Dynamic Systems and Applications 16 (2007) 345-360

OSCILLATION OF SECOND-ORDER DELAY AND NEUTRAL DELAY DYNAMIC EQUATIONS ON TIME SCALES

S. H. SAKER

Department of Mathematics and Statistics, University of Calgary, 2500 Calgary Drive, Calgary Alberta T2N 1N4, Canada (saker@math.ucalgary.ca)

ABSTRACT. In this paper, we consider the second-order linear delay dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x(\tau(t)) = 0,$$

on a time scale $\mathbb{T}$. We will study the properties of the solutions and establish some sufficient conditions for oscillations. In the special case when $\mathbb{T} = \mathbb{R}$ and $\tau(t) = t$, our results include some well-known results in the literature for differential equations. When, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, for $h > 0$ and $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}_0\}$ where $t_n\}$ is the set of the harmonic numbers defined by $t_0 = 0$, $t_n = \sum_{k=1}^{n} \frac{1}{k}$ for $n \in \mathbb{N}_0$ our results are essentially new. The results will be applied on second-order neutral delay dynamic equations in time scales to obtain some sufficient conditions for oscillations. An example is considered to illustrate the main results.

Keywords. Oscillation, delay dynamic equations, neutral delay dynamic equations, time scales.

AMS (MOS) Subject Classification. 34K11, 39A10, 39A99 (34A99, 34C10, 39A11).

1. INTRODUCTION

In recent years, the study of dynamic equations on time scales has become and an area of mathematics and has received a lot of attention. It was created to unify the study of differential and difference equations, and it also extends these classical cases to cases “in between,” e.g., to the so-called q-difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts.

The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of the reals. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus. The books by Bohner and Peterson [4, 5] summarize and organize much of time scale calculus. Dynamic equations on a time scale have an enormous potential for applications such

Received August 22, 2006

On leave from Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt. The author was partially supported by AIF grant.

as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population.

For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\},$$

where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left–dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, is right–dense if $\sigma(t) = t$, is left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon |\sigma(t) - s|, \quad \text{for all } s \in U.$$

In this case, we say $x^{\Delta}(t)$ is the (delta) derivative of x at t and that x is (delta) differentiable at t. Assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.

(i) If g is differentiable at t, then g is continuous at t.

(ii) If g is continuous at t and t is right-scattered, then g is differentiable at t with

$$g^{\Delta}(t) := \frac{g(\sigma(t)) - g(t)}{\mu(t)}.$$

(iii) If g is differentiable and t is right-dense, then

$$g^{\Delta}(t) := \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

(iv) If g is differentiable at t, then $g(\sigma(t)) := g(t) + \mu(t) g^{\Delta}(t)$. In this paper we will refer to the (delta) integral which we can define as follows: If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by

$$\int_a^t g(s)\Delta s := G(t) - G(a).$$

It can be shown (see [4]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. For a more general definition of the delta integral see [4, 5].

In recent years there has been much research activity concerning the qualitative theory of dynamic equations on time scales. One of the main subjects of the qualitative analysis of the dynamic equations is the oscillatory behavior. Recently, some interesting results have been established for oscillation and nonoscillation of dynamic equations on time scales, we refer to the papers [3, 6–12, 16, 17, 21] and the references cited therein. To the best our knowledge the papers that are concerned with the oscillation of delay dynamic equations in time scales are [1, 2, 11, 18, 19, 20, 22, 23].

In this paper, we are concerned with oscillation of the second-order linear delay dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x(\tau(t)) = 0, \tag{1.1}$$

on a time scale $\mathbb{T}$, where the function q is an rd-continuous function such that $q(t) > 0$ for $t \in \mathbb{T}$, $\tau : \mathbb{T} \to \mathbb{T}$, $\tau(t) \le t$ and $\lim_{t\to\infty} \tau(t) = \infty$. Throughout this paper these assumptions will be assumed. Let $T_0 = \min\{\tau(t) : t \ge 0\}$ and $\tau_{-1}(t) = \sup\{s \ge 0 : \tau(s) \le t\}$ for $t \ge T_0$. Clearly $\tau_{-1}(t) \ge t$ for $t \ge T_0$, $\tau_{-1}(t)$ is nondecreasing and coincides with the inverse of τ when the latter exists. By a solution of (1.1) we mean a nontrivial real-valued functions $x(t) \in C_r^2[T_x, \infty)$, $T_x \ge \tau_{-1}(t_0)$ where C_r is the space of rd-continuous functions.

The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_\mathbb{T}$ by $[t_0, \infty)_\mathbb{T} := [t_0, \infty) \cap \mathbb{T}$.

We note that, Equation (1.1) in its general form covers several different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. When $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$, $\mu(t) = 0$,

$$f^\Delta = f', \text{ and } \int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

and (1.1) becomes the second-order delay differential equation

$$x''(t) + q(t)x(\tau(t)) = 0, \tag{1.2}$$

When $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t + 1$, $\mu(t) = 1$,

$$f^\Delta = \Delta f, \text{ and } \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t),$$

and (1.1) becomes the general second-order delay difference equation

$$x(t + 2) - 2x(t + 1) + x(t) + q(t)x(\tau(t)) = 0. \tag{1.3}$$

When $\mathbb{T} = h\mathbb{Z}$, $h > 0$, $\sigma(t) = t + h$, $\mu(t) = h$,

$$f^{\Delta} = \Delta_h f = \frac{f(t+h) - f(t)}{h}, \text{ and } \int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} hf(a+kh)$$

and (1.1) becomes the second-order delay difference equation

$$x(t+2h) - 2x(t+h) + x(t) + h^2 q(t)x(\tau(t)) = 0. \tag{1.4}$$

When $\mathbb{T} = \{t_n : n \in \mathbb{N}_0\}$, where $\{t_n\}$ is the set of harmonic numbers defined by

$$t_0 = 0, \quad t_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N},$$

we have $\sigma(t_n) = t_{n+1}$, $\mu(t_n) = \frac{1}{n+1} < 1$, $x^{\Delta}(t_n) = (n+1)\Delta x(t_n)$ and (1.1) becomes the difference equation

$$x(t_{n+2}) - \left[\frac{n+3}{n+2}\right] x(t_{n+1}) + (n+1)\left(\frac{1}{n+2} + (n+2)q(t_n)\right) x(\tau(t_n)) = 0. \tag{1.5}$$

For oscillation of second-order differential equations, Hille [14] considered the linear equation

$$x''(t) + q(t)x(t) = 0, \tag{1.6}$$

and proved that: If

$$q_* := \liminf_{t\to\infty} t \int_t^{\infty} q(s)ds > \frac{1}{4}, \tag{1.7}$$

or

$$q^* := \limsup_{t\to\infty} t \int_t^{\infty} q(s)ds > 1, \tag{1.8}$$

then every solution of equation (1.6) oscillates. Nehari [15] considered also (1.6) and proved that: If

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} s^2 q(s)ds > \frac{1}{4}, \tag{1.9}$$

then every solution oscillates. For oscillation of dynamic equations on time scales, Erbe, Peterson and Saker [8] established some new oscillation criteria for nonlinear dynamic equations. As a linear version of their results one can easily see that if

$$\liminf_{t\to\infty} t \int_t^{\infty} q(s)\Delta s > \frac{1}{4}, \tag{1.10}$$

then every solution of the dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x^{\sigma} = 0, \tag{1.11}$$

is oscillatory. We note that the condition (1.10) is a time scale analogue of the Hille condition (1.7).

Lomtatidze [13], considered (1.6) when (1.7) and (1.8) are not satisfied, and proved that: If

$$q_* \leq \frac{1}{4}, \text{ and } \lim_{t\to\infty}\sup \frac{1}{t}\int_{t_0}^{t} s^2 q(s)ds > \frac{1}{2}\left(1+\sqrt{1-4q_*}\right), \tag{1.12}$$

then every solution of (1.6) oscillates.

The natural question now is: Can the oscillation condition (1.12) *of Lomtatidze be extended for* (1.1) *on time scales and in the special case when* $\mathbb{T}=\mathbb{R}$ *and* $\tau(t)=t$ *include the condition* (1.12)*, i.e., can we find new oscillation criteria for* (1.1) *when*

$$\lim_{t\to\infty}\inf t\int_{\sigma(t)}^{\infty} \frac{\tau(s)}{s}q(s)\Delta s \leq \frac{1}{4l} \qquad \text{and}$$

$$\lim_{t\to\infty}\inf \frac{1}{t}\int_{t_0}^{t} s^2\left(\frac{\tau(s)}{s}q(s)\right)\Delta s \leq \frac{1}{4}, \tag{1.13}$$

where $l := \lim_{t\to\infty}\frac{t}{\sigma(t)}$.

The purpose of this paper is to give an affirmative answer to this question. The paper, is organized as follows: In Section 2, by analyzing the Riccati dynamic inequality, we establish some properties of the solutions of (1.1) and also establish some new sufficient conditions for oscillation of (1.1). In the special case when $\mathbb{T}=\mathbb{R}$ and $\tau(t)=t$ our results include the oscillation condition (1.12) established by Lomtatidze [13] for second-order differential equation. In the case, when $\mathbb{T}=\mathbb{Z}$, $\mathbb{T}=h\mathbb{Z}$, $h>0$, $\mathbb{T}=\mathbb{T}_n$ our results are essentially new. An example is considered to illustrate the main results. In Section 3, we consider the second-order neutral delay dynamic equation

$$[y(t)+r(t)y(\tau(t))]^{\Delta\Delta}+q(t)y(\delta(t))=0, \tag{1.14}$$

on a time scale $\mathbb{T}$ and extend the results in Section 2 and establish some new sufficient conditions for oscillations. The technique in this paper is different from the techniques considered in [1, 2, 11, 18, 19, 20, 22, 23].

2. MAIN RESULTS

In this section, we study the properties of the solutions of (1.1) and establish some new sufficient conditions for oscillations. In what follows, we will assume that the graininess function $\mu(t)$ satisfies $\max_{t\in\mathbb{T}}\mu(t)=h_0\geq 0$, and

$$\int_{t_0}^{\infty} \tau(s)q(s)\Delta s=\infty. \tag{2.1}$$

In the next results, for simplicity, we will use the notations

$$\begin{array}{ll} P(t) := t\int_{\sigma(t)}^{\infty} \frac{\tau(s)}{s} q(s)\Delta s, & Q(t) := \frac{1}{t}\int_{t_0}^{t} s^2 \left(\frac{\tau(s)}{s} q(s)\right)\Delta s, \\ P_* := \liminf_{t\to\infty} P(t), & Q_* := \liminf_{t\to\infty} Q(t), \\ P^* := \limsup_{t\to\infty} P(t), & Q^* := \limsup_{t\to\infty} Q(t), \end{array}$$

We will need the following lemma in the proof of our main results.

Lemma 2.1 ([11])**.** *Let x be a positive solution of* (1.1) *on* $[t_0, \infty)$ *and* $T = \tau_{-1}(t_0)$. *Then*

(i) $x^{\Delta}(t) \geq 0$, $x(t) \geq tx^{\Delta}(t)$*for* $t \geq T$,
(ii) *x is nondecreasing, while $x(t)/t$ is nonincreasing on* $[T, \infty)$.

Before, we proceed to the formulation of the oscillation results, we establish some properties of the solution of (1.1).

Lemma 2.2. *Let $x(t)$ be a nonoscillatory solution of* (1.1) *such that* $x(\tau(t)) > 0$ *for* $t \geq t_1 \geq \tau^{-1}(t_0)$. *Assume that* $0 \leq P_* \leq 1/4l$. *Define* $w(t) := \frac{x^{\Delta}(t)}{x(t)}$, *then*

$$\lim_{t\to\infty} \inf t^{\lambda} w(t) = 0, \text{ for } \lambda < 1, \tag{2.2}$$

$$\lim_{t\to\infty} \inf t w^{\sigma}(t) \geq \frac{1}{2}(1 - \sqrt{1 - 4P_* l}). \tag{2.3}$$

Proof. From the definition of $w(t)$ and in in view of Lemma 2.1, since $x(t)$ be a nonoscillatory solution of (1.1) such that $x(\tau(t)) > 0$ for $t \geq t_1 \geq \tau^{-1}(t_0)$, we have $w(t) > 0$, and satisfies

$$\begin{aligned} w^{\Delta}(t) &= \left(x^{\Delta}\right)^{\sigma}\left[\frac{1}{x(t)}\right]^{\Delta} + \frac{1}{x(t)} x^{\Delta\Delta}(t) = \left(x^{\Delta}\right)^{\sigma}(t)\frac{-x^{\Delta}(t)}{x(t)x^{\sigma}(t)} + \frac{1}{x(t)} x^{\Delta\Delta}(t) \\ &= -\frac{\left(x^{\Delta}\right)^{\sigma}(t)}{x^{\sigma}(t)}\frac{x^{\Delta}(t)}{x(t)} + \frac{1}{x(t)} x^{\Delta\Delta}(t), \text{ for } t \geq t_1. \end{aligned}$$

In view of (1.1), we get

$$w^{\Delta}(t) + q(t)\frac{x(\tau(t))}{x(t)} + w(t)w^{\sigma}(t) = 0, \quad \text{for } t \geq t_1.$$

From Lemma 2.1, we have $\frac{x(\tau(t))}{x(t)} \geq \frac{\tau(t)}{t}$. This implies that

$$w^{\Delta}(t) + p(t) + w(t)w^{\sigma}(t) \leq 0, \quad \text{for } t \geq t_1. \tag{2.4}$$

where $p(s) := \frac{\tau(s)}{s} q(s) > 0$. This implies that

$$\frac{w^{\Delta}(t)}{w(t)w^{\sigma}(t)} < -1, \qquad \text{for } t \geq t_1.$$

So that

$$\left(\frac{-1}{w(t)}\right)^{\Delta} < -1.$$

Integrating the last inequality from t_1 to t, we have

$$(t-t_1)w(t)<1, \quad \text{for} \;\; t\geq t_1, \tag{2.5}$$

which implies that

$$\lim_{t\to\infty} w(t)=0, \;\; \lim_{t\to\infty} t^{\lambda}w(t)=0, \text{ for } \lambda<1 \text{ and } \lim_{t\to\infty}\frac{1}{t}\int_{t_1}^{t} w(s)\Delta s=0, \tag{2.6}$$

and this proves (2.2). Now, we prove (2.3). If $P_*=0$, then (2.3) is trivial. So, we may assume that $P_*>0$. Integrating (2.4) from $\sigma(t)$ to ∞ ($\sigma(t)\geq t_1$) and using (2.6), we have

$$w^{\sigma}(s)\geq\int_{\sigma(t)}^{\infty} p(s)\Delta s+\int_{\sigma(t)}^{\infty} w(s)w^{\sigma}(s)\Delta s \;\text{ for } t\geq t_1. \tag{2.7}$$

Set $r:=\lim_{t\to\infty}\inf tw^{\sigma}(t)$. By using (2.5), we see that

$$0<r\leq 1, \text{ and } r-r^2>0. \tag{2.8}$$

Then it follows that for any $\epsilon\in(0,r)$ there exists $t_2\geq t_1$, such that

$$r-\epsilon<tw^{\sigma}(t),$$

and from the definition of P_*, we have

$$t\int_{\sigma(t)}^{\infty} p(s)\Delta s\geq P_*-\epsilon, \text{ for } t\geq t_2.$$

From (2.7), we have

$$\begin{aligned} w^{\sigma} &\geq \int_{\sigma(t)}^{\infty} p(s)\Delta s+\int_{\sigma(t)}^{\infty} w(s)w^{\sigma}(s)\Delta s\geq\int_{\sigma(t)}^{\infty} p(s)\Delta s+\int_{\sigma(t)}^{\infty}\frac{sw^{\sigma}(s)sw^{\sigma}(s)}{s^2}\Delta s \\ &\geq \int_{\sigma(t)}^{\infty} p(s)\Delta s+\int_{\sigma(t)}^{\infty}(r-\epsilon)^2\frac{1}{s\sigma(s)}\Delta s \\ &\geq \int_{\sigma(t)}^{\infty} p(s)\Delta s+(r-\epsilon)^2\int_{\sigma(t)}^{\infty}\left(\frac{-1}{s}\right)^{\Delta}\Delta s \\ &= \int_{\sigma(t)}^{\infty} p(s)\Delta s+\frac{1}{\sigma(t)}(r-\epsilon)^2. \end{aligned}$$

This implies that,

$$tw^{\sigma}(t)\geq t\int_{\sigma(t)}^{\infty} p(s)\Delta s+\frac{t}{\sigma(t)}(r-\epsilon)^2.$$

Then

$$r\geq P_*-\epsilon+l(r-\epsilon)^2 \;\text{ for } t\geq t_2. \tag{2.9}$$

Since ϵ is an arbitrary, we have

$$P_*\leq r-lr^2, \tag{2.10}$$

which implies that

$$lr^2-r+P_*\leq 0. \tag{2.11}$$

Then, from (2.11) since $P_* \le 1/4l$, we see that (2.3) holds. The proof is complete. $\square$

Lemma 2.3. *Let* $x(t)$ *be a nonoscillatory solution of* (1.1) *such that* $x(\tau(t)) > 0$ *for* $t \ge t_1 \ge \tau^{-1}(t_0)$. *Assume that* $0 \le Q_* \le 1/4$. *Then*

$$\limsup_{t\to\infty} tw^{\sigma}(t) \le \frac{1}{2}(1+\sqrt{1-4Q_*}). \tag{2.12}$$

Proof. We proceed as in the proof of Lemma 2.2 to get (2.4). From (2.4) we see that $w^{\Delta}(t) \le 0$. This implies that $w(t) \ge w^{\sigma}(t)$, and hence (2.4) becomes

$$w^{\Delta}(t) + p(t) + (w^{\sigma}(t))^2 \le 0, \quad \text{for} \ \ t \ge t_1. \tag{2.13}$$

Multiplying (2.13) by t^2, and integrating from t_1 to t $(t \ge t_1)$ and using the integration by parts, we obtain

$$\begin{aligned}
\int_{t_1}^{t} s^2 p(s)\Delta s &\le -\int_{t_1}^{t} s^2 w^{\Delta}(s)\Delta s - \int_{t_1}^{t} s^2 \left(w^{\sigma}(s)\right)^2 \Delta s \\
&= \left[-t^2 w(t)\right]_{t_1}^{t} + \int_{t_1}^{t} (s^2)^{\Delta} w^{\sigma}(s)\Delta s - \int_{t_1}^{t} s^2 \left(w^{\sigma}(s)\right)^2 \Delta s \\
&= -t^2 w(t) + t_1^2 w(t_1) + \int_{t_1}^{t} (s+\sigma(s)) w^{\sigma}(s)\Delta s - \int_{t_1}^{t} s^2 \left(w^{\sigma}(s)\right)^2 \Delta s \\
&= -t^2 w(t) + t_1^2 w(t_1) + \int_{t_1}^{t} 2s w^{\sigma}(s)\Delta s - \int_{t_1}^{t} s^2 \left(w^{\sigma}(s)\right)^2 \Delta s \\
&\quad + \int_{t_1}^{t} \mu(s) w^{\sigma}(s)\Delta s.
\end{aligned}$$

It follows that

$$\begin{aligned}
tw(t) &\le \frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}(s)\Delta s - \frac{1}{t}\int_{t_1}^{t} s^2 p(s)\Delta s \\
&\quad + \frac{1}{t}\int_{t_1}^{t} \left[2s w^{\sigma}(s) - s^2 \left(w^{\sigma}(s)\right)^2\right] \Delta s.
\end{aligned}$$

Then, we have

$$\begin{aligned}
tw^{\sigma}(t) \le {} & \frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}(s)\Delta s - \frac{1}{t}\int_{t_1}^{t} s^2 p(s)\Delta s \\
& + \frac{1}{t}\int_{t_1}^{t} \left[2s w^{\sigma}(s) - s^2 \left(w^{\sigma}(s)\right)^2\right] \Delta s.
\end{aligned} \tag{2.14}$$

From (2.6), since $\lim_{t\to\infty} \frac{1}{t}\int_{t_1}^{t} w(s)\Delta s = 0$, and $w(t) \ge w^{\sigma}(t)$, we have

$$\lim_{t\to\infty} \left[\frac{t_1^2 w(t_1)}{t} + \frac{h_0 \int_{t_1}^{t} w^{\sigma}(s)\Delta s}{t}\right] = 0.$$

Also, using the inequality $a^2 + b^2 \ge 2ab$, we have

$$\left[2\sigma(s) w^{\sigma}(s) - \sigma^2(s) \left(w^{\sigma}(s)\right)^2\right] \le 1,$$

and this implies that

$$\frac{1}{t}\int_{t_1}^{t}\left[2sw^{\sigma}(s)-s^2\left(w^{\sigma}(s)\right)^2\right]\Delta s\leq 1. \tag{2.15}$$

Set

$$R:=\lim_{t\to\infty}\sup tw^{\sigma}(t). \tag{2.16}$$

Then from (2.14), we have

$$R\leq 1-Q_*.$$

The estimation in (2.12) is valid for $Q_*=0$. We may assume that $Q_*>0$. For an arbitrary $\epsilon\in(R,\,1-Q_*)$, there exists $t_2\geq t_1$ such that

$$R-\epsilon<tw^{\sigma}(t)<R+\epsilon. \tag{2.17}$$

From the definition of Q_*, we see that

$$\frac{1}{t}\int_{t_0}^{t}s^2p(s)\Delta s>Q_*-\epsilon,\ \text{ for } t\geq t_2. \tag{2.18}$$

Then, from (2.14)–(2.18), we obtain

$$\lim_{t\to\infty}\sup tw^{\sigma}(t)\leq -Q_*+\epsilon+(R+\epsilon)(2-R-\epsilon),\ \text{ for } t\geq t_2. \tag{2.19}$$

From (2.16) and (2.19), since ϵ is an arbitrary, we get

$$Q_*\leq R-R^2, \tag{2.20}$$

and therefore

$$R^2-R+Q_*\leq 0. \tag{2.21}$$

Now since $Q_*\leq 1/4$, we see that $R\leq\frac{1}{2}(1+\sqrt{1-4Q_*})$ which is (2.12). The proof is complete. □

From Lemma 2.2 and Lemma 2.3, we have the following properties of the solutions of (1.1).

Lemma 2.4. *Let $x(t)$ be a nonoscillatory solution of* (1.1) *such that $x(\tau(t))>0$ for $t\geq t_1\geq\tau^{-1}(t_0)$. Assume that $0\leq P_*\leq 1/4l$ and $0\leq Q_*\leq 1/4$. Then*

$$\lim_{t\to\infty}\sup t\left(\frac{x^{\Delta}}{x}\right)^{\sigma}(t)\leq\frac{1}{2}(1+\sqrt{1-4Q_*}),$$

and

$$\lim_{t\to\infty}\inf t\left(\frac{x^{\Delta}}{x}\right)^{\sigma}(t)\geq\frac{1}{2}(1-\sqrt{1-4P_*l}).$$

Theorem 2.2. *Assume that*

$$\lim_{t\to\infty}\sup[P(t)+Q(t)]>1, \tag{2.22}$$

then every solution of (1.1) *oscillates.*

Proof. Assume for the sake of contradiction that (1.1) has a nonoscillatory solution. Without loss of generality, we may assume that there is a positive solution $x(t)$ of (1.1) such that $x(\tau(t)) > 0$ for $t \geq t_1 \geq \tau_{-1}(t_0)$. Let $w(t) = \frac{x^{\Delta}(t)}{x(t)}$ and proceeding as in the proofs of Lemmas 2.2 and 2.3, to get

$$t\int_{\sigma(t)}^{\infty} p(s)\Delta s \leq tw^{\sigma}(t) - t\int_{\sigma(t)}^{\infty} w(s)w^{\sigma}(s)\Delta s, \tag{2.23}$$

and

$$\begin{aligned}\frac{1}{t}\int_{t_1}^{t} s^2 p(s)\Delta s &\leq -tw^{\sigma}(t) + \frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}(s)\Delta s \\ &+ \frac{1}{t}\int_{t_1}^{t}\left[2sw^{\sigma}(s) - s^2\left(w^{\sigma}(s)\right)^2\right]\Delta s.\end{aligned} \tag{2.24}$$

From (2.23) and (2.24), we obtain

$$\begin{aligned}P(t)+Q(t) \quad &\leq \quad \frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}(s)\Delta s + \frac{1}{t}\int_{t_1}^{t}\left[2sw^{\sigma}(s) - s^2\left(w^{\sigma}(s)\right)^2\right]\Delta s \\ &\quad -t\int_{\sigma(t)}^{\infty} w(s)w^{\sigma}(s)\Delta s\end{aligned}$$

Using the fact that $\left[2sw^{\sigma}(s) - s^2\left(w^{\sigma}(s)\right)^2\right] \leq 1$, we have

$$\begin{aligned}P(t)+Q(t) \quad &\leq \quad 1 + \frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}(s)\Delta s - t\int_{\sigma(t)}^{\infty} w(s)w^{\sigma}(s)\Delta s \\ &\leq \quad 1 + \frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}(s)\Delta s - t\int_{\sigma(t)}^{\infty}\left(w^{\sigma}(s)\right)^2\Delta s.\end{aligned}$$

Now, since $\int_{\sigma(t)}^{\infty}\left(w^{\sigma}(s)\right)^2 < \infty$, we have

$$\lim_{t\to\infty}\sup[P(t)+Q(t)] \leq 1 + \lim_{t\to\infty}\sup[\frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}(s)\Delta s].$$

Using (2.6) in the last inequality, we get

$$\lim_{t\to\infty}\sup[P(t)+Q(t)] \leq 1,$$

which contradicts (2.22). The proof is complete. □

From Theorem 2.2, we have the following oscillation result.

Corollary 2.2. *Assume that*

$$Q^* > 1, \;\; or \;\; P^* > 1, \tag{2.25}$$

then every solution of (1.1) *oscillates.*

Example 2.1. Consider the second-order delay Euler dynamic equation

$$x^{\Delta\Delta}(t) + \frac{\gamma}{t\tau(t)}x(\tau(t)) = 0, \quad \text{for } t \in [1,\infty)_{\mathbb{T}}, \tag{2.26}$$

where $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$. It is clear that $\int_{t_0}^{\infty} \tau(s)q(s)\Delta s = \int_{t_0}^{\infty} \tau(s)\frac{\gamma}{s\tau(s)} \Delta s = \int_1^{\infty} \frac{\gamma}{s}\Delta s = \infty$, so the condition (2.1) holds. To apply Corollary 2.2, it remains to satisfy condition (2.25). In the case, we have

$$\begin{aligned} Q^* &= \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} s^2 P(s)\Delta s = \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} s^2 \frac{\gamma}{s\tau(s)}\frac{\tau(s)}{s}\Delta s \\ &= \gamma \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \Delta s = \gamma. \end{aligned}$$

So by Corollary 2.2, every solution of (2.26) oscillates if $\gamma > 1$. Also, we note that

$$\begin{aligned} P^* &= \limsup_{t\to\infty} t\int_{\sigma(t)}^{\infty} p(s)\frac{\tau(s)}{s}\Delta s = \gamma \liminf_{t\to\infty} t\int_{\sigma(t)}^{\infty} \frac{1}{s^2}\Delta s \\ &\geq \gamma \liminf_{t\to\infty} t\int_{\sigma(t)}^{\infty} \frac{1}{s\sigma(s)}\Delta s = \gamma \liminf_{t\to\infty} t\int_{\sigma(t)}^{\infty} \left(\frac{-1}{s}\right)^{\Delta} \Delta s = \gamma. \end{aligned}$$

So by Corollary 2.2, every solution of (2.26) oscillates if $\gamma > 1$.

Remark 2.1. Note that the oscillation condition $P^* > 1$ on Corollary 2.2 is the time scale analogue of the condition (1.8) of Hille [14].

Now, we concentrate our work to give an affirmative answer to the question posed in the introduction and consider the case when $Q_* \leq 1/4$, and $P_* \leq 1/4l$.

Theorem 2.3. *Assume that $P_* \leq 1/4l$. Then, every solution of* (1.1) *oscillates if*

$$Q^* > \frac{1}{2}(1 + \sqrt{1 - 4P_* l}). \tag{2.27}$$

Proof. Assume for the sake of contradiction that (1.1) has a nonoscillatory solution. Without loss of generality, we may assume that there is a positive solution $x(t)$ of (1.1) such that $x(\tau(t)) > 0$ for $t \geq t_1 \geq \tau_{-1}(t_0)$. Then from Lemma 2.2, we have

$$tw^{\sigma}(t) > r - \epsilon, \quad \text{for } t \geq t_2 > t_1,$$

where

$$w^{\sigma}(t) = \left(\frac{x^{\Delta}}{x}\right)^{\sigma}(t) \text{ and } r = \frac{1}{2}(1 - \sqrt{1 - 4P_* l}).$$

From Theorem 2.2, we have

$$\begin{aligned} \frac{1}{t}\int_{t_1}^{t} s^2 p(s)\Delta s &\leq -tw^{\sigma}(t) + \frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}(s)\Delta s \\ &\quad + \frac{1}{t}\int_{t_1}^{t} \left[2sw^{\sigma}(s) - s^2\left(w^{\sigma}(s)\right)^2\right]\Delta s \\ &\leq -r + \epsilon + 1 + \frac{t_1^2 w(t_1)}{t} + \frac{h_0}{t}\int_{t_1}^{t} w^{\sigma}(s)\Delta s. \end{aligned}$$

It follows that

$$Q^* \leq \frac{1}{2}(1 + \sqrt{1 - 4P_* l}),$$

which contradicts the condition (2.27). The proof is complete. □

Theorem 2.4. *Assume that* $Q_* \leq 1/4$. *Then, every solution of* (1.1) *oscillates if*

$$P^* > \frac{1}{2}(1+\sqrt{1-4Q_*}). \tag{2.28}$$

Proof. Assume for the sake of contradiction that (1.1) has a nonoscillatory solution. Without loss of generality, we may assume that there is a positive solution $x(t)$ of (1.1) such that $x(\tau(t)) > 0$ for $t \geq t_1 \geq \tau_{-1}(t_0)$. Then from Lemma 2.3, we have

$$tw^\sigma < R + \epsilon, \quad \text{for } t \geq t_2 > t_1,$$

where

$$R = \frac{1}{2}(1+\sqrt{1-4Q_*}).$$

From Lemma 2.2, we have

$$tw^\sigma(t) \geq t\int_{\sigma(t)}^{\infty} p(s)\Delta s + t\int_{\sigma(t)}^{\infty} w(s)w^\sigma(s)\Delta s \geq t\int_{\sigma(t)}^{\infty} p(s)\Delta s, \text{ for } t \geq t_1,$$

which implies that

$$\limsup_{t\to\infty} t\int_{\sigma(t)}^{\infty} p(s)\Delta s \leq R+\epsilon, \quad \text{for } t \geq t_2 > t_1.$$

Since ϵ is an arbitrary, so that

$$P^* \leq \frac{1}{2}(1+\sqrt{1-4Q_*}),$$

which contradicts the assumption (2.28). The proof is complete. □

Remark 2.2. In the special case when $\mathbb{T} = \mathbb{R}$ and $\tau(t) = t$, we see that the condition (2.28) becomes the condition (1.12). So, our results in the special case involve the oscillation results of differential equations established by Lomtatidze [13], and are essentially new for equations (1.3)–(1.5) and can be applied to different types of time scales with $\max_{t\in\mathbb{T}} \mu(t) = h_0 \geq 0$. Also the results can be extended to the nonlinear delay dynamic equations

$$x^{\Delta\Delta}(t) + q(t)f(x(\tau(t))) = 0, \text{ when } |f(u)| \geq K\,|u| \text{ for } K > 0.$$

3. APPLICATIONS ON NEUTRAL DYNAMIC EQUATIONS

In this section, we apply the oscillation results established in Section 2 on the neutral delay dynamic equation (1.14). We assume that the following assumptions are satisfied:

(h_1). $\tau(t) \leq t$, $\delta(t) \leq t$ are defined on the time scale $\mathbb{T}$, and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$,

(h_2). $r(t)$ and $q(t)$ are positive real-valued rd-continuous functions defined on $\mathbb{T}$ and $0 \leq r(t) \leq r < 1$.

(h_3). $\int_{t_0}^{\infty} \delta(s)q(s)(1-r(\delta(s))) = \infty$.

To the best of our knowledge no similar results are given for equation (1.14) even for differential or difference equations.

Lemma 3.1. *Assume that (h_1)–(h_3) hold. Let $y(t)$ be a nonoscillatory solution of (1.14) such that $y(t)$, $y(\tau(t))$ and $y(\delta(t)) > 0$ for $t \geq t_0$ sufficiently large. Let*

$$u(t) := y(t) + r(t)y(\tau(t)), \tag{3.1}$$

and

$$b_* := \liminf_{t\to\infty} t \int_{\sigma(t)}^{\infty} B(s)\Delta s, \ B_* := \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} s^2 B(s)\Delta s,$$

where $B(s) := \frac{\delta(s)}{s} q(s)(1-r(\delta(s)))$. Assume that $0 \leq b_* \leq 1/4l$ and $0 \leq B_* \leq 1/4$, then

$$\liminf_{t\to\infty} t^{\lambda}\left(\frac{u^{\Delta}}{u}\right)(t) = 0, \ \textit{for } \lambda < 1,$$

$$\liminf_{t\to\infty} t\left(\frac{u^{\Delta}}{u}\right)^{\sigma}(t) \geq \frac{1}{2}(1-\sqrt{1-4b_* l}), \ \textit{and} \ \limsup_{t\to\infty} t\left(\frac{u^{\Delta}}{u}\right)^{\sigma}(t) \leq \frac{1}{2}(1+\sqrt{1-4B_*}).$$

Proof. In view of (1.14) and (3.1), we have

$$u^{\Delta\Delta}(t) + q(t)y(\delta(t)) \leq 0, \quad t \geq t_1 > t_0, \tag{3.2}$$

and so $u^{\Delta}(t)$ is an eventually decreasing function. From Lemma 2.1, $u^{\Delta}(t)$ is eventually nonnegative and hence

$$\begin{aligned} y(t) &= u(t) - r(t)y(\tau(t)) = u(t) - r(t)[u(\tau(t)) - r(\tau(t))y(\tau(\tau(t)))] \\ &\geq u(t) - r(t)u(\tau(t)) \geq (1-r(t))u(t). \end{aligned}$$

Then, for $t \geq t_2 = \delta^{-1}(t_1)$, we see that

$$y(\delta(t)) \geq (1-r(\delta(t)))u(\delta(t)). \tag{3.3}$$

Then from (3.2) and (3.3), we have

$$u^{\Delta\Delta}(t) + q(t)(1-r(\delta(t)))u(\delta(t)) \leq 0, \ \text{for } t \geq t_2. \tag{3.4}$$

Define $w(t) := \frac{u^{\Delta}(t)}{u(t)}$ and proceeding as in the proof of Lemma 2.2 by using (3.4) we have $w(t) > 0$ and satisfies the Riccati dynamic inequality

$$w^{\Delta}(t) + B(t) + (w^{\sigma}(t))^2 \leq 0, \quad \text{for } \ t \geq t_2. \tag{3.5}$$

The remainder of the proof is similar to that of the proofs of Lemmas 2.2 and 2.3 by using (3.5) and hence is omitted. □

Now, we are ready to state the main oscillation results for (1.14) based on Lemma 3.1 and the inequality (3.5). The proofs are similar to that of the proofs of Theorems 2.2–2.4 and hence are omitted.

Theorem 3.1. *Assume that* (h_1)–(h_3) *hold. If*

$$b_* > \frac{1}{4}, \text{ or } B_* > \frac{1}{4}, \tag{3.6}$$

then every solution of (1.14) *oscillates.*

Theorem 3.2. *Assume that* (h_1)–(h_3) *hold. If*

$$b^* + B^* > 1, \tag{3.7}$$

where

$$b^* := \lim_{t\to\infty} \sup t \int_{\sigma(t)}^{\infty} B(s)\Delta s, \text{ and } B^* := \lim\sup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} s^2 B(s)\Delta s.$$

then every solution of (1.14) *oscillates.*

Theorem 3.3. *Assume that* (h_1)–(h_3) *hold and* $b_* \leq 1/4l$. *If*

$$B^* > \frac{1}{2}(1 + \sqrt{1 - 4b_* l}), \tag{3.8}$$

then, every solution of (1.15) *oscillates.*

Theorem 3.4. *Assume that* (h_1)–(h_3) *hold and* $B_* \leq 1/4$. *If*

$$b^* > \frac{1}{2}(1 + \sqrt{1 - 4B_*}), \tag{3.9}$$

then every solution of (1.14) *oscillates.*

REFERENCES

[1] R. P. Agarwal, M. Bohner and S. H. Saker, Oscillation of second-order delay dynamic equations, *Canadian Appl. Math. Quart.* (to appear).

[2] R. P. Agarwal, D. O'Regan and S. H. Saker, Oscillation criteria for second-order nonlinear neutral delay dynamic equations, *J. Math. Anal. Appl.* 300 (2004), 203–217.

[3] E. A. Bohner and J. Hoffacker, Oscillation properties of an Emden-Fowler type Equations on Discrete time scales, *J. Diff. Eqns. Appl.* 9 (2003), 603–612.

[4] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, *Birkhäuser, Boston*, 2001.

[5] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, *Birkhäuser, Boston*, 2003.

[6] M. Bohner and S. H. Saker, Oscillation of second order nonlinear dynamic equations on time scales, *Rocky Mountain J. Math.* 34 (2004), 1239–1254.

[7] M. Bohner and S. H. Saker, Oscillation criteria for perturbed nonlinear dynamic equations, *Mathl. Comp. Modelling* 40 (2004), 249–260.

[8] E. Akin-Bohner, M. Bohner and S. H. Saker, Oscillation criteria for a certain class of second order Emden-Fowler dynamic Equations, *Elect. Transc. Numerical. Anal.* (to appear).

[9] L. Erbe and A. Peterson, Positive solutions for a nonlinear differential equation on a measure chain, *Mathl. Comp. Modelling, Boundary Value Problems and Related Topics*, 32 (2000), 571–585.

[10] L. Erbe and A. Peterson, Riccati equations on a measure chain, In G. S. Ladde, N. G. Medhin, and M. Sambandham, editors, *Proceedings of Dynamic Systems and Applications*, volume 3, pages 193–199, Atlanta, 2001. Dynamic publishers.

[11] L. Erbe, A. Peterson and S. H. Saker, Kamenev-type oscillation criteria for second-order linear delay dynamic equations, *Dynamic Syst. & Appl.* 15 (2006), 65–78.

[12] L. Erbe, A. Peterson and S. H. Saker, Oscillation criteria for second-order nonlinear dynamic equations on time scales, *J. London Math. Soc.* 67 (2003), 701–714.

[13] A. Lomatidze, Oscillation and nonoscillation criteria for second-order linear differential equations, *Georgian Math. J.* 4 (1997), 129–138.

[14] E. Hille, Non-oscillation theorems, *Trans. Amer. Math. Soc.* 64 (1948), 234–252.

[15] Z. Nehari, Oscillation criteria for second-order linear differential equations, *Trans. Amer. Math. Soc.* 85 (1957), 428–445.

[16] S. H. Saker, Oscillation of nonlinear dynamic equations on time scales, *Appl. Math. Comp.* 148 (2004), 81–91.

[17] S. H. Saker, Oscillation criteria of second-order half-linear dynamic equations on time scales, *J. Comp. Appl. Math.* 177 (2005), 375–387.

[18] S. H. Saker, Oscillation of second-order neutral delay dynamic equations of Emden-Fowler type, *Dynamic Sys. Appl.* (accepted).

[19] S. H. Saker, Oscillatory behavior of linear neutral delay dynamic equations on time scales, *Kyungpook Math. Journal* (accepted).

[20] S. H. Saker, Oscillation criteria for a certain class of second-order neutral delay dynamic equations, *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms* (accepted).

[21] S. H. Saker, New oscillation criteria for second-order nonlinear dynamic equations on time scales, *Nonlinear Fun. Anal. Appl.* (to appear).

[22] Y. Şahiner, Oscillation of second-order delay differential equations on time scales, *Nonlinear Analysis* 63 (2005), 1073–1080.

[23] B. G. Zhang and Xinghua Deng, Oscillation of delay differential equations on time scales, *Mathl. Comp. Modelling* 36 (2002) 1307–1318.