,u)}{u} = \infty \ \text{ uniformly in } t \in A.$$

Lemma 4.1. *Assume $(C_2), (C_3)$ and (S). There exists $\tilde{\lambda}$ such that if u is a positive solution of (P_λ), then $\lambda \le \tilde{\lambda}$.*

Proof. Take any set of positive measure $B \subset A \cap (\delta, 1-\delta)$ for some positive number δ and put $\delta_1 = \inf B$ and $\delta_2 = \sup B$. Then we can easily check for $t \in B$,

$$u(t) \geq \min\{\delta_1, 1-\delta_2\}\|u\|_\infty.$$

By $(C_2), (C_3)$ and (S), there exists $k > 0$ such that $f(t,u) \geq ku$ for all $t \in B$ and $u \geq 0$. Thus

$$\begin{aligned}
\|u\|_\infty \geq u(\tfrac{\delta_1+\delta_2}{2}) &= \lambda \textstyle\int_0^1 G(\tfrac{\delta_1+\delta_2}{2}, s) f(s, u(s)) ds \\
&\geq \lambda \textstyle\int_B G(\tfrac{\delta_1+\delta_2}{2}, s) f(s, u(s)) ds \\
&\geq \lambda k \textstyle\int_B G(\tfrac{\delta_1+\delta_2}{2}, s) u(s) ds \\
&\geq \lambda k \min\{\delta_1, 1-\delta_2\} \textstyle\int_B G(\tfrac{\delta_1+\delta_2}{2}, s) ds \|u\|_\infty.
\end{aligned}$$

Therefore we get

$$\lambda \leq \left(k \min\{\delta_1, 1-\delta_2\} \int_B G(\frac{\delta_1+\delta_2}{2}, s) ds \right)^{-1}.$$

Taking $\tilde{\lambda} = (k \min\{\delta_1, 1-\delta_2\} \int_B G(\frac{\delta_1+\delta_2}{2}, s) ds)^{-1}$, we completes the proof. □

Lemma 4.2. *Assume (S). Let J be a given compact interval in $(0, \infty)$. Then there exists $M_J > 0$ such that all possible positive solutions u of (P_λ) with $\lambda \in J$ satisfy $\|u\|_\infty \leq M_J$.*

Proof. Suppose on the contrary that there exists a sequence (u_n) of positive solutions for (P_{λ_n}) with $(\lambda_n) \subset J = [a, b]$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. We know that for $t \in B$, $u_n(t) \geq \min\{\delta_1, 1-\delta_2\}\|u_n\|_\infty$, where B, δ_1 and δ_2 are given in the proof of Lemma 4.1. Take

$$R = 2 \left(a \min\{\delta_1, 1-\delta_2\} \int_B G(\frac{\delta_1+\delta_2}{2}, s) ds \right)^{-1}.$$

Then by (S), there exists $l > 0$ such that $f(t,u) > Ru$ for all $t \in B$ and for all $u > l$. Since $\|u_n\|_\infty \to \infty$, there exists N such that $\|u_n\|_\infty > \frac{l}{\min\{\delta_1, 1-\delta_2\}}$ for $n > N$. By the same computation as in the proof of Lemma 4.1, we get

$$\|u_n\|_\infty \geq u_n(\frac{\delta_1+\delta_2}{2}) \geq aR \min\{\delta_1, 1-\delta_2\} \int_B G(\frac{\delta_1+\delta_2}{2}, s) ds \|u_n\|_\infty.$$

This implies

$$\big(a \min\{\delta_1, 1-\delta_2\} \int_B G(\frac{\delta_1+\delta_2}{2}, s) ds\big)^{-1} \geq R$$

and the contradiction completes the proof. □

Now we have the first existence result of problem (P_λ).

Theorem 4.3. *Assume $(C_1), (C_2), (C_3)$ and (S). Then there exist λ_* and λ^* with $\mu_1 \leq \lambda_* \leq \lambda^*$ such that problem (P_λ) has at least one positive solution for $0 < \lambda < \lambda_*$ and no positive solution for $\lambda > \lambda^*$.*

We do not know whether $\lambda_* = \lambda^*$ or not which means the existence in Theorem 4.3 is local with respect to parameter λ. With additional conditions on f, we can figure out the shape of branch more detail. For this purpose, we consider the following two cases.

(S_1) $f(t,u) > r(t)u$ for all $(t,u) \in (0,1) \times (0,\infty)$.
(S_2) There exists $\tilde{u} > 0$ such that $f(t,u) < r(t)u$ for all $(t,u) \in (0,1) \times (0,\tilde{u})$.

Corollary 4.4. *Assume $(C_1), (C_2), (C_3), (S)$ and (S_1). Then (P_λ) has at least one positive solution for $0 < \lambda < \mu_1$ and no positive solution for $\lambda \geq \mu_1$.*

Proof. It suffices to show $\lambda < \mu_1$ if u is a positive solution of (P_λ). From (S_1), we get

$$0 = u''(t) + \lambda f(t,u(t)) > u''(t) + \lambda r(t)u(t). \tag{4.2}$$

Let $\phi(t)$ be a positive eigenfunction of (4.1) for the first eigenvalue μ_1. Multiplying (4.2) by $\phi(t)$ and integrating from 0 to 1, we get

$$\begin{aligned} 0 &> \int_0^1 u''(s)\phi(s)ds + \lambda \int_0^1 r(s)u(s)\phi(s)ds \\ &= -\mu_1 \int_0^1 r(s)u(s)\phi(s)ds + \lambda \int_0^1 r(s)u(s)\phi(s)ds. \end{aligned}$$

This implies $\lambda < \mu_1$ and the proof is done. □

Corollary 4.5. *Assume $(C_1), (C_2), (C_3), (S)$ and (S_2). Then there exist λ_* and λ^* with $\mu_1 < \lambda_* \leq \lambda^*$ such that (P_λ) has at least two positive solutions for $\lambda \in (\mu_1, \lambda_*)$, one positive solution for $\lambda \in (0, \mu_1] \cup \{\lambda_*\}$ or no positive solution for $\lambda \in (\lambda^*, \infty)$.*

Proof. It suffices to show that if u is a positive solution of (P_λ) with $\|u\| < \tilde{u}$, then $\lambda > \mu_1$. If u is a positive solution of (P_λ) with $\|u\| < \tilde{u}$, then

$$0 = u''(t) + \lambda f(t,u(t)) < u''(t) + \lambda r(t)u(t).$$

By the same computation in Corollary 4.4, we can get $\lambda > \mu_1$. □

Example 4.6. Let

$$f(t,u) = [(t(1-t))^{-2+\frac{1}{u+1}}]u + u^2.$$

For any $M > 0$, define $h_M(t) = [(t(1-t))^{-2+\frac{1}{M+1}}]M + M^2$. Then $h_M \in \mathcal{A}$ and $f(t,u) \leq h_M(t)$ for all $(t,u) \in (0,1) \times [0,M]$. And let $r(t) = (t(1-t))^{-1}$. Then $h_M \in \mathcal{A}$ and

$$\lim_{u \to 0^+} \frac{f(t,u)}{u} = r(t) \quad \text{uniformly in } t \in (0,1)$$

and $r \in \mathcal{A}$. Thus f satisfies (C_1) and (C_2) and we can easily show that f satisfies (C_3), (S) and (S_1). Thus by Corollary 4.4, (P_λ) has at least one positive solution for $0 < \lambda < \mu_1$ and no positive solution for $\lambda \geq \mu_1$.

Example 4.7. Let

$$f(t,u) = (t(1-t))^{-1}\Phi(u),$$

where

$$\Phi(u) = \begin{cases} \sin u, \ 0 \le u \le \frac{\pi}{2}, \\ (u-\frac{\pi}{2})^2 + 1, \ \frac{\pi}{2} < u, \end{cases}$$

For $h_M(t) = (t(1-t))^{-1}(2+M^2)$ and $r(t) = (t(1-t))^{-1}$, we can easily check that f satisfies the all hypotheses of Corollary 4.5. Thus there exist λ_* and λ^* with $\mu_1 < \lambda_* \le \lambda^*$ such that (P_λ) has at least two positive solutions for $\lambda \in (\mu_1, \lambda_*)$, one positive solution for $\lambda \in (0, \mu_1] \cup \{\lambda_*\}$ or no positive solution for $\lambda \in (\lambda^*, \infty)$. One may guess $\lambda_* = \lambda^*$ in this example, but Corollary 4.5 cannot provide this delicacy which means some rooms of improvement left.

Now, we consider the sublinear case that f satisfies the following condition.

(B) There exist $q \in \mathcal{A}$ and $\Phi \in C(\mathbf{R}_+, \mathbf{R}_+)$ such that

$$f(t,u) \le q(t)\Phi(u) \ \text{ for all } (t,u) \in (0,1) \times [0,\infty),$$

$$0 < \lim_{u \to 0^+} \frac{\Phi(u)}{u} = \Phi_0 < \infty \ \text{ and } \ \Phi_\infty = \lim_{u \to \infty} \frac{\Phi(u)}{u} = 0.$$

We notice that condition (B) implies condition (C_1) and get the following lemmas.

Lemma 4.8. *Assume (C_2) and (B). Then there exists $\bar{\lambda} < \mu_1$ such that if (P_λ) has a positive solution, then $\lambda \ge \bar{\lambda}$.*

Proof. By (B), we may choose a constant $M_\Phi > \Phi_0$ such that $\Phi(u) \le M_\Phi u$ for all $u \ge 0$. Thus we get

$$\begin{aligned} 0 = u''(t) + \lambda f(t,u(t)) &\le u''(t) + \lambda q(t)\Phi(u(t)) \\ &\le u''(t) + \lambda M_\Phi q(t) u(t). \end{aligned} \tag{4.3}$$

Let μ_Φ be the first eigenvalue of the problem

$$\begin{aligned} &u''(t) + \lambda \Phi_0 q(t) u(t) = 0, \quad t \in (0,\ 1), \\ &u(0) = 0 = u(1) \end{aligned}$$

and let v be the corresponding positive eigenfunction to μ_Φ. Multiplying (4.3) by v and integrating, we get

$$\begin{aligned} 0 \ &\le \int_0^1 u''(s)v(s)ds + \lambda \int_0^1 M_\Phi q(s)u(s)v(s)ds \\ &= -\mu_\Phi \Phi_0 \int_0^1 q(s)u(s)v(s)ds + \lambda M_\Phi \int_0^1 q(s)u(s)v(s)ds. \end{aligned}$$

This implies $\lambda \ge \frac{\Phi_0}{M_\Phi}\mu_\Phi$. From (C_2) and (B), $r(t) \le \Phi_0 q(t)$. Thus by the comparison, we know $\mu_1 \ge \mu_\Phi$ and the proof is done with $\bar{\lambda} = \frac{\Phi_0}{M_\Phi}\mu_\Phi$. ∎

Lemma 4.9. *Assume (B). Let J be a given compact interval in $(0,\infty)$. Then there exists $M_J > 0$ such that for all $\lambda \in J$ and all possible positive solutions u of (P_λ), one has*

$$\|u\|_\infty \le M_J.$$

Proof. Suppose on the contrary that there exists a sequence (u_n) of positive solutions for (P_{λ_n}) with $(\lambda_n) \subset J = [a,b]$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Let $\alpha \in (0, \frac{1}{bQ})$ where $Q = \int_0^1 s(1-s)q(s)ds$. Then by (B), there exists $u_\alpha > 0$ such that $u > u_\alpha$ implies $\Phi(u) < \alpha u$. Let $m_\alpha = \max_{u\in[0,u_\alpha]} \Phi(u)$, $A_n = \{t \in [0,1] \mid u_n(t) \le u_\alpha\}$ and $B_n = \{t \in [0,1] \mid u_n(t) > u_\alpha\}$. Then we have

$$\begin{aligned} u_n(t) &= \lambda_n \int_0^1 G(t,s)f(s,u_n(s))ds \le \lambda_n \int_0^1 s(1-s)q(s)\Phi(u_n(s))ds \\ &= \lambda_n \int_{A_n} s(1-s)q(s)\Phi(u_n(s))ds + \lambda_n \int_{B_n} s(1-s)q(s)\Phi(u_n(s))ds \\ &\le \lambda_n m_\alpha Q + \lambda_n \int_{B_n} s(1-s)q(s)\Phi(u_n(s))ds, \end{aligned}$$

for $t \in [0,1]$. Thus

$$\frac{1}{\lambda_n} \le \frac{m_\alpha Q}{\|u_n\|_\infty} + \int_{B_n} s(1-s)q(s)\frac{\Phi(u_n(s))}{\|u_n\|_\infty}ds.$$

On B_n, $u_n(s) > u_\alpha$ implies $\frac{\Phi(u_n(s))}{\|u_n\|_\infty} < \frac{\Phi(u_n(s))}{u_n(s)} < \alpha$. Thus

$$\frac{1}{b} \le \frac{1}{\lambda_n} \le \frac{m_\alpha Q}{\|u_n\|_\infty} + \alpha\int_{B_n} s(1-s)q(s)ds \le \frac{m_\alpha Q}{\|u_n\|_\infty} + \alpha Q.$$

By the fact $\|u_n\|_\infty \to \infty$ as $n \to \infty$, we get

$$\frac{1}{b} \le \alpha Q < \frac{1}{bQ}Q = \frac{1}{b}.$$

This contradiction completes the proof. $\square$

Now we have another existence result of problem (P_λ).

Theorem 4.10. *Assume $(C_2), (C_3)$ and (B). Then there exist λ_* and λ^* with $\lambda_* \le \lambda^* \le \mu_1$ such that problem (P_λ) has at least one positive solution for all $\lambda > \lambda^*$ and no positive solution for $0 < \lambda < \lambda_*$.*

Like superlinear case, let us consider the following two cases for more detail analysis.

(B_1) $f(t,u) < r(t)u$ for all $(t,u) \in (0,1) \times (0,\infty)$.

(B_2) There exists $\tilde{u} > 0$ such that $f(t,u) > r(t)u$ for all $(t,u) \in (0,1) \times (0,\tilde{u})$.

Corollary 4.11. *Assume $(C_2), (C_3), (B)$ and (B_1). Then (P_λ) has at least one positive solution for $\lambda > \mu_1$ and no positive solution for $\lambda \le \mu_1$.*

Proof. It suffices to show $\lambda > \mu_1$ if u is a positive solution of (P_λ). From (B_1), we get

$$0 = u''(t) + \lambda f(t,u(t)) < u''(t) + \lambda r(t)u(t).$$

By the same computation as in the proof of Corollary 4.4, we can get $\lambda > \mu_1$ and the proof is done. $\square$

Corollary 4.12. *Assume $(C_2), (C_3), (B)$ and (B_2). Then there exist λ_* and λ^* with $\lambda_* \leq \lambda^* < \mu_1$ such that (P_λ) has at least two positive solutions for $\lambda \in (\lambda^*, \mu_1)$, one positive solution for $\lambda \in [\mu_1, \infty] \cup \{\lambda^*\}$ or no positive solution for $\lambda \in (0, \lambda_*)$.*

Proof. If u is a positive solution of (P_λ) with $\|u\|_\infty < \tilde{u}$, then

$$0 = u''(t) + \lambda f(t, u(t)) > u''(t) + \lambda r(t)u(t).$$

By the same computation in Corollary 4.4, we can easily get $\lambda < \mu_1$ and this derives the conclusion. □

Example 4.13. Let

$$f(t, u) = \arctan(q(t)\Phi(u)),$$

where $q(t) = (t(1-t))^{-1}$ and

$$\Phi(u) = \begin{cases} u, \ 0 \leq u \leq 1, \\ \sqrt{u}, \ 1 < u, \end{cases}$$

Then $q \in \mathcal{A}$, $f(t, u) < q(t)\Phi(u)$ for all $(t, u) \in (0, 1) \times (0, \infty)$, $\Phi_0 = 1$ and $\Phi_\infty = 0$. Let $r(t) = q(t)$, then we can see that f satisfies the all hypotheses of Corollary 4.11. Thus (P_λ) has at least one positive solution for $\lambda > \mu_1$ and no positive solution for $\lambda \leq \mu_1$.

Example 4.14. Let

$$f(t, u) = \frac{1}{t(1-t)}\Phi(u),$$

where

$$\Phi(u) = \begin{cases} \tan u, \ 0 \leq u \leq \frac{\pi}{4}, \\ \sqrt{\frac{4}{\pi}u}, \ \frac{\pi}{4} < u, \end{cases}$$

Then $\Phi_0 = 1$ and $\Phi_\infty = 0$. If $r(t) = q(t) = (t(1-t))^{-1}$, then we can see that f satisfies the all hypotheses of Corollary 4.12. Thus there exist λ_* and λ^* with $\lambda_* \leq \lambda^* < \mu_1$ such that (P_λ) has at least two positive solutions for $\lambda \in (\lambda^*, \mu_1)$, one positive solution for $\lambda \in [\mu_1, \infty] \cup \{\lambda^*\}$ or no positive solution for $\lambda \in (0, \lambda_*)$.

We end up with this section to extend our results to the following problem,

$$(S_\lambda) \qquad \begin{aligned} &(p(t)v'(t))' + \lambda g(t, v(t)) = 0, \quad t \in (0, 1), \\ &v(0) = 0 = v(1), \end{aligned}$$

where $p \in C^1((0, 1), (0, \infty))$ and $g \in C((0, 1) \times \mathbf{R}_+, \mathbf{R}_+)$. We denote $c = \int_0^1 \frac{1}{p(s)} ds < \infty$ and for $\rho(t) = \frac{1}{c}\int_0^t \frac{1}{p(y)} dy$, we also denote $\mathcal{B} = \{h \in C((0, 1), (0, \infty)) \mid \int_0^1 \rho(t)(1 - \rho(t))h(t)dt < \infty\}$. A solution of problem (S_λ) means a function $v \in C[0, 1]$ such that $v(0) = 0 = v(1)$, $pv' \in C^1(0, 1)$ and v satisfies the equation (S_λ). Let us assume the following hypotheses.

(D_1) For any $M > 0$, there exists $b_M \in \mathcal{B}$ such that

$$g(t,v) \leq b_M(t) \ \text{ for all } (t,v) \in (0,1) \times [0,M].$$

(D_2) There exists $b \in \mathcal{B}$ such that $\lim_{v\to 0^+} \frac{g(t,v)}{v} = b(t)$ uniformly in $t \in (0,1)$.

(D_3) $g(t,v) > 0$ for all $v > 0$ and all $t \in (0,1)$.

By changing variable, so called Liouville transformation, $s = \rho(t)$ and $u(s) = v(t)$, problem (S_λ) can be equivalently written as

$$(P_\lambda) \qquad \begin{aligned} &u''(s) + \lambda f(s,u(s)) = 0, \quad s \in (0,1), \\ &u(0) = 0 = u(1), \end{aligned}$$

with $f(s,u) = c^2 p(\eta(s))g(\eta(s),u)$, $t = \eta(s)$ is the inverse function of $s = \rho(t)$. Let $r(s) = c^2 p(\eta(s))b(\eta(s))$ and $h_M(s) = c^2 p(\eta(s))b_M(\eta(s))$. Then we can easily check that f satisfies conditions (C_i) if g satisfies (D_i) for $i = 1,2,3$. And also it is known that $H : \mathbf{R} \times C[0,1] \to \mathbf{R} \times C[0,1]$ defined by $H(\lambda,v) = (\lambda,u)$ is homeomorphism and $\|H(\lambda,v)\| = \|(\lambda,u)\|$. Therefore there exists a subcontinuum $\tilde{C}_k$ of solutions of (S_λ) bifurcating form $(\mu_k,0)$ with $H(\tilde{C}_k) = C_k$, where C_k is known to exist in Section 2. We notice that $\tilde{C}_k$ has the same shape with C_k. Applying previous results for problem (P_λ), we obtain the following corollaries.

First, for the superlinear case, let us consider the following hypotheses,

(DS) There is a set $B \subset (0,1)$ of positive measure such that

$$\lim_{v\to\infty} \frac{g(t,v)}{v} = \infty \ \text{ uniformly in } t \in B.$$

(DS_1) $g(t,v) > b(t)v$ for all $(t,v) \in (0,1) \times (0,\infty)$.

(DS_2) There exists $\tilde{v} > 0$ such that

$$g(t,v) < b(t)v \ \text{ for all } (t,v) \in (0,1) \times (0,\tilde{v}).$$

Corollary 4.15. *Assume $(D_1),(D_2),(D_3),(DS)$ and (DS_1). Then the conclusion of Corollary 4.4 is valid for the problem (S_λ).*

Corollary 4.16. *Assume $(D_1),(D_2),(D_3),(DS)$ and (DS_2). Then the conclusion of Corollary 4.5 is valid for the problem (S_λ).*

Finally, for the sublinear case, let us consider the following hypotheses,

(DB) There exist $\beta \in \mathcal{B}$ and $\Phi \in C(\mathbf{R}_+,\mathbf{R}_+)$ such that

$$g(t,v) \leq \beta(t)\Phi(v) \ \text{ for all } (t,v) \in (0,1) \times [0,\infty),$$

$$0 < \lim_{v\to 0^+} \frac{\Phi(v)}{v} = \Phi_0 < \infty \ \text{ and } \ \lim_{v\to\infty} \frac{\Phi(v)}{v} = 0.$$

(DB_1) $g(t,v) < b(t)v$ for all $(t,v) \in (0,1) \times (0,\infty)$.

(DB_2) There exists $\tilde{v} > 0$ such that

$$g(t, v) > b(t)v \text{ for all } (t, v) \in (0, 1) \times (0, \tilde{v}).$$

Corollary 4.17. *Assume $(D_2), (D_3), (DB)$ and (DB_1). Then the conclusion of Corollary 4.11 is valid for the problem (S_λ).*

Corollary 4.18. *Assume $(D_2), (D_3), (DB)$ and (DB_2). Then the conclusion of Corollary 4.12 is valid for the problem (S_λ).*

Remark 4.19. The theorems and corollaries in this section are still valid if we replace condition (C_2) (or (D_2)) by the following type; There exists $r \in \mathcal{A}$ such that

$$0 < \lim_{u \to 0^+} \frac{f(t, u)}{r(t)u} < \infty, \quad \text{uniformly in } t \in (0, 1).$$

5. APPLICATION TO SEMILINEAR ELLIPTIC RADIAL PROBLEMS ON AN EXTERIOR DOMAIN

In this section we apply the results in the previous section to study the existence and multiplicity of positive radial solutions of semilinear elliptic problems on an exterior domain. Let us consider

$$(P_E) \qquad \begin{aligned} &\Delta u + \lambda K(|x|, u) = 0 \quad \text{in } \Omega, \\ &u|_{\partial\Omega} = 0 \text{ and } u \to 0 \text{ as } |x| \to \infty, \end{aligned}$$

where $\Omega = \{x \in \mathbb{R}^n : |x| > r_0\}$, $r_0 > 0$, $n > 2$, λ a positive real parameter and $K \in C([r_0, \infty) \times \mathbf{R}_+, \mathbf{R}_+)$. Let $\mathcal{A}_E = \{k \in C([r_0, \infty), (0, \infty)) \mid \int_{r_0}^{\infty} rk(r)dr < \infty\}$. We assume the followings,

(E_1) For any $M > 0$, there exists $k_M \in \mathcal{A}_E$ such that

$$K(r, u) \le k_M(r) \text{ for all } 0 \le u \le M \text{ and } r \in [r_0, \infty).$$

(E_2) There exists $k \in \mathcal{A}_E$ such that $\lim_{u \to 0^+} \frac{K(r,u)}{u} = k(r)$ uniformly in $r \in [r_0, \infty)$.

(E_3) $K(r, u) > 0$ for all $u > 0$ and all $r \ge r_0$.

Applying $r = |x|$ and $v(r) = u(|x|)$, we can transform problem (P_E) into the following the problem:

$$(5.1) \qquad \begin{aligned} &v''(r) + \frac{n-1}{r} v'(r) + \lambda K(r, v) = 0, \quad t > r_0, \\ &v(r_0) = 0 \text{ and } v(r) \to 0 \text{ if } r \to \infty. \end{aligned}$$

Next, applying $t = 1 - (\frac{r}{r_0})^{2-n}$ and $z(t) = v(r)$, we can transform problem (5.1) into the following the problem;

$$(P_\lambda) \qquad \begin{aligned} &z''(t) + \lambda f(t, z(t)) = 0, \quad t \in (0, 1) \\ &z(0) = 0, \ z(1) = 0, \end{aligned}$$

where f can be explicitly written as

$$f(t,z) = (\frac{r_0}{n-2})^2(1-t)^{\frac{2n-2}{2-n}} K(r_0(1-t)^{\frac{1}{2-n}}, z).$$

We know that f is singular at $t=1$. Taking $r(t) = (\frac{r_0}{n-2})^2(1-t)^{\frac{2n-2}{2-n}} k(r_0(1-t)^{\frac{1}{2-n}})$ and $h_M = (\frac{r_0}{n-2})^2(1-t)^{\frac{2n-2}{2-n}} k_M(r_0(1-t)^{\frac{1}{2-n}})$, we can easily check that f satisfies conditions (C_i) if K satisfies (E_i) for $i=1,2,3$. Therefore, using results in Section 4, we obtain the following corollaries.

First, for the superlinear case, let us consider the following hypotheses;

(ES) There is a set $B \subset (r_0, \infty)$ of positive measure such that

$$\lim_{u\to\infty} \frac{K(r,u)}{u} = \infty \text{ uniformly in } r \in B.$$

(ES_1) $K(r,u) > k(r)u$ for all $(r,u) \in [r_0,\infty) \times (0,\infty)$.

(ES_2) There exists $\tilde{u} > 0$ such that

$$K(r,u) < k(r)u, \quad \text{for all } (r,u) \in [r_0,\infty) \times (0,\tilde{u}).$$

Corollary 5.1. *Assume $(E_1),(E_2),(E_3),(ES)$ and (ES_1). Then the conclusion of Corollary 4.4 is valid for the problem (P_E).*

Corollary 5.2. *Assume $(E_1),(E_2),(E_3),(ES)$ and (ES_2). Then the conclusion of Corollary 4.5 is valid for the problem (P_E).*

Finally, for the sublinear case, let us consider the following hypotheses;

(EB) There exists $p \in \mathcal{A}_E$ and $\Phi \in C(\mathbf{R}_+, \mathbf{R}_+)$ such that

$$K(r,u) \le p(r)\Phi(u), \quad \text{for all } (r,u) \in [r_0,\infty) \times [0,\infty),$$

$$0 < \lim_{u\to 0^+} \frac{\Phi(u)}{u} = \Phi_0 < \infty \quad \text{and} \quad \lim_{u\to\infty} \frac{\Phi(u)}{u} = 0.$$

(EB_1) $K(r,u) < k(r)u$ for all $(r,u) \in [r_0,\infty) \times (0,\infty)$.

(EB_2) There exists $\tilde{u} > 0$ such that

$$K(r,u) > k(r)u, \quad \text{for all } (r,u) \in [r_0,\infty) \times (0,\tilde{u}).$$

Corollary 5.3. *Assume $(E_2),(E_3),(EB)$ and (EB_1). Then the conclusion of Corollary 4.11 is valid for the problem (P_E).*

Corollary 5.4. *Assume $(E_2),(E_3),(EB)$ and (EB_2). Then the conclusion of Corollary 4.12 is valid for the problem (P_E).*

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