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ITERATIVE APPROXIMATIONS OF SOLUTIONS FOR SYSTEMS OF NONLINEAR OPERATOR EQUATIONS IN BANACH SPACES

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ABSTRACT. In this paper, by virtue of the Mann iterative technique, we introduce and study a class of systems of nonlinear equations without any mixed monotone property and continuity, and prove the existence, uniqueness and Mann iterative approximation theorems of solutions for systems of nonlinear operator equations. The results presented in this paper improve and generalize the corresponding results of the earlier and recent works.

Keywords. Nonlinear operator equations; weak order-Lipschitz-condition; Mann iteration; existence; convergence

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we always assume that X is a real Banach space with norm $\|\cdot\|$, θ is the null element of X and $P \subset X$ is a cone on X, and the cone P defines a partial ordering $\leq$ in X by $x \leq y$ if and only if $y - x \in P$ for all $x, y \in X$. A cone P in X is said to be normal if there exists a normal constant $N_p > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq N_p\|y\|$ for all $x, y \in X$. The constant N_p is called the normal constant of the cone P. Without loss of generality, we can assume that the normal constant $N_p = 1$. For any $u_0, v_0 \in X$ with $u_0 \leq v_0$, we define the ordered interval $D = [u_0, v_0] = \{u \in X : u_0 \leq u \leq v_0\}$ (see [5]).

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In [7], Song considered the following system of nonlinear operator equations in X:

$$\begin{cases} x = A(x,x), \\ x = B(x,x), \end{cases} \tag{1.1}$$

where $A, B : D \times D \to X$ need not be mixed monotone operators or continuous operators, and introduced the following assumptions (H_1) and (H_2) to show the existence and uniqueness of the system (1.1):

(H_1) There exist nonnegative real constants M and N with $N < M + 1$ such that

(i) $u_0 + N(v_0 - u_0) \le B(u_0, v_0), \quad A(v_0, u_0) \le v_0 - N(v_0 - u_0)$;
(ii) $A(u_2, v_2) - A(u_1, v_1) \ge -M(u_2 - u_1) - N(v_1 - v_2)$;
(iii) $B(u_2, v_2) - B(v_0, u_0) \ge -M(u_2 - u_1) + N(v_1 - v_2)$;
(iv) $A(u_2, u_1) - B(u_1, u_2) \ge -(M + N)(u_2 - u_1)$,
where $u_0 \le u_1 \le u_2 \le v_0$, $u_0 \le v_2 \le v_1 \le v_0$,

and

(H_2) there exists a positive linear operator $L : X \to X$ with $r(L) < 1$ such that

$$A(v,u) - B(u,v) \le L(v-u), \quad u_0 \le u \le v \le v_0,$$

where $r(L)$ is the spectral radius of L.

Theorem S1. ([7]) *Let $P \subset X$ be a normal cone and $u_0, v_0 \in X$ such that $u_0 \le v_0$ and $D = [u_0, v_0]$. Assume that the conditions (H_1) and (H_2) are satisfied. If $2N + r(L) < 1$, then the system of the nonlinear operator equations (1.1) has a unique solution $\bar{u}$ in D, the iterative sequences $\{u_n\}$ and $\{v_n\}$ generated by*

$$\begin{cases} u_n = \frac{1}{1+M-N}[B(u_{n-1}, v_{n-1}) + Mu_{n-1} - Nv_{n-1}], \\ v_n = \frac{1}{1+M-N}[A(v_{n-1}, u_{n-1}) + Mv_{n-1} - Nu_{n-1}] \end{cases} \tag{1.2}$$

for $n = 1, 2, \cdots$ both converge to the unique solution $\bar{u}$ and there exists a natural number n_0 such that, for any constant c with $r(L) < c_i < 1 - 2N$,

$$\|\bar{u} - u_n\| \le b^n \|v_0 - u_0\| \quad \textit{or} \quad \|\bar{u} - v_n\| \le b^n \|v_0 - u_0\|, \quad n \ge n_0,$$

where $b = \frac{c+M+N}{1+M-N}$.

Theorem S2. ([7]) *Let $P \subset X$ be a normal cone and $u_0, v_0 \in X$ such that $u_0 \le v_0$ and $D = [u_0, v_0]$. Assume that the conditions (H_1) is satisfied. If there exists a constant with $0 < b < 1 - 2N$ such that, for $u_0 \le u \le v \le v_0$,*

(H_3) *$A(v,u) - B(u,v) \ge b(v-u)$, then the system of the nonlinear operator equations (1.1) has a unique solution $\bar{u}$ in D, the iterative sequences $\{u_n\}$ and $\{v_n\}$*

generated by (1.2) *both converge to the unique solution* $\bar{u}$ *and have the following error estimate:*

$$\|\bar{u} - u_n\| \leq \left(\frac{b+M+N}{1+M-N}\right)^n \|v_0 - u_0\|$$

for $n = 1, 2, \cdots$.

Remark 1.1. (1) If $A = B$ in (1.1), then we have the following nonlinear operator equation:

$$x = A(x, x). \tag{1.3}$$

(2) If $A(x, y) = T(x) + F(y)$ for all $x, y \in X$ in (1.3), then we have the following nonlinear operator equation:

$$x = T(x) + F(x). \tag{1.4}$$

(3) If $A(x, y) = T(x)F(y)$ for all $x, y \in X$ in (1.3), then we have the following nonlinear operator equation:

$$x = T(x)F(x). \tag{1.5}$$

In fact, the study of these nonlinear operator equations is motivated by an increasing interest in the equation (1.3) with applications in Banach spaces (see, for example, [1], [2], [4], [6]–[9], [11]–[14] and the references therein).

In 1995, Chen [1] studied the fixed point theorems of T-monotone operator under the condition that A satisfies some continuous condition. Recently, Sun, Li and Luan [8] studied T-mixed monotone operator which generalized T-monotone operator and obtained some new fixed point theorems by using the results of Syan [9] and the condition that A satisfies some non-continuous condition. Very recently, Zhang and Xie [14] discussed the existence of the solution and coupled minimal and maximal quasi-solutions for nonlinear non-monotone operator equation (1.3) under the condition that $A(x, y) + T(x)$ is a mixed monotone operator.

Moreover, by using partial order method, Song [7] discussed the existence, uniqueness and Picard's iterative approximation of solutions for the operator equations (1.1) in Banach space (see Theorems S1, S2 above).

Inspired and motivated by the recent works of [3], [6], [7], [11] and [14], in this paper, we study the Mann iterative approximation problem of solutions for the nonlinear operator equations (1.1), (1.3)–(1.5) in Banach spaces and also give the estimation of rate of convergence. The results presented in this paper improve and extend the corresponding results given by Song [7] and some authors.

2. SOME ITERATIVE ALGORITHMS

In this section, we will give some Algorithms (Mann iterative sequences) for solving the equations (1.1), (1.3)–(1.5) in Section 1.

Algorithms 2.1. The sequences $\{u_n\}$ and $\{v_n\}$ are defined by

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n(1+a_n-b_n)^{-1}[B(u_n,v_n) + a_n u_n - b_n v_n], \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n(1+a_n-b_n)^{-1}[A(v_n,u_n) + a_n v_n - b_n u_n] \end{cases} \tag{2.1}$$

for $n = 0, 1, 2, \cdots$, where $\{\alpha_n\}$ is a real sequence in $(0, 1]$ satisfying some conditions and $\{a_n\}$, $\{b_n\}$ are two nonnegative real sequences with $b_n < a_n+1$ for $n = 0, 1, 2, \cdots$.

If $\alpha_n = 1$ for all $n = 0, 1, 2, \cdots$ in (2.1), then Algorithms 2.1 is reduced to the following algorithm:

Algorithms 2.2. The sequences $\{u_n\}$ and $\{v_n\}$ are defined by

$$\begin{cases} u_{n+1} = (1+a_n-b_n)^{-1}[B(u_n,v_n) + a_n u_n - b_n v_n], \\ v_{n+1} = (1+a_n-b_n)^{-1}[A(v_n,u_n) + a_n v_n - b_n u_n] \end{cases} \tag{2.2}$$

for $n = 0, 1, 2, \cdots$.

If $a_n \equiv a$ and $b_n \equiv b$ for all $n = 0, 1, 2, \cdots$ in (2.2), then Algorithms 2.2 is reduced to the following algorithm:

Algorithms 2.3. The sequences $\{u_n\}$ and $\{v_n\}$ are defined by

$$\begin{cases} u_{n+1} = (1+a-b)^{-1}[B(u_n,v_n) + a u_n - b v_n], \\ v_{n+1} = (1+a-b)^{-1}[A(v_n,u_n) + a v_n - b u_n] \end{cases} \tag{2.3}$$

for $n = 0, 1, 2, \cdots$.

Remark 2.1. The iterative procedures $\{u_n\}$ and $\{v_n\}$ in Algorithms 2.3 were studied by Song [7].

Algorithms 2.4. The sequences $\{u_n\}$ and $\{v_n\}$ are defined by

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n(1+a_n-b_n)^{-1}[A(u_n,v_n) + a_n u_n - b_n v_n], \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n(1+a_n-b_n)^{-1}[A(v_n,u_n) + a_n v_n - b_n u_n] \end{cases} \tag{2.4}$$

for $n = 0, 1, 2, \cdots$.

Algorithms 2.5. The sequences $\{u_n\}$ and $\{v_n\}$ are defined by

$$\begin{cases} u_{n+1} = (1-\alpha_n)u_n + \alpha_n(1+a_n-b_n)^{-1}[(T+a_nI)(u_n) + (F-b_nI)(v_n)], \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n(1+a_n-b_n)^{-1}[(T+a_nI)(v_n) + (F-b_nI)(u_n)] \end{cases} \tag{2.5}$$

for $n = 0, 1, 2, \cdots$, where I is an identical operator from X into itself.

Algorithms 2.6. The sequences $\{u_n\}$ and $\{v_n\}$ are defined by

$$\text{(2.6)} \qquad \begin{cases} u_{n+1} = (1+a_n-b_n)^{-1}[T(u_n)F(v_n)+a_nu_n-b_nv_n], \\ v_{n+1} = (1+a_n-b_n)^{-1}[T(v_n)F(u_n)+a_nv_n-b_nu_n] \end{cases}$$

for $n = 0, 1, 2, \cdots$.

3. THE MAIN RESULTS

First, we list for convenience the following assumptions:

(H_1) There exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $b_n < a_n + 1$ for all $n = 0, 1, 2, \cdots$ such that

(a) $\hat{u} + b_n(\hat{v} - \hat{u}) \le B(\hat{u}, \hat{v}), \quad A(\hat{v}, \hat{u}) \le \hat{v} - b_n(\hat{v} - \hat{u})$;

(b) $A(\hat{u}, \hat{v}) - A(\bar{u}, \bar{v}) \ge -a_n(\hat{u} - \bar{u}) + b_n(\hat{v} - \bar{v})$;

(c) $B(\hat{u}, \hat{v}) - B(\bar{u}, \bar{v}) \ge -a_n(\hat{u} - \bar{u}) + b_n(\hat{v} - \bar{v})$;

(d) $A(\hat{u}, \bar{u}) - B(\bar{u}, \hat{u}) \ge (1 - 2b_n)(\hat{u} - \bar{u})$,

where $u_0 \le \bar{u} \le \hat{u} \le v_0$ and $u_0 \le \hat{v} \le \bar{v} \le v_0$.

(H_2) There exists a positive linear operator $L : X \to X$ with $r(L) < 1$ such that

$$A(v, u) - B(u, v) \le L(v - u), \quad u_0 \le u \le v \le v_0,$$

where $r(L)$ is the spectral radius of L.

Now, we give our main results in this paper.

Theorem 3.1. *Let $P \subset X$ be a normal cone, $u_0, v_0 \in X$ such that $u_0 \le v_0$ and $D = [u_0, v_0]$. Suppose that the conditions (H_1) and (H_2) are satisfied. If $2b_n + r(L) < 1$ for $n = 0, 1, 2, \cdots$, $\{\alpha_n\}$ is a monotone decreasing sequence with $\alpha_n \to \alpha \in [0, 1)$, $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then the iterative sequences $\{u_n\}$ and $\{v_n\}$ generated by (2.1) both converge strongly to the unique solution x^* of the system of the nonlinear operator equations (1.1) and there exists a natural number n_0 such that, for any $i = 0, 1, 2, \cdots, n-1$ and $r(L) < c_i < 1 - 2b_i$,*

$$\|x^* - u_n\| \le \epsilon^n \|v_0 - u_0\| \quad or \quad \|v_n - x^*\| \le \epsilon^n \|v_0 - u_0\|, \quad n \ge n_0,$$

where $\epsilon = (1 - \alpha) + \alpha\sigma$, $\sigma = \max_{0 \le i \le n-1} \{\sigma_i\}$, $\sigma_i = \frac{c_i + a_i + b_i}{1 + a_i - b_i}$ for $i = 0, 1, \cdots, n-1$.

Proof. Firstly, from the condition (H_1) and (2.1), we can know that

$$\text{(3.1)} \qquad u_0 \le u_1 \le u_2 \le \cdots \le u_n \le v_n \le \cdots \le v_1 \le v_0.$$

In fact, when $n = 1$, by (2.1) and the condition (H_1), we have

$$\begin{aligned} u_1 - u_0 &= (1 - \alpha_0)u_0 + \alpha_0(1 + a_0 - b_0)^{-1}[B(u_0, v_0) + a_0u_0 - b_0v_0] - u_0 \\ &= \alpha_0\{(1 + a_0 - b_0)^{-1}[B(u_0, v_0) + a_0u_0 - b_0v_0] - u_0\} \\ &\ge \alpha_0\{(1 + a_0 - b_0)^{-1}[u_0 + b_0(v_0 - u_0) + a_0u_0 - b_0v_0] - u_0\} \end{aligned}$$

$$
\begin{aligned}
&= \theta, \\
v_0 - v_1 &= v_0 - \{(1-\alpha_0)v_0 + \alpha_0(1+a_0-b_0)^{-1}[A(v_0,u_0)+a_0v_0-b_0u_0]\} \\
&= \alpha_0\{v_0 - (1+a_0-b_0)^{-1}[A(v_0,u_0)+a_0v_0-b_0u_0]\} \\
&\geq \alpha_0\{v_0 - (1+a_0-b_0)^{-1}[v_0 - b_0(v_0-u_0)+a_0v_0-b_0u_0]\} \\
&= \theta
\end{aligned}
$$

and

$$
\begin{aligned}
v_1 - u_1 &= (1-\alpha_0)(v_0-u_0) \\
&\quad +\alpha_0(1+a_0-b_0)^{-1}\{[A(v_0,u_0)-B(u_0,v_0)+(a_0+b_0)(v_0-u_0)\} \\
&\geq (1-\alpha_0)(v_0-u_0) \\
&\quad +\alpha_0(1+a_0-b_0)^{-1}[(1-2b_0)(v_0-u_0)+(a_0+b_0)(v_0-u_0)] \\
&= v_0 - u_0 \\
&\geq \theta.
\end{aligned}
$$

Therefore, $u_0 \leq u_1 \leq v_1 \leq v_0$, i.e., (3.1) holds for $n = 1$.

Suppose now that (3.1) holds for $n = k$, i.e., $u_{k-1} \leq u_k \leq v_k \leq v_{k-1}$. We shall show that it holds for $n = k+1$. Indeed, by (2.1), the condition (H_1) and induction hypothesis, we have

$$
\begin{aligned}
u_{k+1} - u_k &= (1-\alpha_k)u_k + \alpha_k(1+a_k-b_k)^{-1}[B(u_k,v_k)+a_ku_k-b_kv_k] - u_k \\
&= \alpha_k\{(1+a_k-b_k)^{-1}[B(u_k,v_k)+a_ku_k-b_kv_k]-u_k\} \\
&\geq \alpha_k\{(1+a_k-b_k)^{-1}[u_k+b_k(v_k-u_k)+a_ku_k-b_kv_k]-u_k\} \\
&= \theta, \\
v_k - v_{k+1} &= v_k - \{(1-\alpha_k)v_k + \alpha_k(1+a_k-b_k)^{-1}[A(v_k,u_k)+a_kv_k-b_ku_k]\} \\
&= \alpha_k\{v_k - (1+a_k-b_k)^{-1}[A(v_k,u_k)+a_kv_k-b_ku_k]\} \\
&\geq \alpha_k\{v_k - (1+a_k-b_k)^{-1}[v_k - b_k(v_k-u_k)+a_kv_k-b_ku_k]\} \\
&= \theta. \\
v_{k+1} - u_{k+1} &= (1-\alpha_k)(v_k-u_k) \\
&\quad +\alpha_k(1+a_k-b_k)^{-1}\{[A(v_k,u_k)-B(u_k,v_k)+(a_k+b_k)(v_k-u_k)\} \\
&\geq (1-\alpha_k)(v_k-u_k) \\
&\quad +\alpha_k(1+a_k-b_k)^{-1}\{[(1-2b_k)(v_k-u_k)+(a_k+b_k)(v_k-u_k)\} \\
&= v_k - u_k \\
&\geq \theta.
\end{aligned}
$$

Thus $u_k \leq u_{k+1} \leq v_{k+1} \leq v_k$ and so (3.1) is true.

Secondly, we prove that the existence of solutions for the systems of operator equations (1.1) in D. In fact, it follows from the condition (H_2) that

$$
\begin{aligned}
\theta &\le v_n - u_n \\
&= (1-\alpha_{n-1})(v_{n-1}-u_{n-1}) \\
&\quad +\alpha_{n-1}(1+a_{n-1}-b_{n-1})^{-1}[A(v_{n-1},u_{n-1}) - B(u_{n-1},v_{n-1}) \\
&\quad +(a_{n-1}+b_{n-1})(v_{n-1}-u_{n-1})] \\
&\le \{(1-\alpha_{n-1})I + \alpha_{n-1}(1+a_{n-1}-b_{n-1})^{-1} \\
&\quad \times[L+(a_{n-1}+b_{n-1})I]\}(v_{n-1}-u_{n-1}) \\
&= Q_{n-1}(v_{n-1}-u_{n-1}) \\
&\le Q_{n-1}(Q_{n-2}(v_{n-2}-u_{n-2})) \\
&\le \cdots \\
&\le \prod_{i=0}^{n-1} Q_i(v_0-u_0), \qquad (3.2)
\end{aligned}
$$

where $Q_i = (1-\alpha_i)I + \alpha_i(1+a_i-b_i)^{-1}[L+(a_i+b_i)I]$ for $i = 0,1,2,\cdots,n-1$. Since $\{\alpha_n\}$ is a monotone decreasing sequence in $(0,1]$ and $\alpha_n \to \alpha \in (0,1]$, for all $i = 0,1,2,\cdots,n-1$ and $c_i \in (r(L), 1-2b_i)$, we have (see [10])

$$
\begin{aligned}
\lim_{n\to\infty} \|Q_i^n\|^{\frac{1}{n}} &= r(Q_i) \\
&\le (1-\alpha_i) + \alpha_i \frac{r(L)+a_i+b_i}{1+a_i-b_i} \\
&\le (1-\alpha_i) + \alpha_i \frac{c_i+a_i+b_i}{1+a_i-b_i} \\
&= (1-\alpha_i) + \alpha_i\sigma_i \\
&\le (1-\alpha_i) + \alpha_i\sigma,
\end{aligned}
$$

where $\sigma = \max\{\sigma_i | \, i = 0,1,\cdots,n-1\}$, $\sigma_i = \frac{c_i+a_i+b_i}{1+a_i-b_i}$. Obviously, the condition $c_i < 1-2b_i$ implies $\sigma_i < 1$ for all $i = 0,1,2,\cdots,n-1$ and so $\sigma < 1$ and it follows from $\alpha_i > \alpha$ for all $i = 0,1,2,\cdots,n-1$ that

$$
\begin{aligned}
\lim_{n\to\infty} \|Q_i^n\|^{\frac{1}{n}} &= 1-\alpha_i(1-\sigma) \\
&\le 1-\alpha(1-\sigma) \\
&= \epsilon, \qquad (3.3)
\end{aligned}
$$

where $\epsilon = (1-\alpha)+\alpha\sigma \in (0,1)$. Thus, from (3.3), it is easy to know that there exists a natural number n_0 such that

$$\|Q_i^n\| \le \epsilon^n, \quad \forall n \ge n_0, \quad i \ge 0,$$

i.e.,

$$\prod_{i=0}^{n-1} \|Q_i\| = \|Q_0\| \cdot \|Q_1\| \cdots \|Q_{n-1}\| \leq \epsilon^n, \quad n \geq n_0. \tag{3.4}$$

From (3.2), (3.4) and the normality of P, we get

$$\|v_n - u_n\| \leq \|v_0 - u_0\|\epsilon^n, \quad n \geq n_0. \tag{3.5}$$

It follows from (3.1) that, for any positive integers m and n,

$$\theta \leq u_{m+n} - u_n \leq v_n - u_n, \quad \theta \leq v_n - v_{m+n} \leq v_n - u_n. \tag{3.6}$$

For any $n, m \geq 1$, (3.5), (3.6) and the normality of P imply

$$\begin{cases} \|u_{n+m} - u_n\| \leq \|v_n - u_n\| \leq \epsilon^n \|v_0 - u_0\|, \\ \|v_n - v_{n+m}\| \leq \|v_n - u_n\| \leq \epsilon^n \|v_0 - u_0\|, \end{cases} \quad n \geq n_0, \tag{3.7}$$

which imply that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in X and hence there exist $u^*, v^* \in X$ such that $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$ and $u_n \leq u^* \leq v^* \leq v_n$ for $n = 0, 1, 2, \cdots$. By the normality of P and (3.5), we have

$$\|v^* - u^*\| \leq \|v_n - u_n\| \to 0 \quad (n \to \infty),$$

and so $u^* = v^* \triangleq x^* \in D$, i.e.,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = x^* \quad \text{and} \quad u_n \leq x^* \leq v_n, \quad n \geq 0. \tag{3.8}$$

On the other hand, from (2.1), (3.8) and the condition (H_1), it follows that

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n(1 + a_n - b_n)^{-1}[B(u_n, v_n) + a_n u_n - b_n v_n] \\ &\leq (1-\alpha_n)u_n + \alpha_n(1 + a_n - b_n)^{-1}[B(x^*, x^*) + (a_n - b_n)x^*] \\ &\leq (1-\alpha_n)u_n + \alpha_n(1 + a_n - b_n)^{-1}[A(x^*, x^*) + (a_n - b_n)x^*] \\ &\leq (1-\alpha_n)v_n + \alpha_n(1 + a_n - b_n)^{-1}[A(v_n, u_n) + a_n v_n - b_n u_n] \\ &= v_{n+1} \end{aligned} \tag{3.9}$$

for all $n = 0, 1, 2, \cdots$. Thus, letting $n \to \infty$ in (3.9), it follows from the normality of P, (3.8), $\alpha_n \to \alpha$, $a_n \to a$ and $b_n \to b$ that

$$\begin{aligned} x^* &\leq (1-\alpha)x^* + \alpha(1 + a - b)^{-1}[B(x^*, x^*) + (a - b)x^*] \\ &\leq (1-\alpha)x^* + \alpha(1 + a - b)^{-1}[A(x^*, x^*) + (a - b)x^*] \leq x^*, \end{aligned}$$

i.e.,

$$\begin{aligned} x^* &= (1-\alpha)x^* + \alpha(1 + a - b)^{-1}[B(x^*, x^*) + (a - b)x^*] \\ &= (1-\alpha)x^* + \alpha(1 + a - b)^{-1}[A(x^*, x^*) + (a - b)x^*]. \end{aligned}$$

Therefore, we get

$$x^* = B(x^*, x^*) = A(x^*, x^*).$$

This implies that x^* is a solution of the system of the nonlinear operator equations (1.1) in D.

Thirdly, we shall show that x^* is the unique solution of the system (1.1). Indeed, suppose that $x \in D$ is also a solution of (1.1), then, from $u_0 \leq x = A(x,x) = B(x,x) \leq v_0$ and the condition (H_1), we have

$$\begin{aligned}
x - u_1 &= x - \{(1-\alpha_0)u_0 + \alpha_0(1+a_0-b_0)^{-1}[B(u_0,v_0) + a_0u_0 - b_0v_0]\} \\
&= (1-\alpha_0)(x-u_0) \\
&\quad +\alpha_0(1+a_0-b_0)^{-1}[B(x,x) - B(u_0,v_0) + a_0(x-u_0) - b_0(x-v_0)] \\
&\geq (1-\alpha_0)(x-u_0) \\
&\quad +\alpha_0(1+a_0-b_0)^{-1}[-a_0(x-u_0) + b_0(x-v_0) + a_0(x-u_0) - b_0(x-v_0)] \\
&= (1-\alpha_0)(x-u_0) \\
&\geq \theta
\end{aligned}$$

and

$$\begin{aligned}
v_1 - x &= (1-\alpha_n)v_n + \alpha_n(1+a_n-b_n)^{-1}[A(v_n,u_n) + a_nv_n - b_nu_n] - x \\
&= (1-\alpha_0)(v_0-x) \\
&\quad +\alpha_0(1+a_0-b_0)^{-1}[A(v_0,u_0) - A(x,x) + a_0(v_0-x) - b_0(u_0-x)] \\
&\geq (1-\alpha_0)(v_0-x) \\
&\quad +\alpha_0(1+a_0-b_0)^{-1}[-a_0(v_0-x) + b_0(u_0-x) + a_0(v_0-x) - b_0(u_0-x)] \\
&= (1-\alpha_0)(v_0-x) \\
&\geq \theta,
\end{aligned}$$

which imply that $u_1 \leq x \leq v_1$. By induction, it is easy to prove that

$$u_n \leq x \leq v_n, \quad n \geq 1. \tag{3.10}$$

Thus, letting $n \to \infty$ in (3.10), it follows from (3.8) and the normality of cone P that $x^* \leq x \leq x^*$, i.e., $x^* = x$. Therefore, x^* is the unique solution of the system of the equations (1.1).

Finally, letting $m \to \infty$ in (3.7), we can obtain the following error estimation:

$$\|x^* - u_n\| \leq \epsilon^n \|v_0 - u_0\| \quad \text{or} \quad \|v_n - x^*\| \leq \epsilon^n \|v_0 - u_0\|.$$

This completes the proof. □

Corollary 3.1. *Let $P \subset X$ be a normal cone, $u_0, v_0 \in X$ be such that $u_0 \leq v_0$ and $D = [u_0, v_0]$. Suppose that the conditions (H_1) is satisfied. Suppose that, for $n = 0, 1, 2, \cdots$, there exists a constant c with $0 < c < 1 - 2b_n$ such that*

(H_2') $A(v,u) - B(u,v) \leq c(v-u)$ for all $u_0 \leq u \leq v \leq v_0$.

If $\{\alpha_n\}$ *is a monotone decreasing sequence with* $\alpha_n \to \alpha \in [0,1)$, $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$, *then the iterative sequences* $\{u_n\}$ *and* $\{v_n\}$ *generated by* (2.1) *both converge strongly to the unique solution* x^* *of the system of the nonlinear operator equations* (1.1) *and*

$$\|x^* - u_n\| \le \epsilon^n \|v_0 - u_0\| \quad or \quad \|v_n - x^*\| \le \epsilon^n \|v_0 - u_0\|, \quad n \ge 1.$$

where $\epsilon = (1-\alpha) + \alpha\sigma$ *and* $\sigma = \max_{0\le i\le n-1}\{\frac{c+a_i+b_i}{1+a_i-b_i}\}$.

Proof. By the same way as stated in Theorem 3.1, we can complete the proof. □

Theorem 3.2. *Let* $P \subset X$ *be a normal cone,* $u_0, v_0 \in X$ *be such that* $u_0 \le v_0$ *and* $D = [u_0, v_0]$. *Suppose that the conditions* (H_1) *and* (H_2) *are satisfied. If* $2b_n + r(L) < 1$ *for* $n = 0, 1, 2, \cdots$, $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$, *then the iterative sequences* $\{u_n\}$ *and* $\{v_n\}$ *generated by Algorithm 2.2 both converge strongly to the unique solution* x^* *of the system of the nonlinear operator equations* (1.1) *and there exists a natural number* n_0 *such that, for any* $i = 0, 1, 2, \cdots, n-1$ *and* $r(L) < c_i < 1 - 2b_i$,

$$\|x^* - u_n\| \le \epsilon^n \|v_0 - u_0\| \quad or \quad \|v_n - x^*\| \le \epsilon^n \|v_0 - u_0\|, \quad n \ge n_0,$$

where $\epsilon = \max_{0\le i\le n-1}\{\frac{c_i+a_i+b_i}{1+a_i-b_i}\}$.

Theorem 3.3. *Let* $P \subset X$ *be a normal cone,* $u_0, v_0 \in X$ *be such that* $u_0 \le v_0$ *and* $D = [u_0, v_0]$. *Let the condition* (H_2) *be satisfied. Assume that* $2b + r(L) < 1$ *and the following conditions hold:*

(H_1') *There exist nonnegative real sequences* a *and* b *with* $b < a + 1$ *such that*

(a) $u_0 + b(v_0 - u_0) \le B(u_0, v_0)$, $\quad A(v_0, u_0) \le v_0 - b(v_0 - u_0)$;

(b) $A(\hat{u}, \hat{v}) - A(\bar{u}, \bar{v}) \ge -a(\hat{u} - \bar{u}) + b(\hat{v} - \bar{v})$;

(c) $B(\hat{u}, \hat{v}) - B(\bar{u}, \bar{v}) \ge -a(\hat{u} - \bar{u}) + b(\hat{v} - \bar{v})$;

(d) $A(\hat{u}, \bar{u}) \ge B(\bar{u}, \hat{u})$,

where $u_0 \le \bar{u} \le \hat{u} \le v_0$, $u_0 \le \hat{v} \le \bar{v} \le v_0$.

Then the iterative sequences $\{u_n\}$ *and* $\{v_n\}$ *generated by Algorithm* 2.3 *both converge strongly to the unique solution* x^* *of the system of the nonlinear operator equations* (1.1) *and there exists a natural number* n_0 *such that, for any* $r(L) < c_i < 1 - 2b$,

$$\|x^* - u_n\| \le (\frac{c+a+b}{1+a-b})^n \|v_0 - u_0\| \quad or \quad \|v_n - x^*\| \le (\frac{c+a+b}{1+a-b})^n \|v_0 - u_0\|, \quad n \ge n_0.$$

Proof. Let $\alpha_n \equiv 1$, $a_n = a$ and $b_n = b$ for all $n = 0, 1, 2, \cdots$ in the proof of Theorem 3.1. By the condition (H_1'), we know that

$$\begin{aligned} v_{n+1} - u_{n+1} &= (1+a-b)^{-1}\{[A(v_n, u_n) - B(u_n, v_n) + (a+b)(v_n - u_n)\} \\ &\ge (1+a-b)^{-1}(a+b)(v_n - u_n)\} \end{aligned}$$

$$\geq \theta, \quad n \geq 0.$$

The rest proof can be obtained by the proof of Theorem 3.1 and so it is omitted. □

Remark 3.1. It is easy to know that the condition (H_1') in Theorem 3.3 implies the condition (H_1) in Song [7] and so our results improve and generalize the corresponding results of Song' recent works [7]. In fact, the conditions (i)-(iii) of (H_1) in [7] is the same as the conditions (a)-(c) of (H_1') in Theorem 3.3. But, from the condition (d) of (H_1'), it follows that $A(\hat{u},\bar{u}) \geq B(\bar{u},\hat{u}) > 0 > -(a+b)(\hat{u}-\bar{u})$ implies $A(\hat{u},\bar{u}) \geq B(\bar{u},\hat{u}) > -(a+b)(\hat{u}-\bar{u})$, which is the condition (iv) of (H_1) in [7]. Thus the condition (H_1') improves the condition (H_1) in [7].

Remark 3.2. In Theorems 3.1, 3.3, if A and B are two mixed monotone operators, then (ii) and (iii) of the condition (H_1) are fulfilled.

From Theorem 3.1, we have the following results:

Theorem 3.4. *Let $P \subset X$ be a normal cone, $u_0, v_0 \in X$ be such that $u_0 \leq v_0$ and $D = [u_0, v_0]$. Suppose that operator $A:\ D \times D \to X$ satisfies the following conditions:*

(h_1) There exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $b_n < a_n + 1$ for all $n = 0, 1, 2, \cdots$ such that

(a) $\hat{u} + b_j(\hat{v}-\hat{u}) \leq A(\hat{u},\hat{v}), \quad A(\hat{v},\hat{u}) \leq \hat{v} - b_j(\hat{v}-\hat{u})$;

(b) $A(\hat{u},\hat{v}) - A(\bar{u},\bar{v}) \geq -a_j(\hat{u}-\bar{u}) + b_j(\hat{v}-\bar{v})$,

where $u_0 \leq \bar{u} \leq \hat{u} \leq v_0$, $u_0 \leq \hat{v} \leq \bar{v} \leq v_0$ and some $a_j \in \{a_0, a_1, a_2, \cdots\}$, $b_j \in \{b_0, b_1, b_2, \cdots\}$.

(h_2) There exists a positive linear operator $L:\ X \to X$ with $r(L) < 1$ such that

$$A(v,u) - A(u,v) \leq L(v-u), \quad u_0 \leq u \leq v \leq v_0,$$

where $r(L)$ is the spectral radius of L.

If $2b_n + r(L) < 1$ for $n = 0, 1, 2, \cdots$, $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and $\{\alpha_n\}$ is a monotone decreasing sequence with $\alpha_n \to \alpha \in [0,1)$, then the nonlinear operator equation (1.3) has a unique solution x^* in D and both the iterative sequences $\{u_n\}$ and $\{v_n\}$ generated by Algorithms 2.4 converge strongly to x^*. Further, there exists a natural number n_0 such that, for any $i = 0, 1, 2, \cdots, n-1$ and $r(L) < c_i < 1 - 2b_i$,

$$\|x^* - u_n\| \leq \epsilon^n \|v_0 - u_0\| \quad \text{or} \quad \|v_n - x^*\| \leq \epsilon^n \|v_0 - u_0\|, \quad n \geq n_0,$$

where $\epsilon = (1-\alpha) + \alpha\sigma$, $\sigma = \max_{0 \leq i \leq n-1} \{\frac{c_i + a_i + b_i}{1 + a_i - b_i}\}$.

Remark 3.3. In Theorems 3.1-3.4, we have not required any mixed monotonicity and continuity condition. In particular, if $b_j = 0$ (resp., $b = 0$), then from (H_1) and (h_1) (resp., (H_1')), we know that u_0 and v_0 are coupled lower and upper solutions of the nonlinear operator equations (1.1) and (1.3). Our results improve and generalize many known corresponding results (see, for example, [2], [4], [8], [12], [14] and the references therein).

Corollary 3.2. *Let $P \subset X$ be a normal cone and $u_0, v_0 \in X$ be such that $u_0 \leq v_0$ and $D = [u_0, v_0]$. Suppose that operator $T, F : D \to X$ satisfies the following conditions:*

(h_1') There exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $b_n < a_n + 1$ for all $n = 0, 1, 2, \cdots$ such that

(a) $\hat{u} + b_j(\hat{v} - \hat{u}) \leq T(\hat{u}) + F(\hat{v}), \quad T(\hat{v}) + F(\hat{u}) \leq \hat{v} - b_j(\hat{v} - \hat{u})$;

(b) $T(\hat{u}) - T(\bar{u}) \geq -a_j(\hat{u} - \bar{u})$;

(c) $F(\hat{v}) - F(\bar{v}) \geq b_j(\hat{v} - \bar{v})$,

where $u_0 \leq \bar{u} \leq \hat{u} \leq v_0$, $u_0 \leq \hat{v} \leq \bar{v} \leq v_0$ and $a_j \in \{a_0, a_1, a_2, \cdots\}$, $b_j \in \{b_0, b_1, b_2, \cdots\}$.

(h_2') There exists a positive linear operator $L_1, L_2 : X \to X$ with $r(L_1) + r(L_2) < 1$ such that

$$T(v) - T(u) \leq L_1(v - u), \quad F(u) - F(v) \leq L_2(v - u), \quad u_0 \leq u \leq v \leq v_0,$$

where $r(L_i)$ is the spectral radius of L_i ($i = 1, 2$).

If $2b_n + r(L_1) + r(L_2) < 1$ for $n = 0, 1, 2, \cdots$, $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and $\{\alpha_n\}$ is a monotone decreasing sequence with $\alpha_n \to \alpha \in [0, 1)$, then the nonlinear operator equation (1.4) *has a unique solution x^* in D and both the iterative sequences $\{u_n\}$ and $\{v_n\}$ generated by Algorithms 2.5 converge strongly to x^*. Further, there exists a natural number n_0 such that, for any $i = 0, 1, 2, \cdots, n-1$ and $r(L) < c_i < 1 - 2b_i$,*

$$\|x^* - u_n\| \leq \epsilon^n \|v_0 - u_0\| \quad or \quad \|v_n - x^*\| \leq \epsilon^n \|v_0 - u_0\|, \quad n \geq n_0,$$

where $\epsilon = (1 - \alpha) + \alpha\sigma$ and $\sigma = \max_{0 \leq i \leq n-1} \{\frac{c_i + a_i + b_i}{1 + a_i - b_i}\}$.

Corollary 3.3. *Let $P \subset X$ be a normal cone and $u_0, v_0 \in X$ be such that $u_0 \leq v_0$ and $D = [u_0, v_0]$. Suppose that operator $T, F : D \to X$ satisfies the following conditions:*

(h_1'') There exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $b_n < a_n + 1$ for all $n = 0, 1, 2, \cdots$ such that

(a) $\hat{u} + b_j(\hat{v} - \hat{u}) \leq T(\hat{u})F(\hat{v}), \quad T(\hat{v})F(\hat{u}) \leq \hat{v} - b_j(\hat{v} - \hat{u})$;

(b) $T(\hat{u})F(\hat{v}) - T(\bar{u})F(\bar{v}) \geq -a_j(\hat{u} - \bar{u}) + b_j(\hat{v} - \bar{v})$,

where $u_0 \leq \bar{u} \leq \hat{u} \leq v_0$, $u_0 \leq \hat{v} \leq \bar{v} \leq v_0$ and $a_j \in \{a_0, a_1, a_2, \cdots\}$, $b_j \in \{b_0, b_1, b_2, \cdots\}$.

(h_2'') There exists a positive linear operator $L : X \to X$ with $r(L) < 1$ such that

$$T(v)F(u) - T(u)F(v) \leq L(v - u), \quad u_0 \leq u \leq v \leq v_0,$$

where $r(L)$ is the spectral radius of L.

If $2b_n + r(L) < 1$ *for* $n = 0, 1, 2, \cdots$, $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ *and* $\{\alpha_n\}$ *is a monotone decreasing sequence with* $\alpha_n \to \alpha \in [0, 1)$*, then the nonlinear operator equation* (1.5) *has a unique solution* x^* *in* D *and both the iterative sequences* $\{u_n\}$ *and* $\{v_n\}$ *generated by Algorithms 2.6 converge strongly to* x^**. Further, there exists a natural number* n_0 *such that, for any* $i = 0, 1, 2, \cdots, n-1$ *and* $r(L) < c_i < 1 - 2b_i$*,*

$$\|x^* - u_n\| \le \epsilon^n \|v_0 - u_0\| \quad or \quad \|v_n - x^*\| \le \epsilon^n \|v_0 - u_0\|, \quad n \ge n_0,$$

where $\epsilon = (1-\alpha) + \alpha\sigma$ *and* $\sigma = \max_{0\le i\le n-1}\{\frac{c_i+a_i+b_i}{1+a_i-b_i}\}$.

Remark 3.4. Corollaries 3.2 and 3.3 show the existence of solutions for the nonlinear operator equations involving the sum and the product of two nonlinear operators, respectively.

REFERENCES

[1] Y.Z. Chen, Fixed points of T-monotone operators, *Nonlinear Anal. TMA* **24(8)** (1995), 1281–1287.

[2] S.S. Chang and W.P. Guo, On the existence and uniqueness theorems of solutions for the systems of mixed monotone operations with applications, *Appl. Math. J. of Chinese Univ.* **8B** (1993), 1–14.

[3] B.C. Dhang, Remarks on two fixed-point theorems involving the sum and the product of two operators, *Comput. Math. Appl.* **46** (2003), 1779–1785.

[4] D.J. Guo, Existence and uniqueness of positive fixed point for mixed monotone operators and applications, *Appl. Anal.* **46** (1992), 91–100.

[5] D.J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Bosten and New York, Academic Press, 1988.

[6] Q.K. Liu and H.Y. Lan, Stable iterative procedures for a class of nonlinear increasing operator equations in Banach spaces, *Nonlinear Funct. Anal. & Appl.* **10(3)** (2005), 345–358.

[7] G.X. Song, Iterative solutions for systems of nonlinear operator equations in Banach space, *Acta Math. Scientia* **23B(4)** (2003), 461–466.

[8] Q.F. Sun, R.G. Li and S.X. Luan, Some fixed point theorem of T-mixed monotone operators, *J. Qufu Normal University* **25(4)** (1999), 13–15.

[9] Y.R. Syan, Some fixed point theorems of T-monotone operators, *J. Math. Anal. Appl.* **205** (1997), 325–329.

[10] A.E. Taylor and D.C. Lay, *Introduction to Functional Analysis*, Second ed. New York: John Wiley & Sons Inc, 1980, 227–281.

[11] L.X. Yu and Y.M. Guo, The convergence of Mann iteration sequences of the unique solution for a class of nonlinear operator equations and applications, *J. Engineering Math.* **20(1)** (2003), 49–54.

[12] Z.T. Zhang, New fixed point theorems of mixed monotone operators and applications, *J. Math. Anal. Appl.* **204** (1996), 307–319.

[13] Z.T. Zhang, Some new results about abstract cones and operators, *Nonlinear Anal. TMA* **37** (1999), 449–455.

[14] K.M. Zhang and X.J. Xie, Solution and coupled minimal-maximal quasi-solutions of nonlinear non-monotone operator equations in Banach Spaces, *J. Math. Research & Exposition* **23(1)** (2003), 47–52.