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NOTE ON ASYMPTOTIC STABILITY OF A MECHANICAL ROBOTICS MODEL WITH DELAY AND NEGATIVE AND POSITIVE DAMPING

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ABSTRACT. In this paper we study the asymptotic stability of a mechanical robotics model with damping and delay. In this paper we deal with a more realistic damping model than that considered in a previous paper [5]. This model yields a certain linear third order delay differential equation. In proving our results we make use of Pontryagin's theory for quasi-polynomials.

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1. INTRODUCTION

It is well known that delay or time lag provides a source of instability in dynamical systems. G. Stépán [1] accounts for various sources of delay in robotics and addresses aspects of stability of the resulting differential systems. Stépán [2] considers a position controlled elastic robot with one degree of freedom. (See Figure 1.)

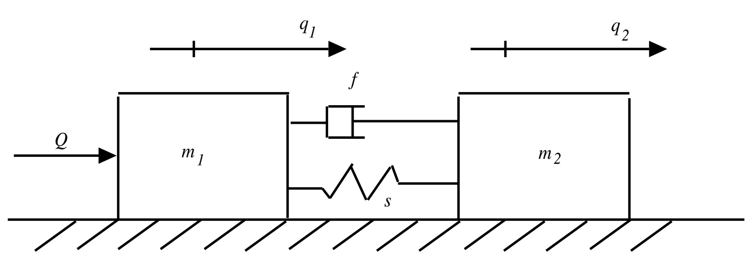

Figure 1

This mechanical system is governed by a system of delay differential equations which interestingly evolves into a third order scalar delay differential equation. See also [3] and [4] for further study of this model and further application of a third order delay

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differential equations. The aim of this paper is to give a complete study and extension of Stépán's stability results for this model. The resulting system is

$$X'(t) = CX(t) + EX(t-\tau) \tag{1.1}$$

where

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \alpha^2 & -\alpha^2 & -2\kappa\alpha \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & -K & 0 \\ 0 & 0 & 0 \\ 0 & -2K\kappa\alpha & 0 \end{pmatrix},$$

$$X(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ v(t) \end{pmatrix},$$

$\alpha = \sqrt{s/m_2}$ is the natural frequency of the undamped, uncontrolled system, and $\kappa = f/(2m_2\alpha)$ is the relative damping factor. We write system (1.1) as a third order delay differential equation in $q_2(t)$

$$q_2'''(t) = -\alpha^2 K q_2(t-\tau) - \alpha^2 q_2'(t) - 2\kappa\alpha q_2''(t) - 2\alpha K\kappa q_2'(t-\tau). \tag{1.2}$$

In this paper we study the asymptotic stability of the zero solution of the following third order delay differential equation

$$y'''(t) = p_1 y''(t) + q_2 y'(t-\tau) + q_1 y'(t) + r_2 y(t-\tau). \tag{1.3}$$

In [5], we studied

$$y'''(t) = p_1 y''(t) + p_2 y''(t-\tau) + q_1 y'(t) + r_2 y(t-\tau). \tag{1.4}$$

which is not equivalent to (1.2). Our paper [5] gives a thorough treatment of stability of the equation (1.4). However, there was an error in passing from (1.2) to (1.4), and as a result (1.4) does not correctly model damping. In the present paper, we address the correct case of damping by studying (1.3).

Very little has been done in the study of delay differential equations of the third order. See [5,6,7,8,9] for some results on asymptotic stability for certain third order delay differential equations. See [10,11,12] for other topics dealing with third order delay differential equations, and see [1,13,14,15,16,17,18,19] for other topics on systems of the form (1.1). One feature of numerous studies of delay differential equations is that delay generally has an unstabilizing effect but that there are rare cases in which delay stabilizes systems that are unstable without delay (see [5,6,18,19]). In this note, we enquire whether delay can stabilize Stépán's system when the damping is "negative" (i.e., $\kappa < 0$). The idea of "negative damping" has arisen recently in some dynamical systems, see for example [20,21]. We consider a more general view of "negative damping" (i.e., $p_1 > 0$ and $q_2 > 0$) and give an algorithmic characterization of asymptotic stability of the zero solution of (1.3) in this case. In the nondelay case (i.e., $\tau = 0$) with general negative damping (i.e., $p_1 > 0$ and $q_2 > 0$), the Routh-Hurwitz criteria yield that the zero solution cannot be stable. We will use our characterization to produce an example where delay does indeed stabilize Stépán system with negative damping. We also give an algorithmic characterization of asymptotic stability of the zero solution of (1.3) in the general "positively damped" case (i.e., $p_1 < 0$ and $q_2 < 0$).

Our papers [5], [19] give general stability criteria for a third order and odd higher order. The stability criteria in this note are idealized for (1.3) and general positive or negative damping. The resulting stability tests have sharper stopping criteria than those in [4], [5] and are thus more efficient.

This paper is organized as follows. In Section 2, we present the tools used in our asymptotic stability analysis. In Section 3 we give our main results. In Section 4 we present some examples.

2. BACKGROUND

In this section, we identify the characteristic function of (1.1) in order to study the asymptotic stability of the zero solution. We also quote the main results of Pontryagin related to asymptotic stability [23,24,25] and the applications of Pontryagin's results [23, 13.7-13.9].

The characteristic function of (1.1) is given by

$$\widehat{H}(s) = s^3 - p_1 s^2 - q_2 s e^{-s\tau} - q_1 s - r_2 e^{-s\tau}. \tag{2.1}$$

Multiplying (2.1) by $e^{s\tau}$ yields

$$e^{s\tau}\widehat{H}(s) = e^{s\tau}s^3 - p_1 s^2 e^{s\tau} - q_2 s - q_1 s e^{s\tau} - r_2. \tag{2.2}$$

Letting $s = \frac{z}{\tau}$, we examine the zeros of

$$H(z) = \tau^3 e^z \widehat{H}(\frac{z}{\tau}) = z^3 e^z - Az^2 e^z - Bze^z - Dz - M \tag{2.3}$$

where

$$A = \tau p_1,\ B = \tau^2 q_1,\ D = \tau^2 q_2 \ \text{ and } M = \tau^3 r_2. \tag{2.4}$$

With regard to the reduced robotic equation (1.2)

$$A = -2\tau\kappa\alpha,\ B = -\tau^2\alpha^2,\ D = -2\tau^2 K\kappa\alpha \ \text{ and } M = -\tau^3\alpha^2 K, \tag{2.5}$$

and with this our study relates to the particular case (1.2) of (1.3).

Theorem 2.1. *In order that all solutions of* (1.1) *approach zero as* $t \to \infty$ *it is necessary and sufficient that all zeros of* (2.1), *or equivalently* (2.3), *have negative real parts.*

See [22]. The function (2.3) is a special function, usually called an exponential polynomial or a quasi-polynomial. The problem of analyzing the distribution of the zeros in the complex plane of such functions has received a great deal of attention.

Definition 2.1. Let $h(z,w)$ be a polynomial in the two variables z and w (with complex coefficients),

$$h(z,w) = \sum_{m,n} a_{mn} z^m w^n, \quad (m,\ n \ \text{nonnegative integers})$$

We call the term $a_{rs}z^r w^s$ the principal term of $h(z,w)$ if $a_{rs} \neq 0$, and for each term $a_{mn}z^m w^n$ with $a_{mn} \neq 0$, we have $r \geq m$ and $s \geq n$.

Note that $H(z) = h(z, e^z)$ where

$$h(z,w) = wz^3 - (Az^2 + Bz)w - Dz - M.$$

It is clear from Definition 2.1 that $h(z,w)$ above has principal term $z^3 w$. We now cite two theorems of Pontryagin, see [22,23,24].

Theorem 2.2. *Let* $H(z) = h(z, e^z)$, *where* $h(z,w)$ *is a polynomial with a principal term. The function* $H(iy)$ *is now separated into real and imaginary parts; that is, we set* $H(iy) = F(y) + iG(y)$. *If all the zeros of the function* $H(z)$ *lie in the open left half plane, then the zeros of the functions* $F(y)$ *and* $G(y)$ *are real, are interlacing, and*

$$\mathbb{D}(y) = G'(y)F(y) - G(y)F'(y) > 0 \tag{2.6}$$

for all real y. *Moreover, in order that all the zeros of the function* $H(z)$ *lie in the open left half plane, it is sufficient that one of the following conditions be satisfied:*

(a) *All the zeros of the functions* $F(y)$ *and* $G(y)$ *are real and interlace, and the inequality* (2.6) *is satisfied for at least one value of* y.
(b) *All the zeros of the function* $F(y)$ *are real and for each of these zeros* $y = y_0$ *condition* (2.6) *is satisfied; that is,* $F'(y_0)G(y_0) < 0$.
(c) *All the zeros of the function* $G(y)$ *are real and for each of these zeros the inequality* (2.7) *is satisfied; that is,* $G'(y_0)F(y_0) > 0$.

In our case,

$$H(iy) = (iy)^3 e^{iy} - (A(iy)^2 + B(iy))e^{iy} - (D(iy) + M) \tag{2.7}$$

or

$$H(iy) = F(y) + iG(y) \tag{2.8}$$

where

$$F(y) = y^3 \sin y + By \sin y + Ay^2 \cos y - M \tag{2.9}$$

and

$$G(y) = -y^3 \cos y - By \cos y + Ay^2 \sin y - Dy. \tag{2.10}$$

In order to study the location of the zeros of $H(z)$ one has to study the zeros of F and G. To do so, we need the following result which is useful in determining whether all roots of F and G are real. Let $f(z, u, v)$ be a polynomial in z, u, and v, which we write in the form

$$f(z, u, v) = \sum_{m,n} z^m \phi_m^{(n)}(u, v) \tag{2.11}$$

where $\phi_m^{(n)}(u, v)$ is a polynomial of degree n, homogeneous in u and v, and let $z^r \phi_r^{(s)}(u, v)$ be the principal term of $f(z, u, v)$, and let $\phi^{*(s)}(u, v)$ denote the coefficient of z^r in $f(z, u, v)$, so that

$$\phi^{*(s)}(u, v) = \sum_{n \leq s} \phi_r^{(n)}(u, v).$$

Also we let

$$\Phi^{*(s)}(z) = \phi^{*(s)}(\cos z, \sin z).$$

Theorem 2.3. *Let $f(z, u, v)$ be a polynomial with principal term $z^r \phi_r^{(s)}(u, v)$. If ϵ is such that $\Phi^{*(s)}(\epsilon + iy) \neq 0$ for all real y, then in the strip $-2\pi k + \epsilon \leq x \leq 2\pi k + \epsilon$, $z = x + iy$, the function $F(z) = f(z, \cos z, \sin z)$ will have, for all sufficiently large values of k, exactly $4sk + r$ zeros. Thus, in order for the function $F(z)$ to have only real roots, it is necessary and sufficient that in the interval $-2\pi k + \epsilon \leq x \leq 2\pi k + \epsilon$, it has exactly $4sk + r$ real roots for all sufficiently large k.*

Note that the functions $F(y)$ and $G(y)$ in (2.9) and (2.10) have principal terms $y^3 \sin y$ and $-y^3 \cos y$, respectively. For the function $F(y)$ and $G(y)$ in (2.9) and (2.10), the number of roots in $-2k\pi + \epsilon \leqslant y \leqslant 2k\pi + \epsilon$ is $4k + 3$ for k sufficiently large. For $F(y)$, we take $0 < \epsilon < \pi$, and for G we take $\epsilon = 0$. We will use Theorem 2.2 and Theorem 2.3 to study the asymptotic stability of (1.1). In the next section we will present the main results of this paper.

3. MAIN RESULTS

In this section we present the main results of this paper. From Theorem 2.2, we have the following necessary condition.

Lemma 3.1. *If the zero solution of* (1.1) *is asymptotically stable, then* $\mathbb{D}(0) = (B + D)M > 0$.

In view of Theorem 2.2, we only consider cases where $\mathbb{D}(0) > 0$. First we start with undamped system, i.e. $\kappa = 0$, or equivalently, $A = 0$ and $D = 0$.

Theorem 3.1. *Assume that $A = 0$ and $D = 0$. Then the zero solution of* (1.1) *is asymptotically stable if and only if $M < 0$, $B < 0$ and there exists $k \in Z^+$ such that*

$$2k\pi + \pi/2 < \sqrt{-B} < (2k+1)\pi + \pi/2 \tag{3.1}$$

and

$$-M < \min(-(2k\pi+\pi/2)((2k\pi+\pi/2)^2+B), ((2k+1)\pi+\pi/2)(((2k+1)\pi+\pi/2)^2+B)). \tag{3.2}$$

Proof. With $D = 0$, this case coincides with a case considered in [5]. See our paper [1] for the proof. This result is the rediscovery of a result given in [2], and a different proof is given in [5]. □

We now state two results from [6].

Lemma 3.2. *For n sufficiently large, the interval $(n\pi, (n+1)\pi)$ contains exactly one zero r_n of G and $\lim_{n\to\infty}[r_n]_\pi = \pi/2$.*

Here $[x]_c$ denotes the unique real number in $[0, c)$ for which $x - [x]_c$ is a multiple of c.

Theorem 3.2. *If the zero solution of* (1.3) *is asymptotically stable, then $M < 0$ and $B + D < 0$.*

We start with the general negative damping case (i.e. $A > 0$ and $D > 0$).

Lemma 3.3. *Assume that $A > 0$ and $D > 0$, $M < 0$ and $B + D < 0$. Let*

$$N_0 = \left[\!\left[\frac{0.5 + 0.5\sqrt{1 + 4(D - B)}}{\pi} \right]\!\right] + 1 \tag{3.3}$$

Then G has all real zeros if and only if G has $N_0 + 1$ zeros in $(0, N_0\pi)$. Furthermore, in this case, G has precisely one zero in $(n\pi, (n+1)\pi)$ for every integer $n \geq N_0$.

Here $[\![\ ]\!]$ denotes the greatest integer function.

Proof. Recall equations (2.9) and (2.10):

$$F(y) = (y^3 + By)\sin y + Ay^2 \cos y - M,$$

$$G(y) = -(y^3 + By)\cos y + Ay^2 \sin y - Dy.$$

By hypotheses, $\mathbb{D}(0) = (B + D)M > 0$. Note that $y = 0$ is a zero of G, and G is an odd function. If y is not a multiple of π, then y is a zero of G if and only if

$$w(y) = \zeta y \tag{3.4}$$

where

$$w(y) = (y^2 + B)\cot y + D\csc y \tag{3.5}$$

and

$$\zeta(y) = Ay. \tag{3.6}$$

As $y \to 0^+$ the function w has limit $-\infty$. As $y \to n\pi^+$ (n a positive integer)the function w has a limit $\operatorname{sgn}((n\pi)^2 + B + (-1)^n D)\infty$ and as $y \to n\pi^-$ w has a limit $\operatorname{sgn}((n\pi)^2 + B + (-1)^n D)(-\infty)$.

For n large the function w has limits $+\infty$ and $-\infty$ at $(n-1)\pi$ and $n\pi$, respectively, when the limits are taken from inside this interval $((n-1)\pi, n\pi)$.

From equation (3.4) we have

$$w'(y) = \frac{y\sin 2y - y^2 - B - D\cos y}{\sin^2 y} \leq \frac{y - y^2 - B + D}{\sin^2 y} < 0$$

when $y > 0.5+0.5\sqrt{1+4(D-B)}$ and y is not a multiple of π. See Figure 1 for typical representation. It follows that if $n\pi > 0.5 + 0.5\sqrt{1+4(D-B)}$, then w is strictly decreasing on $(n\pi, (n+1)\pi)$ and w approaches $-\infty$ and ∞ when y approaches $n\pi$ and $(n+1)\pi$, respectively, from inside the interval $(n\pi, (n+1)\pi)$. It follows $w(y) = \zeta(y)$ has precisely one root in $(n\pi, (n+1)\pi)$ for all $n \geq N_0$. It can be seen that if $y = m\pi$ is a zero of G where m is a positive integer, then $m \leqslant \sqrt{D-B} < N_0$. All parts of the lemma now follow from Theorem 2.3. □

The following remark and Lemma apply to both the the positively damped and the negatively damped cases.

Remark 3.1. In case G has all real zeros, we denote the positive ones as $r_1 < r_2 < r_3 < r_4 \cdots$. By Lemma 3.3 $r_1, r_2, \ldots, r_{N_0+1} \in (0, N_0\pi)$ and for $n \geqslant N_0$, $r_{n+2} \in (n\pi, (n+1)\pi)$. Notice that $G'(0) = -(D+B) > 0$. Since G has all real and simple zeros $\operatorname{sgn}(G'(r_n)) = (-1)^n$ Also $[r_{2j}]_{2\pi} \to \pi/2$ and $[r_{2j+1}]_{2\pi} \to 3\pi/2$. Recall $[x]_c$ denotes the unique real number for which $x - [x]_c$ is a multiple of c. As noted above, the following criterion for asymptotic stability applies to both cases considered of damping in this paper. It involves infinitely many conditions, and we will give efficient stopping criteria that reduce the test to checking only finitely many conditions.

Theorem 3.3. *The zero solution of* (1.3) *is asymptotically stable if and only if*

1. $M < 0$ *and* $B + D < 0$,
2. G *has all real zeros, and*
3. $(-1)^n F(r_n) > 0$, $n = 1, 2, \ldots$

where $r_1 < r_2 < r_3 < \ldots$ *are the positive zeros of* G.

Proof. Necessity of 1 and 2 follows from Theorem 3.2 and Theorems 2.1–2.2. Between consecutive zeros of G, G' must properly change sign, and from Remark 3.1, G' has sign $(-1)^n$ for $n = 1, 2, \dots$ and now 3 follows from Theorems 2.1 and 2.2c. Sufficiency follows in the same fashion. □

With negative damping, $A > 0$, $D > 0$, the following lemma provides an estimate of large positive zeros of G.

Lemma 3.4. *Assume that $A > 0$, $D > 0$, and $B < 0$. If n is a positive integer and*

$$ln > \max\left\{\frac{0.5 + 0.5\sqrt{1 + 4(D - B)}}{\pi}, \frac{A}{\pi} + \frac{4D\sqrt{2}}{\pi^2} - \frac{4B}{\pi^2} - \frac{1}{4}\right\} := \Lambda, \tag{3.7}$$

then the interval $(n\pi, (n + 1)\pi)$ contains exactly one zero r of G and $\pi/2 \le [r]_\pi \le 3\pi/4$.

Proof. Let $y \in (n\pi, (n + 1)\pi)$ be a zero of G. Then $w(y) = \zeta(y)$, and as in Lemma 3.3

$$w'(y) - \zeta'(y) \le \frac{y - y^2 - B + D}{sin^2 y} - A < 0$$

when $y > 0.5 + 0.5\sqrt{0.5 + 4(D - B)}$ and y is not multiple of π. It follows that $w(y) - \zeta(y)$ has exactly one root in $(n\pi, (n + 1)\pi)$. Since $B < 0$ and $D < 0$,

$$\begin{aligned} &w(n\pi + \pi/4) - \zeta(n\pi + \pi/4) = \\ &(n\pi + \pi/4)^2 + B + (-1)^n D\sqrt{2} - A(n\pi + \pi/4) \\ &\ge (n\pi + \pi/4)\{n\pi + \pi/4 + \frac{4B}{\pi} - \frac{4D\sqrt{2}}{\pi} - A\} > 0 \end{aligned} \tag{3.8}$$

by (3.7). Similarly, $w(n\pi + 3\pi/4) - \zeta(n\pi + 3\pi/4) < 0$ when (3.7) holds. The proof is now complete □

Theorem 3.4 (Algorithmic Stability Test I)**.** *Suppose $A > 0$, $D > 0$, $M < 0$, $B + D < 0$. Necessary and sufficient for the zero solution of* (1.3) *to be asymptotically stable is that*

1. *G has $N_0 + 1$ zeros in $(0, N_0\pi)$,*
2. *$F(r_{2\ell}) > 0$ for all $\ell = 1, 2, \dots, n_1$, and*
3. *$F(r_{2\ell+1}) < 0$ for all $\ell = 1, 2, \dots, n_2$*

where N_0 is defined in (3.3) *and n_1 and n_2 are defined in* (3.10–3.12) *and* (3.13) *below.*

Proof. We need only to prove sufficiency. The proof is based on Lemmas 3.3, 3.4, Theorem 3.3 and the observation that $[r_{2j}]_{2\pi} \to \pi/2$ and $[r_{2j+1}]_{2\pi} \to 3\pi/2$. By Remark 3.1, $r_{2n} \in ((2n-2)\pi, (2n-1)\pi)$ and $r_{2n+1} \in ((2n-1)\pi, 2n\pi)$ if $2n-2, 2n-1 \ge N_0$, respectively. From (2.9),

$$F(r_{2\ell}) = r_{2\ell}\left[(r_{2\ell}^2 + B)\sin r_{2\ell} + Ar_{2\ell}\cos r_{2\ell} - \frac{M}{r_{2\ell}}\right]. \tag{3.9}$$

Let n_1 be the first positive integer satisfying

$$2n_1 > \max(N_0, \Lambda) \tag{3.10}$$

$$r_{2n_1}^2 + B > 0 \tag{3.11}$$

$$r_{2n_1} > \frac{A + \sqrt{A^2 - 4B}}{2}. \tag{3.12}$$

By Lemma 3.4, $\sin r_{2n} > \sqrt{2}/2$ and $|\cos r_{2n}| < \sqrt{2}/2$ for all $n \geqslant n_1$, and we have from (3.9) and from $M < 0$ and $A > 0$ that

$$F(r_{2n})/r_{2n} = (r_{2n}^2 + B)\sin r_{2n} + Ar_{2n}\cos r_{2n} - M/r_{2n} >$$
$$\frac{\sqrt{2}}{2}(r_{2n}^2 - Ar_{2n} + B) > 0. \tag{3.13}$$

Thus for all $n > n_1$, $F(r_{2n}) > 0$. Now we choose n_2 to be the first positive integer for which

$$2n_2 + 1 > \max(N_0, \Lambda)$$
$$r_{2n_2+1} > 1,$$
$$r_{2n_2+1}^2 + B > 0.$$
$$r_{2n_2+1} > \frac{A + \sqrt{A^2 + 4(B - M\sqrt{2})}}{2} \text{ if } A^2 + 4(B - M\sqrt{2}) \geq 0.$$

By Lemma 3.4, $\sin r_{2n+1} < -\sqrt{2}/2$ and $|\cos r_{2n}| < \sqrt{2}/2$ for all $n \geqslant n_2$, and by (3.9) for $n > n_2$

(3.14)

$$F(r_{2n+1})/r_{2n+1} = (r_{2n+1}^2 + B)\sin r_{2n+1} + Ar_{2n+1}\cos r_{2n+1} - M/r_{2n+1} <$$
$$\frac{-\sqrt{2}}{2}(r_{2n+1}^2 + B) + \frac{\sqrt{2}}{2}Ar_{2n+1} - M = \frac{\sqrt{2}}{2}(-r_{2n+1}^2 + Ar_{2n+1} + B - M\sqrt{2}) < 0.$$

By (3.14), $F(r_{2n+1}) < 0$ for all $n \geqslant n_2$. By Theorem 3.3 the zero solution of (1.3) is asymptotically stable. □

Now consider equation (1.3) with general positive damping, i.e. $A < 0$ and $D < 0$.

Lemma 3.5. *Assume that $A < 0$, $D < 0$, $M < 0$ and $B < 0$. Let*

$$M_0 = \left[\!\!\left[\frac{0.5 + 0.5\sqrt{1 - 4(D + B + A)}}{\pi} \right]\!\!\right] + 1 \tag{3.15}$$

Then G has all real zeros if and only if G has $M_0 + 1$ zeros in $(0, M_0\pi)$. Furthermore, in this case, G has precisely one zero in $(n\pi, (n+1)\pi)$ for all $n \geq M_0$.

Proof. The proof of Lemma 3.5 is similar to the proof of Lemma 3.3 and we omit it (see Figure 2 for an illustration of $w(y) = \zeta(y)$ when G has all real zeros). The only difference in the proof is that we show $w'(y) < \zeta'(y)$ rather than $w'(y) < 0 < \zeta'(y)$ when $y \in (n\pi, (n+1)\pi)$, $n > M_0$. □

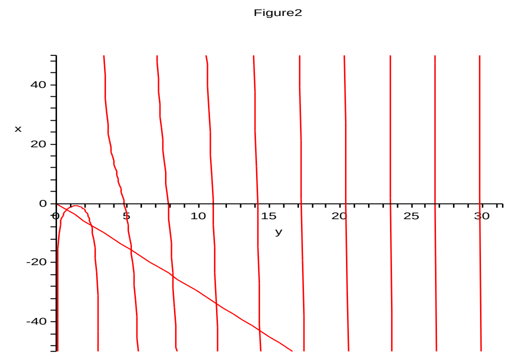

Figure 2. G has all real zeros with negative damping

Lemma 3.6. *Assume $A < 0$, $D < 0$ and $B < 0$. If n is a positive integer and*

$$n > \max\left\{\frac{0.5 + 0.5\sqrt{1 - 4(D+B+A)}}{\pi}, \frac{A}{\pi} - \frac{4D\sqrt{2}}{\pi^2} - \frac{4B}{\pi^2} - \frac{1}{4}\right\} := \Lambda_1, \tag{3.16}$$

then the interval $(n\pi, (n+1)\pi)$ contains exactly one zero r of G and $\pi/2 \le [r]_\pi \le 3\pi/4$.

Proof. The proof of Lemma 3.6 is similar to the proof of Lemma 3.4 and we omit it (see Figure 3 to illustrate that the large zeros of G satisfy $\pi/2 \le [r]_\pi \le 3\pi/4$). □

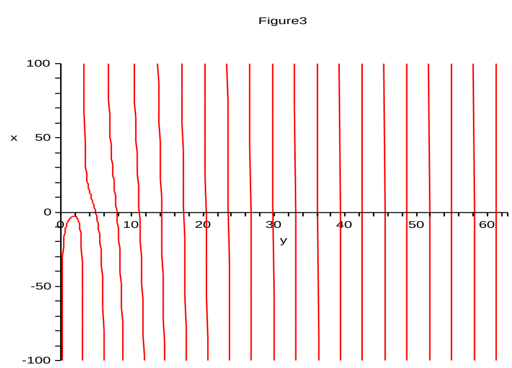

Figure 3. Large zeros of G in $\pi/2 \le [r]_\pi \le 3\pi/4$

Theorem 3.5 (Algorithmic Stability Test II)**.** *Suppose $A < 0$, $D < 0$, $M < 0$, $B < 0$. The zero solution of* (1.3) *is asymptotically stable if and only if*

1. *G has $M_0 + 1$ zeros in* $(0, M_0\pi)$

2. *$F(r_{2\ell}) > 0$ for all $\ell = 1, 2, \ldots, m_1$ where m_1 is the first positive integer satisfying $2m_1 > \max(M_0, \Lambda_1)$, $r_{2m_1}^2 + B > 0$, and $r_{2m_1} > (-A + \sqrt{A^2 - 4B})/2$*
3. *If $F(r_{2\ell+1}) < 0$ for all $\ell = 1, 2, \ldots, m_2$ where m_2 the first positive integer satisfying $2m_2 + 1 > \max(M_0, \Lambda_1)$, $r_{2m_2+1} > 1$, $r_{2m_2+1}^2 + B > 0$, and $r_{2m_2+1} > \big(-A + \sqrt{A^2 - 4(B + M\sqrt{2})}\big)/2$*

where M_0 is defined in (3.15).

Proof. The proof of Theorem 3.5 is similar to the proof of Theorem 3.4 and we omit it. □

4. **EXAMPLES**

Example 4.1. For Stépán's system (1.1), let $\kappa = -1/(200\sqrt{5}) < 0$, $\alpha = 2\sqrt{(5)}$, $K = 1/5$, and $\tau = 1$. With $\kappa < 0$, this example includes negative damping. Equation (1.2) becomes

$$y'''(t) = p_1 y''(t) + q_2 y'(t - \tau) + q_1 y'(t) + r_2 y(t - \tau) \tag{4.1}$$

where

$$A = \tau p_1 = -2\tau\kappa\alpha = 0.02, \; D = \tau^2 q_2 = -2\tau^2 K\kappa\alpha = 0.004,$$
$$B = \tau^2 q_1 = -\tau^2\alpha^2 = -20, \; M = \tau^3 r_2 = -\tau^3\alpha^2 K = -4. \tag{4.2}$$

We apply Algorithmic Stability Test I. Here $N_0 = 2$, and in $[0, 2\pi]$, $w(y) = \zeta(y)$ has $N_0 + 1 = 3$ roots, $r_1 = 1.572362263$, $r_2 = 1.656190448$, $r_3 = 4.653940883$, and thus G has all real zeros. It is easy to see that $\Lambda = 7.864353522$ and $n_2 = n_1 = 4$. We found that $F(r_{2j}) > 0$ for $j = 1, 2, \ldots, n_1$ and $F(r_{2j+1}) < 0$, for $j = 1, 2, \ldots, n_2$. The values are $F(r_1) = -23.55990138$, $F(r_2) = 1.656190448$, $F(r_3) = -3.733778032$. Also we found

$$r_4 = 7.850306032, \quad F(r_4) = 330.7894049,$$
$$r_5 = 10.99335457, \quad F(r_5) = -1104.724709,$$
$$r_6 = 14.13561695, \quad F(r_6) = 2545.808525,$$
$$r_7 = 17.2775045, \quad F(r_7) = -4807.998857,$$
$$r_8 = 20.4193335, \quad F(r_8) = 8109.442002,$$
$$r_9 = 23.5610568, \quad F(r_9) = -12604.07772.$$

Conditions 1-3 are satisfied, and therefore the zero solution of (4.1) is asymptotically stable. Thus it is possible for delay to stabilize (1.1) in the presence of negative damping.

Example 4.2. Consider (1.3)

$$y'''(t) = p_1 y''(t) + q_2 y'(t-\tau) + q_1 y'(t) + r_2 y(t-\tau) \tag{4.3}$$

where

$$A = \tau p_1 = \frac{2}{10},\ D = \tau p_2 = 1,$$
$$B = \tau^2 q_1 = -3.6,\ M = r_2\tau^3 = -1. \tag{4.4}$$

In this example we apply Algorithmic Stability Test I. Here $N_0 = 2$ and in $[0, 2\pi]$, $G(y)$ has $N_0 + 1 = 3$ zeros, $r_1 = 1.202449067$, $r_2 = 2.17490924$, and $r_3 = 4.603295507$. Thus G has all real zeros. Here $\Lambda = 1.845846189$ and $n_1 = 2$, since $r_4 = 7.844163623$, $r_4^2 + B = 57.93090294 > 0$, $r_4 > (A + \sqrt{A^2 - 4B})/2 = 2$ and $F(r_2) = 2.17490246 > 0$, $F(r_4) = 1152.593969 > 0$. For odd zeros $n_2 = 1$ since $r_3 > 1$, $r_3^2 + B > 0$ and $A^2 + 4(B - M\sqrt{2}) = -8.703145752$. For Condition 3 of Algorithmic Stability Test I, we have $F(r_1) = -0.974102994$, $F(r_3) = -28.74053025$, $F(r_5) = -841.8844450$, and therefore the zero solution of (4.1) is asymptotically stable with negative damping. Without the delay the zero solution is not asymptotically stable. [We also examined equation (4.3) with $M = -4$, and we found that interlacing fails and the zero solution is not asymptotically stable.] Although delay generally has an unstabilizing effect, this is another case when the delay stabilize the zero solution.

Example 4.3. Consider (1.3)

$$y'''(t) = p_1 y''(t) + q_2 y'(t-\tau) + q_1 y'(t) + r_2 y(t-\tau) \tag{4.5}$$

where

$$A = \tau p_1 = 4,\ D = \tau p_2 = 1,$$
$$B = \tau^2 q_1 = -100,\ M = r_2\tau^3 = -100. \tag{4.6}$$

In this example $N_0 = 4$, and in $[0, 4\pi]$, $w(y) = \zeta(y)$ has three roots, $r_1 = 1.627310695$, $r_2 = 4.984575786$, $r_3 = 8.852083494$, in $[0, 4\pi]$, the next root is $r_4 = 13.57477560$, which is not in $[0, 4\pi]$. Thus condition (2) of Theorem 3.3 is not valid, G has nonreal zeros. The zero solution of (4.5) is not asymptotically stable. Figure 4 shows the functions $w(y$ and $\zeta(y)$ in this case. It reveals how two real zeros were lost.

Example 4.4. Consider (1.3)

$$y'''(t) = p_1 y''(t) + q_2 y'(t-\tau) + q_1 y'(t) + r_2 y(t-\tau) \tag{4.7}$$

where

$$A = \tau p_1 = -8,\ D = \tau p_2 = -10,$$
$$B = \tau^2 q_1 = -45,\ M = r_2\tau^3 = -6. \tag{4.8}$$

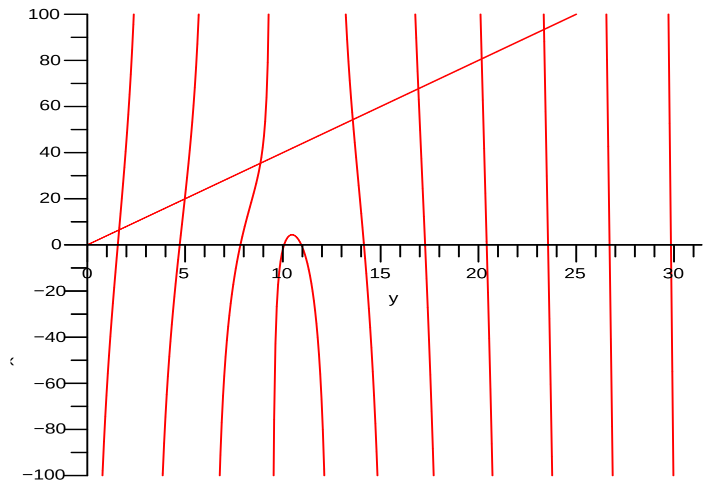

Figure 4. G has non-real zeros

In this example we apply Algorithmic Stability Test II. We found that $r_1 = 1.520451954$ and $r_2 = 3.716633416$. Also $r_3 = 6.522779220$, $r_4 = 8.855923258$, $r_5 = 11.85193114$. In this example $M_0 = 3$, and in $[0, 3\pi]$, $w(y) = \zeta(y)$ has $N_0 + 1 = 4$ roots, thus G has all real zeros. Here $\Lambda_1 = 21.172925664$ and $2n_1 = 2n_0 = 22$, also $r_{22} = 64.52234321 > 1$, and $r_{22}^2 + B = 4118.132773 > 0$. For odd zeros, $n_2 = 11$ since $2n_2 + 1 > \Lambda_1 = 21.17292564$ and $r_{23} > (-A + \sqrt{A^2 - 4(B + M\sqrt{2})})/2$. Conditions (1-3) are satisfied and thus the zero solution is asymptotically stable by Algorithmic Stability Test II.

In Table I we present the values of r_j, $F(r_j)$, and $\sin r_j$, $j = 1, \ldots, 25$. By Algorithmic Stability Test II the zero solution of (4.7) is asymptotically stable.

Table I

$r_1 = 1.520451954$	$F(r_1) = -59.75384590$	$\sin(r_1) = 0.9987329897$
$r_2 = 3.716633416$	$F(r_2) = 161.7735712$	$\sin(r_2) = -0.5438689977$
$r_3 = 6.522779220$	$F(r_3) = -328.4477971$	$\sin(r_3) = 0.2373081589$
$r_4 = 8.855923258$	$F(r_4) = 694.0739859$	$\sin(r_4) = -0.5386674682$
$r_5 = 11.85193114$	$F(r_5) = -1584.287170$	$\sin(r_5) = -0.6551940646$
$r_6 = 14.52768533$	$F(r_6) = 2879.482436$	$\sin(r_6) = 0.9247118502$
$r_7 = 17.68140408$	$F(r_7) = -5327.719645$	$\sin(r_7) = -0.9200279635$
$r_8 = 20.73647753$	$F(r_8) = 8663.418259$	$\sin(r_8) = 0.9504471462$
$r_9 = 23.87855121$	$F(r_9) = -13331.54648$	$\sin(r_9) = -0.9502974918$
$r_{10} = 26.96351754$	$F(r_{10}) = 19273.10224$	$\sin(r_{10}) = 0.9663951235$
$r_{11} = 30.10329059$	$F(r_{11}) = -26910.92607$	$\sin(r_{11}) = -0.9668612729$
$r_{12} = 33.20570284$	$F(r_{12}) = 36202.52591$	$\sin(r_{12}) = 0.9761195408$
$r_{13} = 36.34520605$	$F(r_{13}) = -47557.14584$	$\sin(r_{13}) = -0.9765713080$
$r_{14} = 39.45834120$	$F(r_{14}) = 60942.62430$	$\sin(r_{14}) = 0.9822989656$
$r_{15} = 42.59803668$	$F(r_{15}) = -76759.72312$	$\sin(r_{15}) = -0.9826525760$

Table I: continued

$r_{16} = 45.71812796$	$F(r_{16}) = 94982.94616$	$\sin(r_{16}) = 0.9864126909$
$r_{17} = 48.85809309$	$F(r_{17}) = -1.160075213 \times 10^5$	$\sin(r_{17}) = -0.9866787642$
$r_{18} = 51.98291500$	$F(r_{18}) = 1.!39812432 \times 10^5$	$\sin(r_{18}) = 0.9892681606$
$r_{19} = 55.12313029$	$F(r_{19}) = -1.667891332 \times 10^5$	$\sin(r_{19}) = -0.9894684705$
$r_{20} = 58.25129952$	$F(r_{20}) = 1.969197374 \times 10^5$	$\sin(r_{20}) = 0.9913222895$
$r_{21} = 61.39172565$	$F(r_{21}) = -2.30590257 \times 10^5$	$\sin(r_{21}) = -0.9914749590$
$r_{22} = 64.52234321$	$F(r_{22}) = 2.677933727 \times 10^5$	$\sin(r_{22}) = 0.9928452438$
$r_{23} = 67.66294201$	$F(r_{23}) = -3.089076017 \times 10^5$	$\sin(r_{23}) = -0.9929634278$

Notice that the values of $sin(r_j)$ which reveals that $[r_{2j}]_{2\pi} \to \pi/2$ and $[r_{2j+1}]_{2\pi} \to 3\pi/2$.

In this example for $A > -0.8578732450$ G has nonreal zeros and the zero solution of (4.7) is not asymptotically stable. Figure 5 shows the functions $w(y)$ and $\zeta(y)$ for $A = -8$. When $A > -0.8578732450$, the two zeros in the "third branch" of $w(y)$ are lost.

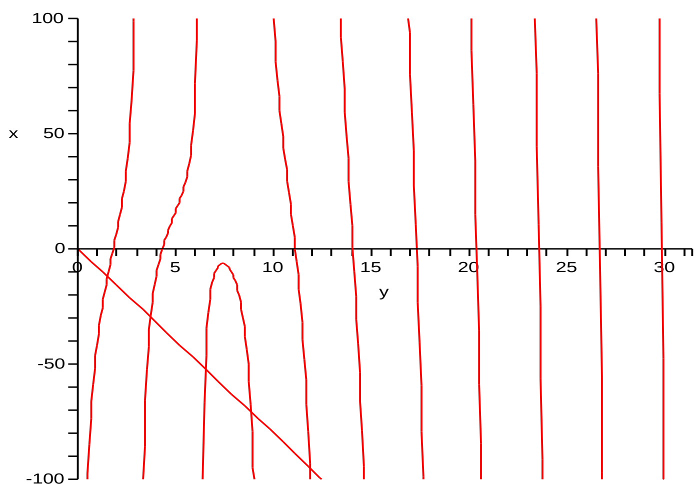

Figure 5. G has all real zeros

Example 4.5. Consider (1.3)

$$y'''(t) = p_1 y''(t) + q_2 y'(t-\tau) + q_1 y'(t) + r_2 y(t-\tau) \tag{4.9}$$

where

$$\begin{aligned} &A = \tau p_1 = -8, \ D = \tau p_2 = -8, \\ &B = \tau^2 q_1 = -4\pi^2, \ M = r_2 \tau^3 = -6. \end{aligned} \tag{4.10}$$

In this example we apply Theorem 3.3. We found that $r_1 = 1.561186405$ and $r_2 = 2.960693918$, $r_3 = 4.727703217$. Also $r_4 = 7.8361690369$,$r_5 = 10.07808985$, and $r_5 = 10.92099762$, and $N_0 = 2$ and G has three real zeros, in $(0, 2\pi]$ thus G has all real zeros but $F(r_1) = 83.99999206 > 0$, and by Theorem 3.3 the zero solution is not asymptotically stable.

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