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OSCILLATION OF LINEAR SECOND ORDER MATRIX DIFFERENTIAL SYSTEMS WITH DAMPING

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ABSTRACT. In this paper, sufficient conditions have been obtained for the oscillation of a class of linear second order matrix differential systems with damping.

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1. INTRODUCTION

In this paper, we consider oscillatory properties for the linear second order matrix differential system with damping

$$X'' + P(t)X' + Q(t)X = 0, \quad t \in [t_0, \infty), \tag{1}$$

where $P(t)$, $Q(t)$ and $X(t)$ are $n \times n$ continuous matrix-valued functions, $P(t)$ and $Q(t)$ are symmetric. When $P(t) \equiv 0$, system (1) reduces to the linear second order matrix differential system

$$X'' + Q(t)X = 0, \quad t \in [t_0, \infty). \tag{2}$$

By M^* we mean the transpose of the matrix M, for any $n \times n$ symmetric matrix M, its eigenvalues are real numbers. We always denote by $\lambda_1[M] \geq \lambda_2[M] \geq \ldots \geq \lambda_n[M]$, and as usual, $tr[M] = \sum_{i=1}^{n} \lambda_i[M]$. A solution $X(t)$ of (1) (or(2)) is said to be nontrivial solution if $\det X(t) \neq 0$ for at least one point $t \in [t_0, \infty)$, and a solution $X(t)$ of (1) (or(2)) is said to be prepared if

$$X^*(t)X'(t) - (X^*(t))'X(t) \equiv 0, \quad t \in [t_0, \infty).$$

System (1) (or (2)) is said to be oscillatory on $[t_0, \infty)$ in case the determinant of every nontrivial prepared solution vanishes for at least one point on $[T, \infty)$ for each $T \geq t_0$.

The oscillation and nonoscillation of system (2) have been extensively studied by many authors (see [1 – 9, 11, 12, 16 – 18, 22]). A discrete version of (2) is studied in [19].

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We recall the following concept from [3]. For any subset E of the real line R, $\mu(E)$ denote the Lebesgue measure of E. If $f : [t_0, \infty) \longrightarrow R$ is continuous and if $l,\ m$ satisfy $-\infty \le l,\ m \le \infty$, then lim approxinf$_{t\to\infty}f(t) = l$ if and only if $\mu\{t \in [t_0, \infty) \mid f(t) \le l_1\} < +\infty$ for all $l_1 < l$ and $\mu\{t \in [t_0, \infty) \mid f(t) \le l_2\} = +\infty$ for all $l_2 > l$. Similarly, lim approxsup$_{t\to\infty}f(t) = m$ if and only if $\mu\{t \in [t_0, \infty) \mid f(t) \ge m_1\} = +\infty$ for all $m_1 < m$ and $\mu\{t \in [t_0, \infty) \mid f(t) \ge m_2\} < +\infty$ for all $m_2 > m$. We define lim approx$_{t\to\infty}f(t) = \lambda$ in case

$$\lim_{t\to\infty} \text{approxsup} f(t) = \lim_{t\to\infty} \text{approxinf} f(t) = \lambda.$$

In general,

$$\liminf_{t\to\infty} f(t) \le \lim_{t\to\infty} \text{approxinf} f(t) \le \lim_{t\to\infty} \text{approxsup} f(t) \le \limsup_{t\to\infty} f(t).$$

The motivation for the present work has come chiefly from [3]. Here we list the main results of [3] as follows:

TheoremA[3,Theorem2.1]. Assume that

$$\liminf_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} \int_{t_0}^{t} tr\,(Q(s))\, \mathrm{d}s\mathrm{d}t > -\infty.$$

Then system (2) is oscillatory in case any of the following conditions holds:

$$\limsup_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} \lambda_1 \left(\int_{t_0}^{t} Q(s)\mathrm{d}s \right) \mathrm{d}t = +\infty,$$

$$\limsup_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} \left[\lambda_1 \left(\int_{t_0}^{t} Q(s)\mathrm{d}s \right) \right]^2 \mathrm{d}t = \infty,$$

$$\lim \text{approxsup}_{T\to\infty} \lambda_1 \left(\int_{t_0}^{T} Q(s)\mathrm{d}s \right) = +\infty,$$

$$\lim \text{approxinf}_{T\to\infty} \lambda_1 \left(\int_{t_0}^{T} Q(s)\mathrm{d}s \right) = -\infty.$$

Nevertheless, oscillation of system with damping has drawn less attention [22], so we are concerned with extending oscillation criteria for system (2) to that of the damped linear second order matrix differential system (1). The purpose of this paper is to establish some new oscillation criteria for system (1). The criteria extend and improve the main results of Butler, Erbe and Mingarelli [3] and Parhi and Praharaj [18] for system (2).

2. MAIN RESULTS

Let $f(t)$ be a smooth and real-valued function on $[t_0, \infty)$, and let

$$a(t) = \exp(-2\int_{t_0}^{t} f(s)ds).$$

If a prepared solution $X(t)$ of (1) is nonoscillatory, then $X(t)$ is nonsingular for all sufficiently large t, without loss of generality, say $t \geq t_0$. Let

$$V(t) = a(t)[X'(t)X^{-1}(t) + f(t)I], \quad t \geq t_0. \tag{3}$$

Where I is the $n \times n$ identity matrix. Then $V(t)$ is symmetric and for any $t \geq t_0$, from (1), we get

$$V'(t) = -\frac{1}{a(t)}V^2(t) - P(t)V(t) - R(t), \tag{4}$$

where

$$R(t) = a(t)\left\{Q(t) + f^2(t)I - f(t)P(t) - f'(t)I\right\}.$$

From (4) and the definition of prepared solution, we have PV is symmetric. Denote

$$W(t) = V(t) + \frac{a(t)}{2}P(t). \tag{5}$$

Then we obtain

$$W'(t) = -\frac{1}{a(t)}W^2(t) + \frac{a(t)}{4}P^2(t) - R(t) + \left(\frac{a(t)}{2}P(t)\right)', \tag{6}$$

or

$$W(t) = W(t_0) - \int_{t_0}^{t}\left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \int_{t_0}^{t} a^{-1}(s)W^2(s)\mathrm{d}s$$

$$+\frac{a(t)}{2}P(t) - \frac{a(t_0)}{2}P(t_0). \tag{7}$$

Further, $W^*(t) = W(t)$ due to (5).

In the sequel, we need the following lemmas.

Lemma1[22]. Suppose A, B and C are $n \times n$-matrices, and A is symmetric. Then

(i) $[\lambda_1(\mathrm{A})]^2 \leq \lambda_1[\mathrm{A}^2] \leq \mathrm{tr}[\mathrm{A}^2]$

(ii) $[\mathrm{tr}(\mathrm{A})]^2 \leq \mathrm{ntr}[\mathrm{A}^2]$

(iii) $\dfrac{\mathrm{tr}(\mathrm{B})}{\mathrm{n}} \leq \lambda_1(\mathrm{B})$

(iv) $\mathrm{tr}[(\mathrm{B}+\mathrm{C})^2] \leq 2\left(\mathrm{tr}[\mathrm{B}^2] + \mathrm{tr}[\mathrm{C}^2]\right)$

Lemma2. Assume that (1) is nonoscillatory on $[t_0, \infty)$, and $a(t) \leq m^*$ ($m^* > 0$ is a constant), for $t \in [t_0, \infty)$. Then

$$0 < \lim_{T\to\infty}\int_{t}^{T} a^{-1}(s)W^2(s)\mathrm{d}s < +\infty, \qquad t \geq t_0, \tag{8}$$

if and only if

$$\liminf_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left(tr\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]\right)\mathrm{d}t>-\infty. \tag{9}$$

Proof. Suppose (8) holds. Thus,

$$0<\int_{t}^{\infty}a^{-1}(s)W^2(s)\mathrm{d}s<\infty,\quad t\geq t_0. \tag{10}$$

From (7), we obtain

$$tr[W(t)]=tr\left[W(t_0)-\frac{a(t_0)}{2}P(t_0)\right]-tr\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]$$

$$-\int_{t_0}^{t}a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s, \tag{11}$$

that is,

$$tr\left[W(t)\right]-M(t)=-tr\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]+L, \tag{12}$$

where

$$L=tr\left[W(t_0)-\frac{a(t_0)}{2}P(t_0)\right]-\int_{t_0}^{\infty}a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s$$

and

$$M(t)=\int_{t}^{\infty}a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s.$$

Since $a(t)\leq m^*$, from (10) and Lemma1, it follows that

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left(tr\left[W(s)\right]\right)^2\mathrm{d}s\leq nm^*\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s=0.$$

That is,

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left(tr\left[W(s)\right]\right)^2\mathrm{d}s=0. \tag{13}$$

Moreover, by (10), we observe that $\lim_{t\to\infty}M(t)=0$, i.e. for every $\epsilon>0$, it is possible to find a $t_1>t_0$, such that for $t\geq t_1$, $M(t)<\epsilon$. Hence,

$$\frac{1}{T}\int_{t_0}^{T}M^2(t)\mathrm{d}t=\frac{1}{T}\int_{t_0}^{t_1}M^2(t)\mathrm{d}t+\frac{1}{T}\int_{t_1}^{T}M^2(t)\mathrm{d}t\leq\frac{1}{T}\int_{t_0}^{t_1}M^2(t)\mathrm{d}t+\frac{T-t_1}{T}\epsilon^2.$$

Thus,

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}M^2(t)\mathrm{d}t\leq\epsilon^2.$$

Since ϵ is arbitrary, then

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}M^2(t)\mathrm{d}t=0. \tag{14}$$

From (13) and (14), we have

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left(tr\left[W(t)\right]-M(t)\right)^2\mathrm{d}t\leq 2\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left\{\left(tr\left[W(t)\right]\right)^2+M^2(t)\right\}\mathrm{d}t=0.$$

Thus, from (12) it follows that

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left\{-tr\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]+L\right\}^2\mathrm{d}t=0. \tag{15}$$

By the Cauchy-Schwartz inequality,

$$\left|\frac{1}{T}\int_{t_0}^{T}\left\{-tr\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]+L\right\}\mathrm{d}t\right|$$

$$\leq\left[\frac{1}{T}\int_{t_0}^{T}\left\{-tr\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]+L\right\}^2\mathrm{d}t\right]^{\frac{1}{2}}\times\left[\frac{T-t_0}{T}\right]^{\frac{1}{2}}.$$

Hence, by (15),

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left\{-tr\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]+L\right\}\mathrm{d}t=0,$$

that is,

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}tr\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]\mathrm{d}t=L>-\infty,$$

so that (9) holds.

Conversely, suppose that (9) holds. From (7), we have

$$\frac{1}{T}\int_{t_0}^{T}W(t)\mathrm{d}t+\frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t}a^{-1}(s)W^2(s)\mathrm{d}s\mathrm{d}t=\frac{T-t_0}{T}\left(W(t_0)-\frac{a(t_0)}{2}P(t_0)\right)-$$
$$\frac{1}{T}\int_{t_0}^{T}\left\{\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right\}\mathrm{d}t,$$

so that from (9), we obtain

$$\limsup_{T\to\infty}\left[\frac{1}{T}\int_{t_0}^{T}tr\left[W(t)\right]\mathrm{d}t+\frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t}a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s\mathrm{d}t\right]<\infty. \tag{16}$$

Since $a^{-1}(t)tr\left[W^2(t)\right]\geq 0$, for $t\geq t_0$, it follows that

$$\lim_{t\to\infty}\int_{t_0}^{t}a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s\quad\text{exists, finite or infinite.}$$

Suppose that

$$\lim_{t\to\infty}\int_{t_0}^{t}a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s=+\infty.$$

Hence,

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t}a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s\mathrm{d}t=+\infty.$$

Then (16) yields

$$\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}tr\left[W(t)\right]\mathrm{d}t=-\infty.$$

So for large T we have, again using (16),

$$\frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t}a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s\mathrm{d}t\leq-\frac{2}{T}\int_{t_0}^{T}tr\left[W(t)\right]\mathrm{d}t. \tag{17}$$

Now by the Cauchy-Schwartz inequality and Lemma1, we have

$$\left|\frac{1}{T}\int_{t_0}^{T} tr\,[W(s)]\,\mathrm{d}s\right| \le \left\{\frac{1}{T}\int_{t_0}^{T} a(s)a^{-1}(s)\,(tr\,[W(s)])^2\,\mathrm{d}s\right\}^{\frac{1}{2}} \times \left\{\frac{T-t_0}{T}\right\}^{\frac{1}{2}}$$

$$\le \left\{\frac{m^*}{T}\int_{t_0}^{T} a^{-1}(s)\,(tr\,[W(s)])^2\,\mathrm{d}s\right\}^{\frac{1}{2}}$$

$$\le \left\{\frac{nm^*}{T}\int_{t_0}^{T} a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s\right\}^{\frac{1}{2}}.$$

So that (17) gives

$$\left\{\frac{1}{T}\int_{t_0}^{T}\int_{t_0}^{t} a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s\mathrm{d}t\right\}^2 \le \frac{4nm^*}{T}\int_{t_0}^{T} a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s, \tag{18}$$

for large T, say, for $T \ge T_1$. Setting, for $T \ge T_1$,

$$H(T) = \int_{t_0}^{T}\int_{t_0}^{t} a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s\mathrm{d}t > 0,$$

We obtain

$$H'(T) = \int_{t_0}^{T} a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s.$$

Thus, (18) yields $H^2(T) \le 4nm^*TH'(T)$, for $T \ge T_1$. Integrating this inequality from T_1 to T and noting that $H(T) > 0$, for $T \ge T_1$, we get

$$\frac{1}{4nm^*}\left[\log T - \log T_1\right] \le \frac{1}{H(T_1)}.$$

A contradiction is obtained as $T \to \infty$. Thus,

$$\lim_{t\to\infty}\int_{t_0}^{t} a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s \quad \text{exists as a finite limit.}$$

We see that this implies the existence of $\lim_{t\to\infty}\int_{t_0}^{t} a^{-1}(s)W^2(s)\mathrm{d}s$, as follows: let the (operator) norm of a matrix A be denoted by $|A|$. For $t_0 \le s \le t$, define $A(s,t)$ by $A(s,t) = \int_s^t a^{-1}(\sigma)W^2(\sigma)\mathrm{d}\sigma$. Then $A(s,t)$ is a nonnegative definite matrix and $|A(s,t)| = \lambda_1\left[A(s,t)\right] \le tr\left[A(s,t)\right] = \int_s^t a^{-1}(\sigma)tr\left[W^2(\sigma)\right]\mathrm{d}\sigma$. This last integral converges to zero as $s,\ t \longrightarrow \infty$, and so we have $|A(s,t)| \longrightarrow 0$ as $s,\ t \longrightarrow \infty$, i.e. $\int_s^t a^{-1}(\sigma)W^2(\sigma)\mathrm{d}\sigma \longrightarrow 0$ as $s,\ t \longrightarrow \infty$, yielding the existence of $\lim_{t\to\infty}\int_{t_0}^{t} a^{-1}(s)W^2(s)\mathrm{d}s$ as asserted. This completes the proof of Lemma2.

Now we give the main results of this paper.

Theorem1. Assume there exist a smooth and real-valued function $f(t)$ on $[t_0,\infty)$ and $a(t) \le m^*$ ($m^* > 0$ is a constant), where $a(t) = \exp(-2\int_{t_0}^{t} f(s)\mathrm{d}s)$, and

$$\liminf_{T\to\infty}\frac{1}{T}\int_{t_0}^{T} tr\left[\int_{t_0}^{t}\left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right]\mathrm{d}t > -\infty.$$

If one of the conditions

$$\limsup_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\lambda_1\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]\mathrm{d}t=+\infty, \tag{19}$$

$$\limsup_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}\left(\lambda_1\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]\right)^2\mathrm{d}t=+\infty, \tag{20}$$

$$\lim\ \mathrm{approxsup}_{T\to\infty}\lambda_1\left[\int_{t_0}^{T}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]=+\infty, \tag{21}$$

$$\lim\ \mathrm{approxinf}_{T\to\infty}\lambda_1\left[\int_{t_0}^{T}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]=-\infty, \tag{22}$$

holds, where $R(t)=a(t)\{Q(t)+f^2(t)I-f(t)P(t)-f'(t)I\}$, then (1) is oscillatory.

Proof. Suppose that (1) is nonoscillatory.Then there exists a prepared solution $X(t)$ of (1) which is not oscillatory. Without loss of generality, we may suppose that $\det X(t)\neq 0$ for $t\geq t_0$. Denote $V(t)$ by (3); then we have that (7) holds. By (9) and Lemma2, it follows that (8) holds.

Suppose that (19) holds. From (7), we obtain

$$\lambda_1\left[W(t_0)-\frac{a(t_0)}{2}P(t_0)-W(t)\right]=\lambda_1\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right.$$
$$\left.+\int_{t_0}^{t}a^{-1}(s)W^2(s)\mathrm{d}s\right]. \tag{23}$$

By the convexity of λ_1 and the fact that $\int_{t_0}^{t}a^{-1}(s)W^2(s)\mathrm{d}s\geq 0$, we have from (23) that

$$\lambda_1\left[W(t_0)-\frac{a(t_0)}{2}P(t_0)\right]+\lambda_1\left[-W(t)\right]\geq\lambda_1\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right],$$

and hence,

$$\frac{1}{T}\int_{t_0}^{T}\lambda_1\left[-W(s)\right]\mathrm{d}s+\frac{T-t_0}{T}\lambda_1\left[W(t_0)-\frac{a(t_0)}{2}P(t_0)\right]$$
$$\geq\frac{1}{T}\int_{t_0}^{T}\lambda_1\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]\mathrm{d}t. \tag{24}$$

So that from hypothesis (19), there exists a sequence $T_n\to\infty$ with

$$\frac{1}{T_n}\int_{t_0}^{T_n}\lambda_1\left[-W(s)\right]\mathrm{d}s\longrightarrow+\infty,\qquad \text{as }\ T_n\to\infty. \tag{25}$$

By the Cauchy-Schwartz inequality and from Lemma1, we have

$$\left|\frac{1}{T_n}\int_{t_0}^{T_n}\lambda_1\left[-W(s)\right]\mathrm{d}s\right|\leq\left(\frac{1}{T_n}\int_{t_0}^{T_n}\left(\lambda_1\left[-W(s)\right]\right)^2\mathrm{d}s\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{T_n} \int_{t_0}^{T_n} \lambda_1 \left[W^2(s) \right] \mathrm{d}s \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{T_n} \int_{t_0}^{T_n} tr \left[W^2(s) \right] \mathrm{d}s \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{m^*}{T_n} \int_{t_0}^{T_n} a^{-1}(s) tr \left[W^2(s) \right] \mathrm{d}s \right)^{\frac{1}{2}},$$

thus from (25), we get

$$\lim_{n\to\infty} \frac{1}{T_n} \int_{t_0}^{T_n} a^{-1}(s) tr \left[W^2(s) \right] \mathrm{d}s = +\infty,$$

which in turns implies that

$$\lim_{n\to\infty} \int_{t_0}^{T_n} a^{-1}(s) tr \left[W^2(s) \right] \mathrm{d}s = +\infty.$$

On the other hand, Lemma2 implies that

$$\lim_{n\to\infty} \int_{t_0}^{T_n} a^{-1}(s) tr \left[W^2(s) \right] \mathrm{d}s < +\infty.$$

This contradiction completes the proof of the part under the assumption (19) of the theorem.

Let (20) be true. From (7), we have

$$\int_{t_0}^{t} \left(R(s) - \frac{a(s)}{4} P^2(s) \right) \mathrm{d}s - \frac{a(t)}{2} P(t)$$

$$= -W(t) + W(t_0) - \frac{a(t_0)}{2} P(t_0) - \int_{t_0}^{\infty} a^{-1}(s) W^2(s) \mathrm{d}s + \int_{t}^{\infty} a^{-1}(s) W^2(s) \mathrm{d}s. \tag{26}$$

Hence, by (26) and Lemma1 we have

$$\left\{ \lambda_1 \left[\int_{t_0}^{t} \left(R(s) - \frac{a(s)}{4} P^2(s) \right) \mathrm{d}s - \frac{a(t)}{2} P(t) \right] \right\}^2$$

$$= \left\{ \lambda_1 \left[-W(t) + W(t_0) - \frac{a(t_0)}{2} P(t_0) - \int_{t_0}^{\infty} a^{-1}(s) W^2(s) \mathrm{d}s + \int_{t}^{\infty} a^{-1}(s) W^2(s) \mathrm{d}s \right] \right\}^2$$

$$\leq \lambda_1 \left(\left[-W(t) + W(t_0) - \frac{a(t_0)}{2} P(t_0) - \int_{t_0}^{\infty} a^{-1}(s) W^2(s) \mathrm{d}s + \int_{t}^{\infty} a^{-1}(s) W^2(s) \mathrm{d}s \right]^2 \right)$$

$$\leq tr \left(\left[-W(t) + W(t_0) - \frac{a(t_0)}{2} P(t_0) - \int_{t_0}^{\infty} a^{-1}(s) W^2(s) \mathrm{d}s + \int_{t}^{\infty} a^{-1}(s) W^2(s) \mathrm{d}s \right]^2 \right)$$

$$\leq 2tr \left[\left(-W(t) + \int_{t}^{\infty} a^{-1}(s) W^2(s) \mathrm{d}s \right)^2 \right]$$

$$+2tr \left[\left(W(t_0) - \frac{a(t_0)}{2} P(t_0) - \int_{t_0}^{\infty} a^{-1}(s) W^2(s) \mathrm{d}s \right)^2 \right]$$

$$\leq 4tr\left[W^2(t)\right] + 4tr\left[\left(\int_t^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right)^2\right]$$

$$+2tr\left[\left(W(t_0) - \frac{a(t_0)}{2}P(t_0) - \int_{t_0}^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right)^2\right]. \tag{27}$$

Let $2tr\left[\left(W(t_0) - \frac{a(t_0)}{2}P(t_0) - \int_{t_0}^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right)^2\right] = C$. Using (27), we obtain

$$\left\{\lambda_1\left[\int_{t_0}^t \left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right]\right\}^2$$

$$\leq \quad 4tr\left[W^2(t)\right] + 4tr\left[\left(\int_t^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right)^2\right] + C.$$

Therefore, we get

$$\frac{1}{T}\int_{t_0}^T \left\{\lambda_1\left[\int_{t_0}^t \left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right]\right\}^2 \mathrm{d}t$$

$$\leq \frac{4}{T}\int_{t_0}^T tr\left[W^2(s)\right]\mathrm{d}s + \frac{4}{T}\int_{t_0}^T tr\left[\left(\int_t^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right)^2\right]\mathrm{d}t + \frac{T-t_0}{T}C. \tag{28}$$

If we set

$$B(t) = \int_t^\infty a^{-1}(s)W^2(s)\mathrm{d}s,$$

then $tr\,[B(t)] \longrightarrow 0(t \longrightarrow \infty)$ and $B(t) \geq 0$. So that $\lambda_1\,[B(t)] \longrightarrow 0(t \longrightarrow \infty)$ and hence $\lambda_1\,[B^2(t)] \longrightarrow 0(t \longrightarrow \infty)$. Therefore $tr\,[B^2(t)] \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, the second integral on the right side of (28) tend to 0 as $T \longrightarrow \infty$. From Lemma2, we get

$$\frac{1}{T}\int_{t_0}^T tr\left[W^2(s)\right]\mathrm{d}s \leq \frac{m^*}{T}\int_{t_0}^T a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s \longrightarrow 0, \quad \text{as } T \longrightarrow \infty.$$

Thus,

$$\frac{1}{T}\int_{t_0}^T tr\left[W^2(s)\right]\mathrm{d}s \longrightarrow 0, \quad \text{as } T \longrightarrow \infty.$$

Therefore, the first integral on the right side of (28) tend to 0 as $T \longrightarrow \infty$. However, condition (20) implies that the left side of (28) is not bounded and this contradiction completes the proof of the part under the assumption (20) of the theorem.

Let us assume that (21) holds. From (26), we have

$$\lambda_1\left[\int_{t_0}^t \left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right] \leq \lambda_1\left[-W(t)\right] + \lambda_1\left[W(t_0) - \frac{a(t_0)}{2}P(t_0)\right]$$

$$-\lambda_n\left[\int_{t_0}^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right] + \lambda_1\left[\int_t^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right]. \tag{29}$$

Using (29), we obtain

$$\lambda_1 [-W(t)] + \lambda_1 \left[\int_t^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right]$$

$$\geq \lambda_1 \left[\int_{t_0}^t \left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right]$$

$$+\lambda_n \left[\int_{t_0}^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right] - \lambda_1 \left[W(t_0) - \frac{a(t_0)}{2}P(t_0)\right]. \tag{30}$$

By (21), for any $k \geq 1$,

$$\mu\left\{t \in [t_0, \infty) : \lambda_1 \left[\int_{t_0}^t \left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right] \geq k\right\} = +\infty.$$

So that if

$$k \geq 1 + \left|-\lambda_1 \left[W(t_0) - \frac{a(t_0)}{2}P(t_0)\right] + \lambda_n \left[\int_{t_0}^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right]\right|,$$

then from (30),

$$\mu\left\{t \in [t_0, \infty) : \lambda_1[-W(t)] + \lambda_1 \left[\int_t^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right] \geq 1\right\} = +\infty. \tag{31}$$

Setting $D(t) = \int_t^\infty a^{-1}(s)W^2(s)\mathrm{d}s$, by Lemma2, we get $tr[D(t)] \longrightarrow 0$ as $t \longrightarrow \infty$ and $D(t) \geq 0$, thus, $\lambda_1[D(t)] \longrightarrow 0 (t \longrightarrow \infty)$. So that there exists a $T_0 > t_0$, such that $t \geq T_0$ implies that $\lambda_1[D(t)] < \frac{1}{2}$, therefore, we see that

$$\mu\left\{t \in [t_0, \infty) : \lambda_1 [-W(t)] \geq \frac{1}{2}\right\} = +\infty.$$

That is,

$$\int_{E_k} (\lambda_1 [-W(t)])^2 \mathrm{d}t \ \geq \ \frac{1}{4}\mu(E_k) = +\infty,$$

where

$$E_k = \left\{t \in [t_0, \infty) : \lambda_1 [-W(t)] \geq \frac{1}{2}\right\}.$$

On the other hand,

$$\begin{aligned}\int_{E_k} (\lambda_1 [-W(t)])^2 \mathrm{d}t &\leq \int_{E_k} \lambda_1 \left[W^2(t)\right] \mathrm{d}t \\ &\leq \int_{E_k} tr \left[W^2(t)\right] \mathrm{d}t \\ &\leq \int_{E_k} m^* a^{-1}(s) tr \left[W^2(s)\right] \mathrm{d}s \\ &\leq \int_{t_0}^\infty m^* a^{-1}(s) tr \left[W^2(s)\right] \mathrm{d}s < +\infty\end{aligned}$$

due to (8) and Lemma1. It is a contradiction. Hence, the proof of the part under the assumption (21) of the theorem is complete.

Suppose that (22) holds. From (26), we have

$$\lambda_1\left[-W(t)\right] \le \lambda_1\left[-W(t)+\int_t^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right]$$
$$\le \lambda_1\left[\int_{t_0}^t\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]$$
$$-\lambda_n\left[W(t_0)-\frac{a(t_0)}{2}P(t_0)\right]+\lambda_1\left[\int_{t_0}^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right]. \tag{32}$$

By (22), for any $k>0$,

$$\mu\left\{t\in[t_0,\infty):\lambda_1\left[\int_{t_0}^t\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]<-k\right\}=+\infty.$$

So that if

$$k>1+\left|-\lambda_n\left[W(t_0)-\frac{a(t_0)}{2}P(t_0)\right]+\lambda_1\left(\int_{t_0}^\infty a^{-1}(s)W^2(s)\mathrm{d}s\right)\right|,$$

then from (32), we get

$$\mu\left\{t\in[t_0,\infty):\lambda_1\left[-W(t)\right]\le -1\right\}=+\infty,$$

therefore,

$$\int_{t_0}^\infty\left(\lambda_1\left[-W(t)\right]\right)^2\mathrm{d}t\ge\int_{E_k}\left(\lambda_1\left[-W(t)\right]\right)^2\mathrm{d}t\ge\mu(E_k)=+\infty,$$

where,

$$E_k=\left\{t\in[t_0,\infty):\lambda_1\left[-W(t)\right]\le -1\right\}.$$

However,

$$\int_{t_0}^\infty\left(\lambda_1\left[-W(t)\right]\right)^2\mathrm{d}t\le\int_{t_0}^\infty\lambda_1\left[W^2(t)\right]\mathrm{d}t$$
$$\le\int_{t_0}^\infty tr\left[W^2(t)\right]\mathrm{d}t$$
$$\le\int_{t_0}^\infty m^*a^{-1}(s)tr\left[W^2(s)\right]\mathrm{d}s<+\infty$$

due to Lemma2. This contradiction completes the proof of the part under the assumption (22) of the theorem.

This completes the proof of Theorem1.

Example1. Consider the following 2-dimensional system

$$X''+P(t)X'+Q(t)X=0,\quad t\ge\frac{\pi}{2}, \tag{33}$$

where

$$P(t)=\begin{bmatrix}2d & 0\\ 0 & -2d\end{bmatrix}\ (d\ge 0 \text{ is a constant}),$$

$$Q(t)=\begin{bmatrix}\sin t+2t\cos t-\frac{1}{2}t^2\sin t+d^2+1 & 0\\ 0 & \sin t+2t\cos t-\frac{1}{2}t^2\sin t+d^2+1\end{bmatrix}.$$

Let $f(t)=0$, then $a(t)=1$ and $R(t)=Q(t)$, hence,

$$\int_{\frac{\pi}{2}}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s$$

$$= \begin{bmatrix} t(1+\sin t)+\frac{1}{2}t^2\cos t-\pi & 0 \\ 0 & t(1+\sin t)+\frac{1}{2}t^2\cos t-\pi \end{bmatrix}, \tag{34}$$

from (34), we have

$$\int_{\frac{\pi}{2}}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)$$

$$= \begin{bmatrix} t(1+\sin t)+\frac{1}{2}t^2\cos t-\pi-d & 0 \\ 0 & t(1+\sin t)+\frac{1}{2}t^2\cos t-\pi+d \end{bmatrix},$$

and then

$$tr\left[\int_{\frac{\pi}{2}}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]=2t(1+\sin t)+t^2\cos t-2\pi.$$

Thus,

$$\int_{\frac{\pi}{2}}^{T}\left(tr\left[\int_{\frac{\pi}{2}}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]\right)\mathrm{d}t=T^2(1+\sin T)-2\pi T+\frac{\pi^2}{2}$$

$$\geq -2\pi T.$$

Consequently,

$$\liminf_{T\to\infty}\frac{1}{T}\int_{\frac{\pi}{2}}^{T}\left(tr\left[\int_{\frac{\pi}{2}}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]\right)\mathrm{d}t\geq -2\pi>-\infty,$$

and

$$\limsup_{T\to\infty}\frac{1}{T}\int_{\frac{\pi}{2}}^{T}\left(\lambda_1\left[\int_{\frac{\pi}{2}}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]\right)\mathrm{d}t$$

$$=\limsup_{T\to\infty}\frac{1}{T}\int_{\frac{\pi}{2}}^{T}\left[t(1+\sin t)+\frac{1}{2}t^2\cos t-\pi+d\right]\mathrm{d}t$$

$$=\limsup_{T\to\infty}\left[\frac{1}{2}T(1+\sin T)-\pi+\frac{\pi^2}{4T}+d-\frac{\pi d}{2T}\right]=+\infty.$$

Thus, from Theorem 1 (19), it follows that (33) is oscillatory.

Remark1. It is easy to see that when $d\neq 0$ the results in [1–21] can not be applied to this case. However, if we let $d=0$, (33) reduces to (2).

Theorem2. Assume $a(t)\leq m^*$, for $t\geq t_0$,

$$\liminf_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}tr\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]\mathrm{d}s\mathrm{d}t=-\infty, \tag{35}$$

Then (1) is oscillatory if

$$\lim \text{approxsup}_{T\to\infty}\lambda_n\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]=\beta>-\infty,$$

where $a(t)$ is the same as in Theorem 1.

Proof. Suppose that (1) is not oscillatory. Then there exists a prepared solution $X(t)$ of (1). Such that $X(t)$ is nonsingular. Without loss of generality, we assume that $\det X(t)\neq 0$ for $t\geq t_0$. Denote V(t) by (3) then we have (7) holds. From (35) and Lemma2 it follows that

$$\int_{t_0}^{t}a^{-1}(s)tr\left(W^2(s)\right)\mathrm{d}s\longrightarrow+\infty,\quad t\longrightarrow+\infty,$$

and hence,

$$\lambda_1\left[\int_{t_0}^{t}a^{-1}(s)W^2(s)\mathrm{d}s\right]\longrightarrow+\infty,\quad t\longrightarrow+\infty.$$

Since

$$\begin{aligned}&\lambda_1\left[-\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s+\frac{a(t)}{2}P(t)\right]\\&=-\lambda_n\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right],\end{aligned}$$

it is easy to see that

$$\begin{aligned}&\lambda_1\left[-W(t)-\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s+\frac{a(t)}{2}P(t)\right]\\&\leq\lambda_1\left[-W(t)\right]+\lambda_1\left[-\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s+\frac{a(t)}{2}P(t)\right]\\&=\lambda_1\left[-W(t)\right]-\lambda_n\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right].\end{aligned}$$

Now for any $\epsilon>0$,

$$\mu\left\{t\in[t_0,\infty):\lambda_n\left[\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s-\frac{a(t)}{2}P(t)\right]\geq\beta-\epsilon\right\}=+\infty.$$

From (7), we have

$$\begin{aligned}&\frac{1}{n}tr\left[\int_{t_0}^{t}a^{-1}(s)W^2(s)\mathrm{d}s\right]\\&=\frac{1}{n}tr\left[-W(t)+W(t_0)-\frac{a(t_0)}{2}P(t_0)-\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s+\frac{a(t)}{2}P(t)\right]\\&=\frac{1}{n}tr\left[-W(t)-\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s+\frac{a(t)}{2}P(t)\right]+\frac{1}{n}tr\left[W(t_0)-\frac{a(t_0)}{2}P(t_0)\right]\\&\leq\lambda_1\left[-W(t)-\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s+\frac{a(t)}{2}P(t)\right]+\frac{1}{n}tr\left[W(t_0)-\frac{a(t_0)}{2}P(t_0)\right]\\&\leq\lambda_1\left[-W(t)\right]+\lambda_1\left[-\int_{t_0}^{t}\left(R(s)-\frac{a(s)}{4}P^2(s)\right)\mathrm{d}s+\frac{a(t)}{2}P(t)\right]+\frac{1}{n}tr\left[W(t_0)-\frac{a(t_0)}{2}P(t_0)\right]\end{aligned}$$

(36)

$$= \lambda_1[-W(t)] - \lambda_n\left[\int_{t_0}^t \left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right] + \frac{1}{n}tr\left[W(t_0) - \frac{a(t_0)}{2}P(t_0)\right],$$

and since

$$\frac{1}{n}tr\left[\int_{t_0}^t a^{-1}(s)W^2(s)\mathrm{d}s\right] \geq \frac{1}{n}\int_{t_0}^t \lambda_1\left[a^{-1}(s)W^2(s)\right]\mathrm{d}s$$

from (36), we have that

$$\frac{1}{n}\int_{t_0}^t \lambda_1\left[a^{-1}(s)W^2(s)\right]\mathrm{d}s - \frac{1}{n}tr\left[W(t_0) - \frac{a(t_0)}{2}P(t_0)\right]$$

$$\leq \lambda_1[-W(t)] - \lambda_n\left[\int_{t_0}^t \left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right], \quad t \geq t_0. \tag{37}$$

Hence, we have that for any $\epsilon > 0$,

$$\mu\left\{t \in [t_0, \infty) : \frac{1}{n}\int_{t_0}^t \lambda_1\left[a^{-1}(s)W^2(s)\right]\mathrm{d}s - \frac{1}{n}tr\left[W(t_0) - \frac{a(t_0)}{2}P(t_0)\right]\right.$$
$$\left.\leq \lambda_1[-W(t)] - \beta + \epsilon\right\} = +\infty,$$

and since

$$\int_{t_0}^t \lambda_1\left[a^{-1}(s)W^2(s)\right]\mathrm{d}s \geq \lambda_1\left[\int_{t_0}^t a^{-1}(s)W^2(s)\mathrm{d}s\right] \longrightarrow +\infty, \text{ as } t \longrightarrow +\infty,$$

we see that if E is defined by

$$E = \left\{t : \frac{1}{2n}\int_{t_0}^t \lambda_1\left[a^{-1}(s)W^2(s)\right]\mathrm{d}s \leq \lambda_1[-W(t)]\right\} \subseteq [t_0 + 1, \infty),$$

then $\mu(E) = +\infty$. But now with $G(t) = \int_{t_0}^t \lambda_1\left[a^{-1}(s)W^2(s)\right]\mathrm{d}s$, we have

$$G'(t) = \lambda_1\left[a^{-1}(t)W^2(t)\right] \geq \frac{1}{n}tr\left[a^{-1}(t)W^2(t)\right] \geq \frac{a^{-1}(t)}{n}\lambda_1\left[W^2(t)\right],$$

and so

$$nm^*G'(t) \geq na(t)G'(t) \geq \lambda_1\left[W^2(t)\right] \geq (\lambda_1[-W(t)])^2 \geq \frac{1}{4n^2}G^2(t), t \in E,$$

which in turn implies that

$$G'(t)/G^2(t) \geq 1/(4n^3m^*), \quad t \in E,$$

and now

$$\int_E G'(t)/G^2(t)\mathrm{d}t \geq \frac{1}{4n^3m^*}\mu(E) = +\infty,$$

a contradiction since the integral on the left is $\leq 1/G(t_0 + 1)$.

This completes the proof of Theorem 2.

Remark2. Theorem 1 and Theorem 2 are generalizations and improvements of [3,Theorem2.1 and Theorem2.2], respectively. In fact, when $P(t) \equiv 0$, we let $f(t) = 0,\ a(t) = 1$ then Theorem 1 and Theorem 2 are reduced to Theorem2.1 and Theorem2.2 of [3],respectively.

In order to illustrate our theorems, we consider the following example.

Example2. Consider the following 2-dimensional system

$$X'' + P(t)X' + Q(t)X = 0, \quad t \geq t_0, \tag{38}$$

where

$$P(t) = \mu(t)\begin{bmatrix} p_1(t) & 0 \\ 0 & p_2(t) \end{bmatrix}, \quad Q(t) = \begin{bmatrix} q_1(t) & 0 \\ 0 & q_2(t) \end{bmatrix},$$

and $q_i(t),\ p_i(t)$ are continuous functions of t on $[t_0, +\infty)$, for $i = 1,\ 2$. If we let $f(t) = 0$, $a(t) = 1$ for $t \geq t_0$, then we have

$$R(t) = Q(t),$$

and

$$\int_{t_0}^{t}\left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)$$

$$= \begin{bmatrix} \int_{t_0}^{t}\left(q_1(s) - \frac{\mu^2(s)}{4}p_1^2(s)\right)\mathrm{d}s - \frac{1}{2}\mu(t)p_1(t) & 0 \\ 0 & \int_{t_0}^{t}\left(q_2(s) - \frac{\mu^2(s)}{4}p_2^2(s)\right)\mathrm{d}s - \frac{1}{2}\mu(t)p_2(t) \end{bmatrix}.$$

Set

$$U(t) = tr\left[\int_{t_0}^{t}\left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right],$$

and

$$V(t) = \lambda_1\left[\int_{t_0}^{t}\left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right].$$

Now let us consider the following two cases.

Case1. If

$$\liminf_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}U(t)\mathrm{d}t > -\infty,$$

and one of the following conditions holds:

$$\limsup_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}V(t)\mathrm{d}t = +\infty,$$

$$\limsup_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}V^2(t)\mathrm{d}t = +\infty,$$

$$\lim\ \mathrm{approxsup}_{t\to\infty}V(t) = +\infty,$$

$$\lim\ \mathrm{approxinf}_{t\to\infty}V(t) = -\infty,$$

then (38) is oscillatory by Theorem 1.

Case2. If

$$\liminf_{T\to\infty}\frac{1}{T}\int_{t_0}^{T}U(t)\mathrm{d}t = -\infty,$$

and

$$\lim\ \mathrm{approxsup}_{t\to\infty}\lambda_n\left[\int_{t_0}^{t}\left(R(s) - \frac{a(s)}{4}P^2(s)\right)\mathrm{d}s - \frac{a(t)}{2}P(t)\right] > -\infty,$$

then (38) is oscillatory by Theorem 2.

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