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ON DELAY DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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ABSTRACT. Some existence results are formulated for delay differential problems with boundary conditions assuming monotonicity of functions on the right hand side of our problem. It is shown that two monotone sequences converge to corresponding limit functions.

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1. INTRODUCTION

The monotone iterative method combined with the method of lower and upper solutions is well known. It can be applied successfully to nonlinear differential problems to obtain some existence results usually in some segments generated by lower and upper solutions. In this paper we apply this method for quite general delay problems with boundary conditions of the form

$$(1.1)\quad \begin{cases} x'(t) &= f(t,x(t),x(\alpha(t))) - g(t,x(t),x(\beta(t))) \equiv Fx(t) - Gx(t), \quad t \in J, \\ x(0) &= \lambda x(T) + k, \end{cases}$$

where $J = [0,T],\ 0 < T < \infty,\ \lambda, k \in \mathbb{R}$ and

$H_1:\ f, g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ \alpha, \beta \in C(J,J),\ 0 \le \alpha(t) \le t,\ 0 \le \beta(t) \le t$ on J.

Note that the boundary condition in (1.1) contains, as special cases, initial condition ($\lambda = 0$), periodic condition ($\lambda = 1$) and anti–periodic condition ($\lambda = -1$) when $k = 0$. Indeed, differential equation in (1.1) contains ordinary differential equations (without delayed arguments) as a special case. Corresponding results for above mentioned special cases of problem (1.1) are obtained, for example, in papers [10, 11, 16, 17]. There are only a few papers when the iterative method is applied to problems of type (1.1) for $g = 0$, see for example, [5, 8, 9, 12, 14, 15], see also [4]. Recently, ordinary differential equations with antiperiodic boundary conditions have been considered in [2, 13], and also with nonlinear boundary conditions in [3, 6, 7]. In paper [5], there are some existence results for delay problems of type (1.1) (with $g = 0$) when f satisfies one–sided Lipschitz condition with corresponding constants.

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In this paper, the monotonicity of f and g is assumed to formulate some existence results. Under such assumption the convergence of two monotone sequences is proved. This paper extends some results of [1] in which initial problems without delays have been considered. Some examples are added to support theoretical results.

2. MAIN RESULTS

We will say that functions $u, v \in C^1(J, \mathbb{R})$ are coupled lower and upper solutions (CLUS for short) of problem (1.1) if

$$\begin{cases} u'(t) & \leq & Fu(t) - Gv(t), \ t \in J, \\ v'(t) & \geq & Fv(t) - Gu(t), \ t \in J, \end{cases}$$

with boundary conditions

$$u(0) \leq \lambda u(T) + k, \quad v(0) \geq \lambda v(T) + k \quad \text{if} \quad \lambda \geq 0, \tag{2.1}$$

or with conditions

$$u(0) \leq \lambda v(T) + k, \quad v(0) \geq \lambda u(T) + k \quad \text{if} \quad \lambda < 0. \tag{2.2}$$

Functions $y, z \in C^1(J, \mathbb{R})$ are coupled extremal quasi–solutions of problem (1.1) i.e. if $U \in C^1(J, \mathbb{R})$ is any solution of (1.1) such that $y_0(t) \leq U(t) \leq z_0(t), \ t \in J$, then $y_0(t) \leq y(t) \leq U(t) \leq z(t) \leq z_0(t), \ t \in J$, where y, z are solutions of system

$$\begin{cases} y'(t) & = & Fy(t) - Gz(t), \ \ t \in J, \\ z'(t) & = & Fz(t) - Gy(t), \quad t \in J \end{cases} \tag{2.3}$$

with boundary conditions

$$y(0) = \lambda y(T) + k, \quad z(0) = \lambda z(T) + k \quad \text{if} \quad \lambda \geq 0, \tag{2.4}$$

or with conditions

$$y(0) = \lambda z(T) + k, \quad z(0) = \lambda y(T) + k \quad \text{if} \quad \lambda < 0. \tag{2.5}$$

Remark 2.1. Let $Gx(t) = 0, \ t \in J$ and $\lambda \geq 0$. In this case, the functions of CLUS of problem (1.1) are known as lower and upper solutions of (1.1). The notion of coupled extremal quasi–solutions of (1.1) denotes extremal solutions of problem (1.1), see, for example [5, 8, 14, 15], see also [10, 11].

Let $Gx(t) = 0, \ t \in J$, and $\lambda < 0$. In this case, the notion of CLUS of problem (1.1) denotes weakly coupled lower and upper solutions of (1.1). Similarly, the notion of coupled extremal quasi–solutions of (1.1) means weakly coupled quasi–solutions of (1.1), see, for example, [4, 5], see also [16, 17].

Theorem 2.2. *Let Assumption H_1 hold. Suppose that*

$H_2:$ $y_0, z_0 \in C^1(J, \mathbb{R})$ *are CLUS of problem* (1.1) *and* $y_0(t) \leq z_0(t)$ *on* J,

H_3 : *functions* f *and* g *are nondecreasing with respect to the last two variables in the sector* $[y_0, z_0] = \{w \in C^1(J, I\!R) : y_0(t) \le w(t) \le z_0(t),\ t \in J\}$.

Then problem (1.1) *has, in the sector* $[y_0, z_0]$, *coupled extremal quasi–solutions.*

Proof. First we consider the case when $\lambda \ge 0$. For $n = 0, 1, \cdots$, we construct two sequences by

$$\begin{cases} y'_{n+1}(t) &= Fy_n(t) - Gz_n(t),\ t \in J, \quad y_{n+1}(0) = \lambda y_n(T) + k, \\ z'_{n+1}(t) &= Fz_n(t) - Gy_n(t),\ t \in J, \quad z_{n+1}(0) = \lambda z_n(T) + k. \end{cases}$$

Note that y_1, z_1 are well defined and $y_1, z_1 \in C^1(J, \mathbb{R})$. Let $p = y_0 - y_1$, so $p(0) \le 0$, and

$$p'(t) \le Fy_0(t) - Gz_0(t) - Fy_0(t) + Gz_0(t) = 0.$$

It yields $y_0(t) \le y_1(t)$ on J. Similarly, we get $z_1(t) \le z_0(t),\ t \in J$. Now we put $q = y_1 - z_1$. Then $q(0) \le 0$, and

$$q'(t) \le Fy_0(t) - Gz_0(t) - Fz_0(t) + Gy_0(t) \le 0,$$

in view of Assumption H_3. Hence $y_1(t) \le z_1(t),\ t \in J$, so

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t),\ \ t \in J.$$

Moreover, by Assumption H_3, we get

$$\begin{aligned} y'_1(t) &= Fy_0(t) - Gz_0(t) \le Fy_1(t) - Gz_1(t), \quad t \in J \quad \text{and} \quad y_1(0) \le \lambda y_1(T) + k, \\ z'_1(t) &= Fz_0(t) - Gy_0(t) \ge Fz_1(t) - Gy_1(t), \quad t \in J \quad \text{and} \quad z_1(0) \ge \lambda z_1(T) + k. \end{aligned}$$

It proves that y_1, z_1 are CLUS of problem (1.1).

Basing on the above, we can prove the relations

$$y_0(t) \le \cdots \le y_{n-1}(t) \le y_n(t) \le z_n(t) \le z_{n-1}(t) \le \cdots \le z_0(t),\ \ n = 0, 1, \cdots,$$

by mathematical induction.

By standard arguments, $y_n \to y,\ z_n \to z$ as $n \to \infty,\ y(t) \le z(t)$ and $y, z \in C^1(J, \mathbb{R})$ are solutions of system (2.3),(2.4). Now, we need to show that y, z are coupled extremal quasi–solutions of problem (1.1). Let u be any solution of (1.1) such that $y_0(t) \le u(t) \le z_0(t),\ t \in J$. Assume that

$$y_m(t) \le u(t) \le z_m(t),\ \ t \in J$$

for some positive m. Put $p = y_{m+1} - u$ on J. Then $p(0) \le 0$, and

$$p'(t) = Fy_m(t) - Gz_m(t) - Fu(t) + Gu(t) \le 0,$$

by Assumption H_3. It gives $y_{m+1}(t) \le u(t)$ on J. Similarly, we can obtain the relation $u(t) \le z_{m+1}(t),\ t \in J$. It results

$$y_0(t) \le y_{m+1}(t) \le u(t) \le z_{m+1}(t) \le z_0(t),\ \ t \in J.$$

Hence, by mathematical induction, we obtain

$$y_n(t) \le u(t) \le z_n(t), \quad t \in J, \quad n = 0, 1, \cdots .$$

Now, if $n \to \infty$, then

$$y_0(t) \le y(t) \le u(t) \le z(t) \le z_0(t), \; t \in J$$

showing that the assertion holds.

In case when $\lambda < 0$, we construct two sequences by

$$\begin{cases} y'_{n+1}(t) &= Fy_n(t) - Gz_n(t), \; t \in J, \quad y_{n+1}(0) = \lambda z_n(T) + k, \\ z'_{n+1}(t) &= Fz_n(t) - Gy_n(t), \; t \in J, \quad z_{n+1}(0) = \lambda y_n(T) + k. \end{cases}$$

The proof is similar to the previous case and therefore it is omitted. This ends the proof. □

Remark 2.3. Assume that $Gx(t) = 0, \; t \in J$. Let $\lambda \ge 0$. In this case, y, z are solutions of problem (1.1). Moreover, y, z are extremal solutions of problem (1.1).

Remark 2.4. Assume that $Fx(t) = 0, \; t \in J$. Then we have a result when $-g(t, x_1, x_2)$ is nonincreasing with respect to the last two variables.

Remark 2.5. Theorem 2.2 deals with the case when f and g are nondecreasing. It is also true when

(i) f and g are nonincreasing,
(ii) f is nonincreasing and g is nondecreasing,
(iii) f is nondecreasing and g is nonincreasing.

Case (i). Note that

$$x'(t) = Fx(t) - Gx(t) = \mathcal{F}x(t) - \mathcal{G}x(t),$$

where

$$\mathcal{F}x(t) = -Gx(t), \qquad \mathcal{G}x(t) = -Fx(t).$$

Changing the order, we see that $-g$ and $-f$ are nondecreasing.

In Case (ii), $f_1 = f - g, \; g_1 = 0$ are nonincreasing; and in Case (iii), $f_2 = f - g, \; g_2 = 0$ are nondecreasing.

Remark 2.6. Assume that there exist nonnegative constants M, N such that

$$f(t, x_1, x_2) - f(t, \bar{x}_1, \bar{x}_2) \ge -M(x_1 - \bar{x}_1) - N(x_2 - \bar{x}_2)$$

if $x_i \ge \bar{x}_i, \; i = 1, 2$.

Consider the problem

$$\begin{cases} x'(t) &= f(t, x(t), x(\alpha(t))) = \tilde{F}x(t) - \tilde{G}x(t), \quad t \in J, \\ x(0) &= \lambda x(T) + k, \end{cases} \tag{2.6}$$

where

$$\begin{aligned}\tilde{F}x(t) &\equiv \tilde{f}(t,x(t),x(\alpha(t))) = f(t,x(t),x(\alpha(t))) + Mx(t) + Nx(\alpha(t)),\\ \tilde{G}x(t) &\equiv \tilde{g}(t,x(t),x(\alpha(t))) = Mx(t) + Nx(\alpha(t)).\end{aligned}$$

Indeed, functions $\tilde{f}, \tilde{g}$ are nondecreasing with respect to 2nd and 3rd variables. We see that Theorem 2.2 can be applied to problems of type (2.6).

Remark 2.7. Suppose that f and g do not depend on the last variable and $\lambda = 0$. Then Theorem 2.2 reduces to Theorem 2.1 of [1].

Remark 2.8. Suppose that f and g do not depend on the last variable. Moreover, we assume that there exists an integrable function $W : J \to \mathbb{R}$ such that

$$|\lambda| e^{\int_0^T W(s)ds} < 1 \tag{2.7}$$

and

$$f(t,u_1) + g(t,u_1) - f(t,u_2) - g(t,u_2) \le W(t)(u_1 - u_2)$$

for $y_0(t) \le u_2 \le u_1 \le z_0(t)$.

Then problem (1.1) has, in the sector $[y_0, z_0]$, a unique solution.

To show it we consider the case when $\lambda \ge 0$. It results from Theorem 2.2 that $y_n \to y,\ z_n \to z$ on J, $y(t) \le z(t),\ t \in J$, and y, z are solutions of system (2.3),(2.4). To show that $y = z$, we put $p = z - y$. Then $p(0) = \lambda p(T)$, and

$$p'(t) = Fz(t) - Fy(t) - Gy(t) + Gz(t) \le W(t)p(t).$$

It yields

$$p(t) \le e^{\int_0^t W(s)ds} p(0),\ t \in J.$$

This, boundary condition and (2.7) say that $z = y$ on J. It shows that y, z are solutions of (1.1). Moreover, y, z are coupled extremal quasi–solutions of problem (1.1). Since $y = z$ on J, it proves that the assertion holds. By the similar way we can prove this result when $\lambda < 0$.

Theorem 2.9 (see Theorem 4 of [3])**.** *Let $Gx(t) = 0,\ t \in J$ and $\lambda \ge 0$. Let Assumptions H_1 and H_2 hold. Suppose that*

H_4 : *there exist nonnegative constants M, N,*

$$\begin{array}{lll}(i) & N(e^{MT} - 1) \le M & \textit{only if } M > 0 \textit{ and } N > 0,\\ (ii) & NT \le 1 & \textit{only if } N > 0 \textit{ and } M = 0\end{array}$$

and such that

$$f(t,\bar{x}_1,\bar{x}_2) - f(t,x_1,x_2) \le M(x_1 - \bar{x}_1) + N(x_2 - \bar{x}_2) \ \textit{ if } x_1 \ge \bar{x}_1,\ x_2 \ge \bar{x}_2.$$

Then problem (1.1) *has extremal solutions in the sector* $[y_0, z_0]$.

Remark 2.10. Condition (i) can be improved by the following

$$N\int_0^T e^{M(t-\alpha(t))}dt \le 1,$$

see Theorem 2.4 of [15].

3. EXAMPLES

Example 3.1. Consider the problem

$$\left\{\begin{array}{rcl} x'(t) & = & x(t) - bx\left(\frac{1}{2}t\right) + c \equiv F_1x(t), \ t \in J = [0, \ln 5], \\ x(0) & = & k \quad \text{for} \quad k = 1, \end{array}\right. \tag{3.1}$$

where

$$5b \le c \le 1 + b, \ b, c > 0. \tag{3.2}$$

Note that $M = 0,\ N = b$, from Theorem 2.9. Put $y_0(t) = e^t,\ \ z_0(t) = e^{2t}$. Indeed,

$$\begin{array}{rcl} F_1y_0(t) & = & e^t - be^{\frac{1}{2}t} + c \ge e^t - \sqrt{5}b + c > e^t = y_0'(t), \\ F_1z_0(t) & = & e^{2t} - be^t + c = 2e^{2t} - e^{2t} - be^t + c \le 2e^{2t} + c - (1 + b) \le 2e^{2t} = z_0'(t), \end{array}$$

and $y_0(0) \le k,\ z_0(0) \ge k$. In view of condition (3.2), $b \le \frac{1}{4}$. It yields

$$b \ln 5 \le \frac{1}{4}\ln 5 < 1,$$

so condition (ii) of Theorem 2.9 holds. By Theorem 2.9, problem (3.1) has, in the sector $[e^t, e^{2t}]$, extremal solutions.

Now, we consider again problem (3.1) using another approach. Indeed, problem (3.1) is identical with the following

$$\left\{\begin{array}{rcl} x'(t) & = & Fx(t) - Gx(t), \quad t \in J, \\ x(0) & = & k, \end{array}\right.$$

where

$$Fx(t) = x(t) + c, \qquad Gx(t) = bx\left(\frac{1}{2}t\right).$$

Note that F and G are nondecreasing. Keep y_0, z_0 as above. We see, that

$$\begin{array}{rcl} Fy_0(t) - Gz_0(t) & = & e^t + c - be^t \ge e^t + c - 5b \ge e^t = y_0'(t), \\ Fz_0(t) - Gy_0(t) & = & e^{2t} + c - be^{\frac{1}{2}t} \le 2e^{2t} + c - (1 + b) \le 2e^{2t} = z_0'(t). \end{array}$$

By Theorem 2.2, problem (3.1) has, in the segment $[e^t, e^{2t}]$, coupled extremal quasi–solutions $y, z \in C^1(J, \mathbb{R})$. Functions y, z are solutions of the system

$$\left\{\begin{array}{rcl} y'(t) & = & y(t) - bz\left(\frac{1}{2}t\right) + c, \quad t \in J, \quad y(0) = k, \\ z'(t) & = & z(t) - by\left(\frac{1}{2}t\right) + c, \quad t \in J, \quad z(0) = k. \end{array}\right.$$

We need to show that $y = z$ on J. To do it we put $p = z - y$. Then

$$\left\{\begin{array}{rcl} p'(t) & = & p(t) + bp\left(\frac{1}{2}t\right), \quad t \in J, \\ p(0) & = & 0. \end{array}\right. \tag{3.3}$$

By Theorem 2.2 of [5], problem (3.3) has a unique solution. We see that $p(t) = 0,\ t \in J$ is this unique solution. It proves that $z(t) = y(t),\ t \in J$, so functions y, z are solutions of problem (3.1). Again, by Theorem 2.2, functions y, z are extremal solutions of problem (3.1). If we take, for example, $b = \frac{1}{4},\ c = \frac{5}{4}$, then condition (3.2) holds.

Example 3.2. Consider the problem

$$\tag{3.4} \begin{cases} x'(t) &= \cos^2 x(t) + x(t) - bx\left(\frac{1}{2}t\right) \equiv Fx(t) - Gx(t), \quad t \in J = [0, T], \\ x(0) &= \lambda x(T), \quad 0 \le \lambda \le e^{-2T}, \quad 0 < b \le e^{-T}, \end{cases}$$

where $Fx(t) = \cos^2 x(t) + x(t),\ \ Gx(t) = bx\left(\frac{1}{2}t\right),\ 0 < T < \infty$.

Let $y_0(t) = 0,\ z_0(t) = e^{2t}$. Then

$$\begin{aligned} Fy_0(t) - Gz_0(t) &= 1 - be^t \ge 1 - be^T \ge 0 = y_0'(t), \\ Fz_0(t) - Gy_0(t) &= \cos^2 e^{2t} + e^{2t} \le 1 + 2e^{2t} - e^{2t} \le 2e^{2t} = z_0'(t), \end{aligned}$$

and

$$y_0(0) - \lambda y_0(T) = 0, \qquad z_0(0) - \lambda z_0(T) = 1 - \lambda e^{2T} \ge 0.$$

By Theorem 2.2, problem (3.4) has, in the sector $[0, e^{2t}]$, coupled extremal quasi–solutions $y, z \in C^1(J, \mathbb{R})$. Functions y, z are solutions of system (2.3),(2.4).

If we are going to apply Theorem 2.9 to problem (3.4), then we see that $M = 0,\ N = b$, so problem (3.4) has extremal solutions in $[0, e^{2t}]$ provided that $bT \le 1$.

Example 3.3. Consider the problem

$$\tag{3.5} \begin{cases} x'(t) &= Fx(t) - Gx(t), \quad t \in J = [0, 1], \\ x(0) &= \lambda x(1) + 1, \quad -e^{-1} \le \lambda < 0, \end{cases}$$

where

$$Fx(t) = Ae^{e^{-1}x(t)}, \quad Gx(t) = e^{-2}x\left(\frac{1}{2}t\right), \quad e^{-1.5} \le A \le e^{-1}.$$

Put $y_0(t) = 0,\ z_0(t) = e^t$. Then

$$\begin{aligned} Fy_0(t) - Gz_0(t) &= A - e^{-2}e^{\frac{1}{2}t} \ge A - e^{-1.5} \ge 0 = y_0'(t), \\ Fz_0(t) - Gy_0(t) &= Ae^{e^{-1}e^t} \le Ae \le 1 = z_0'(t), \end{aligned}$$

and

$$\lambda z_0(1) + 1 = \lambda e + 1 \ge 0 = y_0(0), \quad \lambda y_0(1) + 1 = 1 = z_0(0).$$

By Theorem 2.2, problem (3.5) has, in the sector $[y_0, z_0]$, coupled extremal quasi–solutions.

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