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LEGENDRE, JACOBI, AND RICCATI TYPE CONDITIONS FOR TIME SCALE VARIATIONAL PROBLEM WITH APPLICATION

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ABSTRACT. A time scale quadratic problem $\mathcal{J}$ with piecewise right-dense continuous coefficients and one varying endpoint is considered. Such problems are "hybrid", since they include mixing of continuous- and discrete-time problems. A new notion of a generalized conjugate point involving "dynamic" (hybrid) systems and comprising as special cases those known for the continuous- and discrete-time settings is introduced. A type of a strengthened Legendre condition is identified and used to establish characterizations of the nonnegativity and positivity of $\mathcal{J}$ in terms of (i) the nonexistence of such conjugate points, (ii) the natural conjoined basis of the associated time scale Jacobi equation, and (iii) a solution of the corresponding time scale Riccati equation. These results furnish second order necessary optimality conditions for a nonlinear time scale variational problem. Furthermore, we present an example of an optimal impulsive control problem and we show how this problem can be reduced to a variational problem over a time scale.

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1. INTRODUCTION

Let the time scale $\mathbb{T}$ be a nonempty compact subset of $\mathbb{R}$ that is *not* necessarily connected. Set $a := \min \mathbb{T}$, $b := \max \mathbb{T}$ and denote $\mathbb{T}$ by $[a, b]$. Thus, in this paper $[a, b]$ denotes the *intersection* of $\mathbb{T}$ with the the connected real interval whose endpoints are respectively a and b.

Define $\sigma(t)$ as the "forward jump" at t that reduces to t when t is a "right-dense" point. Similarly, the "backward jump" $\rho(t)$ is defined. For a given function $\eta : [a, b] \to \mathbb{R}^n$, the function $\eta^\sigma(t)$ is $\eta\big(\sigma(t)\big)$. A notion of a time scale Δ-derivative of η at time t, denoted by $\eta^\Delta(t)$, was introduced by Hilger in [12]. A precise definition of this notion will be recalled in the next section. In particular, in the continuous-time case, i.e. when $\mathbb{T}$ is a connected real interval, then $\sigma(t) = t$ and $\eta^\Delta(t) = \eta'(t)$, which is the usual derivative, while in the discrete-time case, i.e. when $\mathbb{T} = \{0, 1, \ldots, N + 1\}$,

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we have $\sigma(t) = t + 1$ and $\eta^\Delta(t) = \Delta\eta(t) := \eta(t+1) - \eta(t)$, that is the usual forward difference.

Let P, Q, and R be $n \times n$ matrix functions, $\hat{\Gamma}_a$ be an $n \times n$ matrix, and the matrix $\mathcal{M}_a$ be a projection in $\mathbb{R}^n$. Consider the quadratic functional over the time scale $\mathbb{T}$

$$\mathcal{J}(\eta) := \eta^T(a)\,\hat{\Gamma}_a\,\eta(a) + \int_a^b \big\{(\eta^\sigma)^T P\,\eta^\sigma + 2\,(\eta^\sigma)^T Q\,\eta^\Delta + (\eta^\Delta)^T R\,\eta^\Delta\big\}(t)\,\Delta t$$

subject to $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ satisfying the boundary conditions

$$\mathcal{M}_a\eta(a) = 0, \quad \eta(b) = 0. \tag{1.1}$$

The notation $\mathrm{C}^1_{\mathrm{prd}}$ stands for the space of piecewise right-dense (shortly rd-) continuously Δ-differentiable functions (see next section for more details).

Since $\mathcal{M}_a\eta(a) = 0$, then $\hat{\Gamma}_a$ can be assumed to be invariant under the projection $I - \mathcal{M}_a$, that is, $(I - \mathcal{M}_a)\,\hat{\Gamma}_a\,(I - \mathcal{M}_a) = \hat{\Gamma}_a$, where I is the identity matrix. Note that the constraint of the form $M_a\eta(a) = 0$ where M_a is an $r \times n$ full-rank matrix is equivalent to the constraint $\mathcal{M}_a\eta(a) = 0$ with $\mathcal{M}_a := M_a^T(M_aM_a^T)^{-1}M_a$.

It is shown in [19] that the nonlinear variational problem over the *time scale* $\mathbb{T}$

$$\text{minimize} \quad \mathcal{F}(x) := K\big(x(a)\big) + \int_a^b L\big(t, x^\sigma(t), x^\Delta(t)\big)\,\Delta t \tag{P}$$

$$\varphi\big(x(a)\big) = 0, \quad x(b) = B \tag{1.2}$$

has its accessory problem (the second variation) at an arc $\hat{x}$ satisfying (1.2) of the form of $\mathcal{J}(\eta)$ subject to (1.1), where $P(t) := \hat{L}_{xx}(t)$, $Q(t) := \hat{L}_{xv}(t)$, and $R(t) := \hat{L}_{vv}(t)$, with $\hat{L}(t)$ standing for the evaluation of $L(t,x,v)$ at $\big(t, \hat{x}^\sigma(t), \hat{x}^\Delta(t)\big)$. Hence, the study of the nonnegativity ($\mathcal{J} \geq 0$) and positivity ($\mathcal{J} > 0$) of the quadratic functional $\mathcal{J}(\eta)$ is directly related to obtaining second order necessary and sufficient conditions for optimality in (P).

When the time scale $\mathbb{T}$ is a *connected* interval or is the discrete set of values $\{0, 1, \dots, N+1\}$ the problem (P) and its corresponding accessory problem involving the functional $\mathcal{J}(\eta)$ reduce respectively to the continuous-time and discrete-time settings, that are intensively studied in the literature. Thus, variational problems over time scales unify both continuous-time and discrete-time problems under one form. More importantly, this class of variational problems includes a *large* spectrum of other problems where the time scale $\mathbb{T}$ could be, for example, a union of disjoint connected time-intervals with some discrete instances. Such problems are known also under the name "hybrid" since, as stated in [23], they are a "mixing of two fundamentally different types of problems". Here we are mixing the discrete- and the continuous-time problems.

The problem (P) falls naturally in the class of optimal control problems over the time scale $\mathbb{T}$ of the form

$$\text{(C)} \qquad \text{minimize} \quad \mathcal{G}(x,u) := K\big(x(a)\big) + \int_a^b f_0\big(t, x^\sigma(t), u(t)\big)\,\Delta t$$

$$\text{(1.3)} \qquad x^\Delta(t) = f\big(t, x^\sigma(t), u(t)\big), \quad t \in [a, \rho(b)],$$
$$\varphi\big(x(a)\big) = 0, \quad x(b) = B.$$

The problem (C) is hybrid and the *time scale* system (1.3) is a hybrid control system, whose paradigm is particularily useful in modelling applications where high-level decision making is used to supervise process behavior. Therefore, these systems appear in many important applications stemming for instance from aerospace and power systems, where the system has to switch between various setpoints or operational modes to extend its effective operating range. Hybrid systems embrace a diverse set of applications from engineering to biology, see e.g. [22, 23] and the references therein. Over the last ten years there has been considerable activitity in this area.

The mathematical description of these systems can be characterized by *impulsive* differential equations. In the case where the resetting events of the equations are defined by a prescribed sequence of times that are independent of the state of the system, the system of impulsive equations is known as *time-dependent*. See for example [2, 3, 8, 10, 11, 22]. In those references the time interval $[t_0, t_f]$ is connected and the state function x is *discontinuous* but left continuous at the resetting instances $\{t_1, t_2, \dots\}$. The impulsive differential equations have been studied there by splitting them into continuous-time and discrete-time systems. However, one can find a *time scale* of the form

$$\mathbb{T} = [t_0, t_1] \cup [t_1 + \varepsilon_1, t_2] \cup [t_2 + \varepsilon_2, t_3] \cup \cdots \subseteq [t_0, t_f + \varepsilon]$$

such that the system of impulsive differential equations is equivalent to a *time scale* system over $\mathbb{T}$. Therefore, time-dependent impulsive control systems can be viewed as special cases of the time scale control systems (1.3). Hence any result obtained for the variational problems (P) and (C) over time scales could be applied to *impulsive* variational problems.

In this paper we focus our attention on the variational problem (P). The corresponding time scale *Jacobi equation* is defined to be

$$\text{(J)} \qquad \big[R(t)\,\eta^\Delta + Q^T(t)\,\eta^\sigma\big]^\Delta = P(t)\,\eta^\sigma + Q(t)\,\eta^\Delta.$$

In the *continuous-time* setting a characterization of the nonnegativity of $\mathcal{J}(\eta)$ was obtained in terms of the conjugate points theory in [21, 27] and in terms of the Riccati differential equation corresponding to (J) in [25]. However, a characterization of the positivity of $\mathcal{J}(\eta)$ in terms of the conjugate points, the conjoined basis of (J), and the Riccati equation were derived in [21, 24, 26]. These results are valid under the strengtened Legendre condition

$$R(t) \geq \alpha I. \tag{1.4}$$

For the *discrete-time* setting the nonnegativity and positivity of $\mathcal{J}(\eta)$ was characterized in terms of the conjugate "intervals", the natural conjoined basis of (J), and the Riccati difference equation in [4, 18], see also the survey paper [20] and the references therein. As opposed to the continuous-time case, no discrete analog of (1.4) is needed in the discrete-time case. However, it is required that $R(t) + Q^T(t)$ be invertible. Moreover, for the derivation of the Riccati difference equation results, the invertibility of $R(t)$ is also needed.

For the general time scale or hybrid quadratic functional with rd-continuous coefficients, a characterization of the *positivity* of $\mathcal{J}(\eta)$ in terms of the principal solution of (J) is obtained in [1, 5, 13, 14, 16]. Moreover, such a result in terms of the time scale Riccati equation is known only for the case of fixed endpoints [15]. However, no concept playing the role of conjugate points or intervals is known for the general time scale quadratic functional . Furthermore, the characterization of the *nonnegativity* of $\mathcal{J}(\eta)$ in this general setting in completely untouched. In addition, the equivalence between the *positivity* of $\mathcal{J}(\eta)$ and conditions involving the Riccati equation (in the case of variable endpoint(s)) remains unknown.

The purpose of this paper is to answer the above mentioned problems that are open even for the case when the coefficients of $\mathcal{J}(\eta)$ are *piecewise* rd-continuous. More specifically, we introduce in Section 3 a concept of "conjugate points" for the *time scale* variational problem (P). In Theorem 5.1 (Section 5) we establish one of the main results of this paper, namely, the characterization of the nonnegativity of $\mathcal{J}(\eta)$ over (1.1) in terms of the nonexistence of "conjugate points", the natural conjoined basis of the hybrid Jacobi equation (J), and the existence of a solution to the time scale (or hybrid) Riccati equation. The strengthening of the conditions in Theorem 5.1 yields a characterization of the positivity of $\mathcal{J}(\eta)$ over (1.1) and is displayed in Theorem 6.1 (Section 6). The underlying hypotheses for these results are

$$S(t) := R(t) + \mu(t)\, Q^T(t) \;\text{ and } R(t) \;\text{ are invertible for all } t \in [a, \rho(b)], \tag{1.5}$$

where $\mu(t)$ is the "graininess" of $\mathbb{T}$ at t and $\rho(b)$ stands for the "backward jump" function at b, and the *strengthened Legendre condition*

$$R(t^{\pm}) > 0 \;\text{ for all dense points } t \in [\sigma(a), \rho(b)], \tag{1.6}$$

where $R(t^+)$ and $R(t^-)$ are the corresponding right-hand and left-hand limits at t. These hypotheses are natural extension of the corresponding conditions required for each of the continuous-time and discrete-time settings. At the end of the paper we display an example illustrating how the results of this paper apply to a hybrid problem that is neither on a continuous time, nor on a discrete-time interval, but is defined on a mixed set. Furthermore, we present an example of an optimal *impulsive* control problem (IC) and we show how this problem can be reduced to a variational problem over a time scale. Then we use the results developed for this latter case to find the optimal solution for (IC).

2. SOME PREREQUISITIES ABOUT TIME SCALES

Let $\mathbb{T}$ be a time scale, i.e. a nonempty closed subset of $\mathbb{R}$. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\}$ (together with $\inf \emptyset := \sup \mathbb{T}$). The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} \mid s < t\}$ (together with $\sup \emptyset := \inf \mathbb{T}$). A point $t \in \mathbb{T} \setminus \{\sup \mathbb{T}\}$ is *right-dense* or *right-scattered* if $\sigma(t) = t$ or $\sigma(t) > t$, respectively. A point $t \in \mathbb{T} \setminus \{\inf \mathbb{T}\}$ is *left-dense* or *left-scattered* if $\rho(t) = t$ or $\rho(t) < t$, respectively. A point $t \in \mathbb{T}$ is *dense* if it is either left-dense or right-dense. The *graininess* function μ is defined by $\mu(t) := \sigma(t) - t$. The set $\mathbb{T}^\kappa$ is defined as $\mathbb{T}$ without the left-scattered maximum of $\mathbb{T}$ (in case it exists).

A function f on $\mathbb{T}$ (with values in a Banach space) is *regulated* if the right-hand limit $f(t^+)$ exists (finite) at all right-dense points $t \in \mathbb{T}$ and the left-hand limit $f(t^-)$ exists (finite) at all left-dense points $t \in \mathbb{T}$. A function f is *rd-continuous* (we write $f \in \mathrm{C}_{\mathrm{rd}}$) if it is regulated and if it is continuous at all right-dense points $t \in \mathbb{T}$. A function f is *piecewise rd-continuous* (we write $f \in \mathrm{C}_{\mathrm{prd}}$) if it is regulated and if it is rd-continuous at all, except possibly at finitely many, right-dense points $t \in \mathbb{T}$.

Remark 2.1. At the right-dense points $\{t_1, \cdots, t_k\}$ where a given $\mathrm{C}_{\mathrm{prd}}$-function f is not continuous, the statements and conditions involving the values $f(t_i)$, $i \in \{1, \ldots, k\}$, simply mean that these statements and conditions hold when the value $f(t_i)$ is replaced by $f(t_i^+)$. This convention will be assumed throughout the paper without further recall.

It is a known fact that a composition of a continuous function g with $f \in \mathrm{C}_{\mathrm{rd}}$ or $f \in \mathrm{C}_{\mathrm{prd}}$ is respectively rd-continuous or piecewise rd-continuous, i.e. $g \circ f \in \mathrm{C}_{\mathrm{rd}}$ or $g \circ f \in \mathrm{C}_{\mathrm{prd}}$.

A matrix-function F is *regressive* if $I + \mu(t)\, F(t)$ is invertible for all $t \in \mathbb{T}^\kappa$.

Remark 2.2. (i) Recall that if $f, g \in \mathrm{C}_{\mathrm{rd}}$ then $f+g \in \mathrm{C}_{\mathrm{rd}}$, $fg \in \mathrm{C}_{\mathrm{rd}}$. To the contrary, even if $g(t) \neq 0$ for all $t \in \mathbb{T}$, then f/g does not need to be rd-continuous, because even if g never vanishes, $1/g$ may not be regulated. As an example we can take $\mathbb{T} = [-1, 0] \cup \{1, 2\}$, $f(t) \equiv 1$, $g(t) = -t$ for $t \in [-1, 0)$ and $g(t) = 1$ for $t \in \{0, 1, 2\}$. Then $f, g \in \mathrm{C}_{\mathrm{rd}}$ but $1/g \notin \mathrm{C}_{\mathrm{rd}}$, because $(1/g)(0^-) = \infty$ is not finite. In this paper we will consider inverses of certain (piecewise) rd-continuous functions and, because of the above example, one has to be careful as to when such an inverse is actually (piecewise) rd-continuous.

(ii) On the other hand, if a matrix function $F \in \mathrm{C_{rd}}$ is regressive, then $(I+\mu F)^{-1}$ is rd-continuous. Similarly, if matrix functions F, G, and G^{-1} are (piecewise) rd-continuous, then $(G+\mu F)^{-1}$ is (piecewise) rd-continuous whenever this inverse exists.

(iii) Let G be a regulated matrix function with $G(t^{\pm}) > 0$ for all dense points t. Then, for some $\alpha > 0$, $G(t^{\pm}) \geq \alpha I$ for all dense points t. Hence, a nonsingular regulated ($\mathrm{C_{rd}}$, $\mathrm{C_{prd}}$) matrix function G with $G(t^{\pm}) > 0$ for all dense points t has a regulated ($\mathrm{C_{rd}}$, $\mathrm{C_{prd}}$) inverse for all t. Consequently, the strengthened Legendre condition (1.6) and

$$R(t^{\pm}) \geq \alpha I \quad \text{for all dense points } t \in [\sigma(a), \rho(b)],$$

for some constant $\alpha > 0$, are equivalent.

(iv) On the other hand, the condition

$$R(t) > 0 \quad \text{for all dense points } t \in [\sigma(a), \rho(b)]$$

with $R \in \mathrm{C_{rd}}$ is not appropriate for the theory on time scales, since it does not necessarily yield the boundedness of R^{-1}. For instance, let $\mathbb{T} := \{-1/n\}_{n=1}^{\infty} \cup \{0\}$ and define $R(t)$ on $\mathbb{T}$ by $R(t) := -t$ for $t < 0$ and $R(0) := 1$.

The time scale Δ-derivative of a function f was introduced by Hilger in [12] and is defined by

$$f^{\Delta}(t) := \lim_{s \to t} \frac{f\big(\sigma(t)\big) - f(s)}{\sigma(t) - s}, \quad \text{where } s \to t,\ s \in \mathbb{T} \setminus \{\sigma(t)\}.$$

It is understood by definition that $\lim_{s\to t} f(s) = f(t)$ when t is an isolated point. When $t := \max \mathbb{T}$ exists and is left-scattered, then $f^{\Delta}(t)$ is not well-defined. When f^{Δ} exists it is shown in [12] that

$$f^{\Delta}(t) = \{f^{\sigma}(t) - f(t)\} / \mu(t). \tag{2.1}$$

A function f is *rd-continuously* Δ*-differentiable* (we write $f \in \mathrm{C}^1_{\mathrm{rd}}$) if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$ and $f^{\Delta} \in \mathrm{C_{rd}}$. A continuous function f is *piecewise rd-continuously* Δ*-differentiable* (we write $f \in \mathrm{C}^1_{\mathrm{prd}}$) if f is continuous and f^{Δ} exists at all, except possibly at finitely many, $t \in \mathbb{T}^{\kappa}$ and $f^{\Delta} \in \mathrm{C_{prd}}$. Note that if $f \in \mathrm{C}^1_{\mathrm{prd}}$ then the points t_i where $f^{\Delta}(t_i)$ does not exist (but we know that $f^{\Delta}(t_i^{+})$ and $f^{\Delta}(t_i^{-})$ exist since f^{Δ} is regulated) are necessarily left-dense and right-dense at the same time.

For $c, d \in \mathbb{T}$, the time scale integral is denoted by $\int_c^d f(t)\,\Delta t$ and is defined as the Cauchy integral associated with the Δ-differentiation. It is known [6, Theorem 1.74] that whenever $f \in \mathrm{C_{rd}}$ ($f \in \mathrm{C_{prd}}$) this integral is well-defined. In general, a function f is said to be Δ*-integrable on* $[c, d]$ if $\int_c^d f(t)\,\Delta t$ exists (finite). Note that it can be shown that

$$\int_c^{\sigma(c)} f(t)\,\Delta t = \mu(c)\, f(c).$$

We shall denote by f^σ and f^ρ the compositions $f \circ \sigma$ and $f \circ \rho$ of a function f with σ and ρ, respectively.

Remark 2.3. If f and g are in $\mathrm{C}_{\mathrm{prd}}$ and $f(t) = g(t)$ except at finitely many right-dense points, then $\int_a^b f(t)\,\Delta t = \int_a^b g(t)\,\Delta t$. In particular, when $f(t) > 0$ except at finitely many right-dense points, then $\int_a^b f(t)\,\Delta t > 0$.

3. CONCEPTS AND BASIC PROPERTIES

The coefficients $P(t)$, $Q(t)$, and $R(t)$ are assumed to be piecewise rd-continuous on $[a, \rho(b)]$, i.e. $P, Q, R \in \mathrm{C}_{\mathrm{prd}}$, and the matrices $P(t)$, $R(t)$, and $\hat{\Gamma}_a$ are assumed to be symmetric. Furthermore, we suppose throughout the paper that (1.5) and (1.6) hold.

If a function f is defined at the points a and b, then we will use a common notation $f(t)\,|_a^b := f(b) - f(a)$.

A time scale *linear Hamiltonian system* equivalent to (J) is

$$\text{(H)} \qquad \eta^\Delta = A(t)\,\eta^\sigma + B(t)\,q, \quad q^\Delta = C(t)\,\eta^\sigma - A^T(t)\,q,$$

where

$$A(t) := -R^{-1}(t)\,Q^T(t), \quad B(t) := R^{-1}(t), \quad C(t) := P(t) - Q(t)\,R^{-1}(t)\,Q^T(t),$$

and

$$\text{(3.1)} \quad q(t) := R(t)\,\eta^\Delta(t) + Q^T(t)\,\eta^\sigma(t), \quad \text{i.e.} \quad \mu(t)\,q(t) = S(t)\,\eta^\sigma(t) - R(t)\,\eta(t).$$

Note that, under (1.5) and (1.6), the functions $R^{-1}(t)$, $S(t)$, $S^{-1}(t)$, $A(t)$, $B(t)$, $C(t)$, and $\tilde{A}(t) := [I - \mu(t)\,A(t)]^{-1} = S^{-1}(t)\,R(t)$ are piecewise rd-continuous, by Remark 2.2.

For a given function $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ we always associate the corresponding $q \in \mathrm{C}_{\mathrm{prd}}$ by (3.1), so that, equivalently, the pair (η, q) solves the first equation of (H). Such a pair (η, q) is called *admissible*. In particular, η solves (J) if and only if (η, q) solves (H).

With q given by (3.1), the functional $\mathcal{J}$ takes the form $\mathcal{J}(\eta) = \mathcal{I}(\eta, q)$, where

$$\mathcal{I}(\eta, q) := \eta^T(a)\,\hat{\Gamma}_a\,\eta(a) + \int_a^b \big\{(\eta^\sigma)^T C\,\eta^\sigma + q^T B\,q\big\}(t)\,\Delta t.$$

In particular, the integrand of $\mathcal{I}$ is a piecewise rd-continuous and hence, a regulated function, so that this integral is well defined.

Definition 3.1 (Nonnegativity and positivity)**.** The quadratic functional $\mathcal{J}$ is *nonnegative* (or *nonnegative definite*), we write $\mathcal{J} \geq 0$, if $\mathcal{J}(\eta) \geq 0$ for all $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ satisfying (1.1). The quadratic functional $\mathcal{J}$ is *positive* (or *positive definite*), we write $\mathcal{J} > 0$, if $\mathcal{J}(\eta) > 0$ for all $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ satisfying (1.1) and $\eta \not\equiv 0$.

Remark 3.2. Equivalently, the quadratic functional $\mathcal{I}$ is *nonnegative* (or *nonnegative definite*), we write $\mathcal{I} \geq 0$, if $\mathcal{I}(\eta, q) \geq 0$ for all admissible (η, q) satisfying (1.1). The quadratic functional $\mathcal{I}$ is *positive* (or *positive definite*), we write $\mathcal{I} > 0$, if $\mathcal{I}(\eta, q) > 0$ for all admissible (η, q) satisfying (1.1) and $\eta \not\equiv 0$.

First we establish the *Legendre condition* for the quadratic functional $\mathcal{J}$.

Lemma 3.3 (Legendre condition)**.** *If $\mathcal{J} \geq 0$, then the Legendre condition holds:*

$$R(t^{\pm}) \geq 0, \quad \textit{for all dense points } t \in [\sigma(a), \rho(b)]. \tag{3.2}$$

Proof. It is similar to the proof of [5, Result 1.3] and is therefore omitted. □

Remark 3.4. If the Legendre condition (3.2) holds and $R(t)$ is invertible on $[a, b]$, then we do not necessarily obtain the strengthened Legendre condition (1.6).

An $n \times n$ matrix solution X of (J), or equivalently (X, U) of (H), is a *conjoined basis* if $X^T(t)U(t)$ is symmetric and $\operatorname{rank}(X^T(t), U^T(t)) = n$ at some (and hence at any) $t \in [a, b]$. Here U is defined similarly to q in (3.1), that is, by

$$U(t) := R(t)\, X^{\Delta}(t) + Q^T(t)\, X^{\sigma}(t), \text{ i.e. } \mu(t)\, U(t) = S(t)\, X^{\sigma}(t) - R(t)\, X(t). \tag{3.3}$$

For any two conjoined bases X and $\tilde{X}$ of (J) the expression $\{X^T\tilde{U} - U^T\tilde{X}\}(t)$ (called the *Wronskian*) is constant on $[a, b]$. This is verified by showing $\{X^T\tilde{U} - U^T\tilde{X}\}^{\Delta} = 0$. Two conjoined bases X and $\tilde{X}$ of (J) are *normalized* if their Wronskian is the identity matrix. A *natural conjoined basis* of (J) is the matrix solution X of (J) given by the inital conditions

$$X(a) = I - \mathcal{M}_a, \quad U(a) = \hat{\Gamma}_a + \mathcal{M}_a.$$

This and (3.3) imply that

$$X^{\sigma}(a) = S^{-1}(a) \left\{ [R(a) + \mu(a)\, \hat{\Gamma}_a]\, X(a) + \mu(a)\, \mathcal{M}_a \right\}. \tag{3.4}$$

When the left endpoint is zero, i.e. when $\mathcal{M}_a = I$, the natural conjoined basis reduces to the *principal solution* $\hat{X}$, which is given by the initial conditions

$$\hat{X}(a) = 0, \quad \hat{U}(a) = I.$$

Note that if a is right-scattered, then $\hat{X}^{\sigma}(a) = \mu(a)\, S^{-1}(a)$ is invertible.

Lemma 3.5. *Let X be the natural conjoined basis of* (J)*. A solution η of* (J) *satisfies*

$$\mathcal{M}_a \eta(a) = 0, \quad q(a) = \hat{\Gamma}_a \eta(a) + \mathcal{M}_a \gamma_a, \tag{3.5}$$

if and only if $(\eta, q) = (X\alpha, U\alpha)$ on $[a, b]$ for some vector $\alpha \in \mathbb{R}^n$.

Proof. Sufficiency is trivial, while necessity follows from the uniqueness of solutions of (H) upon taking $\alpha = \eta(a) + \mathcal{M}_a\gamma_a$. □

In the next lemma we characterize the invertibility of $X^\sigma(a)$ in (3.4). When a is right-dense, i.e. when $\mu(a)=0$, this result is trivial and says that $X(a)$ is invertible if and only if $\mathcal{M}_a=0$.

Lemma 3.6. *Let X be the natural conjoined basis of* (J). *The matrix $X^\sigma(a)$ in* (3.4) *is invertible if and only if*

$$\operatorname{Ker}(I-\mathcal{M}_a)\,[\,R(a)+\mu(a)\,\hat{\Gamma}_a\,]\cap\operatorname{Ker}\mu(a)\,\mathcal{M}_a\;=\;\{0\}.$$

Proof. "$\Rightarrow$" Suppose that there is a vector $\alpha\in\mathbb{R}^n$ such that $(I-\mathcal{M}_a)\,[R(a)+\mu(a)\,\hat{\Gamma}_a]\,\alpha=0$ and $\mu(a)\,\mathcal{M}_a\alpha=0$. Then

$$[S(a)\,X^\sigma(a)]^T\alpha=\big\{X(a)\,[R(a)+\mu(a)\,\hat{\Gamma}_a]+\mu(a)\,\mathcal{M}_a\big\}\alpha=0,$$

i.e. $[X^\sigma(a)]^T S^T(a)\,\alpha=0$. Hence, $\alpha=0$, since both $X^\sigma(a)$ and $S(a)$ are invertible.

"$\Leftarrow$" Suppose that $X^\sigma(a)$ is not invertible so that $[X^\sigma(a)]^T$ is not invertible as well. Then $[X^\sigma(a)]^T\alpha=0$ for some nonzero vector $\alpha\in\mathbb{R}^n$, i.e.

$$\big\{X(a)[R(a)+\mu(a)\,\hat{\Gamma}_a]+\mu(a)\,\mathcal{M}_a\big\}\beta=0, \tag{3.6}$$

where $\beta:=S^{T-1}(a)\,\alpha\neq0$. Multiplying (3.6) by $I-\mathcal{M}_a$, we get $(I-\mathcal{M}_a)\,[R(a)+\mu(a)\,\hat{\Gamma}_a]\,\beta=0$, while multiplying (3.6) by $\mathcal{M}_a$ we get $\mu(a)\,\mathcal{M}_a\beta=0$. Hence, $\beta\in\operatorname{Ker}(I-\mathcal{M}_a)\,[R(a)+\mu(a)\,\hat{\Gamma}_a]\cap\operatorname{Ker}\mu(a)\,\mathcal{M}_a$. This implies by the assumption $\beta=0$, which is a contradiction. ☐

Lemma 3.7. *Let η be a solution of* (J) *satisfying* (3.5). *For a left-dense point $c\in(a,b]$ define the pair $(\tilde\eta,\tilde q)$ by*

$$\tilde\eta(t):=\begin{cases}\eta(t) & \text{for } t\in[a,c],\\ 0 & \text{for } t\in(c,b],\end{cases}\qquad \tilde q(t):=\begin{cases}q(t) & \text{for } t\in[a,c),\\ 0 & \text{for } t\in[c,b].\end{cases} \tag{3.7}$$

For a left-scattered point $c\in[\sigma(a),b]$ define the pair $(\tilde\eta,\tilde q)$ by

$$\tilde\eta(t):=\begin{cases}\eta(t) & \text{for } t\in[a,\rho(c)],\\ 0 & \text{for } t\in[c,b],\end{cases}\quad \tilde q(t):=\begin{cases}q(t) & \text{for } t\in[a,\rho(c)),\\ -\Big\{\frac{1}{\mu}\,R\eta\Big\}^\rho(c) & \text{for } t=\rho(c),\\ 0 & \text{for } t\in[c,b].\end{cases} \tag{3.8}$$

Then $(\tilde\eta,\tilde q)$ is admissible, satisfies $\mathcal{M}_a\tilde\eta(a)=0$, $\tilde\eta(b)=0$, and

$$\mathcal{J}(\tilde\eta)=\mathcal{I}(\tilde\eta,\tilde q)=\begin{cases}\eta^T(c)\,q(c) & \text{if } c \text{ is left-dense},\\ \Big\{\frac{1}{\mu}\,\eta^T S\,\eta^\sigma\Big\}^\rho(c) & \text{if } c \text{ is left-scattered.}\end{cases}$$

Proof. Let $c\in(a,b]$ be left-dense and define $(\tilde\eta,\tilde q)$ by (3.7). Then

$$\begin{aligned}\mathcal{J}(\tilde\eta)=\mathcal{I}(\tilde\eta,\tilde q)&=\tilde\eta^T(a)\,\hat{\Gamma}_a\,\tilde\eta(a)+\bigg\{\int_a^c+\int_c^b\bigg\}\big\{(\tilde\eta^\sigma)^TC\,\tilde\eta+\tilde q^TB\,\tilde q\big\}(t)\,\Delta t\\ &=\eta^T(a)\,\hat{\Gamma}_a\,\eta(a)+\eta^T(t)\,q(t)\,\big|_a^c=\eta^T(c)\,q(c).\end{aligned}$$

Let $c \in [\sigma(a), b]$ be left-scattered and define $(\tilde{\eta}, \tilde{q})$ by (3.8). Then

$$\begin{aligned} \mathcal{J}(\tilde{\eta}) = \mathcal{I}(\tilde{\eta}, \tilde{q}) &= \tilde{\eta}^T(a)\, \hat{\Gamma}_a\, \tilde{\eta}(a) + \left\{ \int_a^{\rho(c)} + \int_{\rho(c)}^{c} + \int_c^b \right\} \{(\tilde{\eta}^\sigma)^T C\, \tilde{\eta} + \tilde{q}^T B\, \tilde{q}\}(t)\, \Delta t \\ &= \eta^T(a)\, \hat{\Gamma}_a\, \eta(a) + \eta^T(t)\, q(t)\, \Big|_a^{\rho(c)} + \{\mu\, (\tilde{\eta}^\sigma)^T C\, \tilde{\eta}^\sigma + \mu\, \tilde{q}^T B\, \tilde{q}\}^\rho(c) \\ &= \left\{ \eta^T (R\eta^\Delta + Q^T \eta^\sigma) + \tfrac{1}{\mu}\, \eta^T R \eta \right\}^\rho (c) = \left\{ \tfrac{1}{\mu}\, \eta^T S\, \eta^\sigma \right\}^\rho (c). \end{aligned}$$

The proof is complete. □

The above result motivates the concept of conjugate points to a.

Definition 3.8 (Conjugate point)**.** Let $c \in (a, b]$. We say that c is *conjugate to* a if there exists a nontrivial solution η of (J) satisfying condition (3.5) for some vector $\gamma_a \in \mathbb{R}^n$, and

$$\begin{aligned} &\eta(c) = 0 \quad \text{if } c \text{ is left-dense}, \\ &\eta^\rho(c) \neq 0, \quad [\eta^\rho(c)]^T S^\rho(c)\, \eta(c) \leq 0 \quad \text{if } c \text{ is left-scattered}. \end{aligned} \tag{3.9}$$

A (left-scattered) point c is *strictly conjugate to* a if it is conjugate to a and the associated solution η satisfies the strict inequality in (3.9).

Remark 3.9. (i) If c is left-scattered, then one could adopt the notion of "$(\rho(c), c]$ conjugate to a", which is equivalent to saying that "c is conjugate to a", since the set $(\rho(c), c]$ is in fact $\{c\}$ in this case. While this conjugate *interval* notion is more consistent with the corresponding notion in the discrete-time setting, it would not represent the continuous-time case where $(\rho(c), c] = \emptyset$.

(ii) When $\mathcal{M}_a = I$ ((P) has fixed endpoints), (3.5) reduces to $\eta(a) = 0$. When $\mathcal{M}_a = 0$ ((P) has free left endpoint), (3.5) reduces to $q(a) = \hat{\Gamma}_a\, \eta(a)$.

4. NEEDED PREVIOUSLY KNOWN RESULTS

The results recalled in this section will be needed in our work. The time scale *Riccati matrix equation* associated with (J) is defined to be

$$R[W](t) := W^\Delta - P(t) + [W^\sigma - \mu(t)\, P(t) - Q(t)]\, S^{-1}(t)\, [W - Q^T(t)] = 0. \tag{R}$$

This implies the identity

$$\mu(t) R[W](t) = [W^\sigma - \mu(t)\, P(t) - Q(t)]\, S^{-1}(t)\, [R(t) + \mu(t)\, W] + Q(t) - W,$$

from which it follows that W^σ can be explicitly calculated from W only if $R + \mu W$ is invertible. If this is the case, then the Riccati equation (R) can be written in the symmetric form

$$W^\Delta = P(t) - [W - Q(t)]\, [R(t) + \mu(t)\, W]^{-1}\, [W - Q^T(t)].$$

With any symmetric matrix function $W(t)$ we associate the symmetric matrix function

$$\begin{aligned} \mathcal{D}(t) :&= \big\{B - \mu B \tilde{A}^T (W^\sigma - \mu C)\, \tilde{A} B\big\}(t) \\ &= \big\{S^{T-1}\,(R + \mu Q + \mu Q^T - \mu W^\sigma + \mu^2 P)\, S^{-1}\big\}(t). \end{aligned} \tag{4.1}$$

We have the following result. Note that compared to [13, Lemma 2] we do not require here $\eta(a) \in \operatorname{Im} X(a)$, since $W(t)$ satisfies $R[W](t) = 0$ on $[a, \rho(b)]$.

Proposition 4.1 (Picone's identity)**.** [13, Lemma 2, Proposition 2] *Let $W(t)$, $t \in [a, b]$, be a symmetric solution of* (R) *on $[a, \rho(b)]$ and let (η, q) be admissible, i.e.* (3.1) *holds. Then*

$$\int_a^b \big\{(\eta^\sigma)^T C\, \eta^\sigma + q^T B\, q\big\}(t)\Delta t = \eta^T(t)\, W(t)\, \eta(t)\,\big|_a^b + \int_a^b \big\{z^T \mathcal{D} z\big\}(t)\, \Delta t,$$

where $z(t) := q(t) - W(t)\, \eta(t)$ on $[a, b]$, and $\mathcal{D}(t)$ is the function corresponding to W in (4.1)*. Furthermore, the following identity holds*

$$\eta(t) + \mu(t)\, \mathcal{D}(t)\, z(t) = \mathcal{D}(t)\, S(t)\, \eta^\sigma(t) \quad \textit{for all } t \in [a, \rho(b)].$$

Let X and $\tilde{X}$ be any normalized conjoined bases of (J). Define the $n \times n$ matrices $W(t)$, $\tilde{W}(t)$ and the $2n \times 2n$ matrix $W^*(t)$ by (we skip the argument (t) below)

$$W := U X^\dagger + (U X^\dagger \tilde{X} - \tilde{U})(I - X^\dagger X) U^T, \tag{4.2}$$

$$\tilde{W} := X^\dagger + X^\dagger \tilde{X}(I - X^\dagger X) U^T, \tag{4.3}$$

$$W^* := \begin{pmatrix} -X^\dagger \tilde{X} X^\dagger X & \tilde{W} \\ \tilde{W}^T & W \end{pmatrix}, \tag{4.4}$$

From [4, Lemma 2] it follows that W in (4.2) is symmetric and satisfies $WX = UX^\dagger X$.

Remark 4.2. Suppose that X and $\tilde{X}$ are normalized conjoined bases of (J) such that $X(t)$ is *invertible* for all $t \in (a, b)$, resp. $t \in (a, b]$. Then formulas (4.2), (4.3), (4.4), and (4.1) yield for $t \in (a, b)$, resp. $t \in (a, b]$, that

$$W(t) = U(t)\, X^{-1}(t), \quad \tilde{W}(t) = X^{-1}(t), \quad W^*(t) = \begin{pmatrix} -X^{-1}\tilde{X} & X^{-1} \\ X^{T-1} & W \end{pmatrix}(t), \tag{4.5}$$

$$\mathcal{D}(t) = X(t)\, [X^\sigma(t)]^{-1} S^{-1}(t) = [R(t) + \mu(t)\, W(t)]^{-1}.$$

Furthermore, W satisfies $R[W](t) = 0$ on $(a, \rho(b))$, resp. on $(a, \rho(b)]$, and hence, using this W, Proposition 4.1 applies on a subinterval $[c, d] \subseteq (a, b)$, resp. $[c, d] \subseteq (a, b]$. If a is right-scattered then [13, Lemma 3] yields that $R[W](a)\, X(a) = 0$.

For convenience we set for $\alpha, \eta \in \mathbb{R}^n$

$$F(t, \alpha, \eta) := \begin{pmatrix} \alpha \\ \eta \end{pmatrix}^T W^*(t) \begin{pmatrix} \alpha \\ \eta \end{pmatrix} \tag{4.6}$$

and recall that $F\big(t,\alpha,\eta(t)\big)\,\big|_a^b := F\big(b,\alpha,\eta(b)\big) - F\big(a,\alpha,\eta(a)\big)$.

Proposition 4.3 (Extended Picone's identity)**.** [13, Proposition 3] *Suppose that X and $\tilde{X}$ are normalized conjoined bases of* (J) *such that $X(t)$ is invertible for all $t \in (a,b]$. Let (η,q) be admissible, i.e.* (3.1) *holds, and $\alpha \in \mathbb{R}^n$ with $\alpha + U^T(a)\,\eta(a) \in \operatorname{Im} X^T(a)$. Then*

$$\int_a^b \big\{(\eta^\sigma)^T C\,\eta^\sigma + q^T B\,q\big\}(t) = F\big(t,\alpha,\eta(t)\big)\,\big|_a^b + \int_a^b \big\{z^T \mathcal{D} z\big\}(t)\,\Delta t,$$

where $W(t)$, $\tilde{W}(t)$, and $W^*(t)$ are defined by (4.2), (4.3), *and* (4.4) *on $[a,b]$, $z(t) := q(t) - W(t)\,\eta(t) - \tilde{W}^T(t)\,\alpha$ on $[a,b]$, and $\mathcal{D}(t)$ is given by* (4.1).

The following result is Corollary 1.2.4 of [21]. There, the proof is purely algebraical (it uses the properties of normalized conjoined bases) and requires a limit argument as $\varepsilon \to 0^+$ at a. For the time scale version we need the same limit argument at the right-dense point a.

Proposition 4.4. [21, Corollary 1.2.4] *Let a be right-dense and let $a_n \searrow a$ be a right-sequence for a. Suppose that X and $\tilde{X}$ are any normalized conjoined bases of* (J) *with $X(b)$ and $X(a_n)$ invertible for all n sufficiently large. Let $\eta(t)$ be (rd-)continuous at $t = a$ with $\eta(a) = X(a)\,d$ for some $d \in \mathbb{R}^n$. Then for*

$$\alpha_n := -U^T(a_n)\,\eta(a_n) + X^T(a_n)\,U(a)\,d$$

we have

$$\lim_{n\to\infty} F\big(t,\alpha_n,\eta(t)\big)\,\big|_{a_n}^b = \eta^T(b)\,W(b)\,\eta(b) - d^T X^T(a)\,U(a)\,d,$$

where $F(t,\alpha,\eta)$ is defined by (4.6) *with matrices $W(t)$ and $W^*(t)$ as in* (4.5).

In Proposition 4.4, the matrix $X(b)$ is invertible while $X(a)$ could be singular. In order to treat the opposite situation, namely the case when $X(a)$ is invertible while $X(b)$ could be singular, a time-reversed version of Proposition 4.4 is obtained by the transformation $t \mapsto a + b - t$.

Proposition 4.5. *Let b be left-dense and let $b_n \nearrow b$ be a left-sequence for b. Suppose that X and $\tilde{X}$ are any normalized conjoined bases of* (J) *with $X(a)$ and $X(b_n)$ invertible for all n sufficiently large. Let $\eta(t)$ be continuous at $t = b$ with $\eta(b) = X(b)\,d$ for some $d \in \mathbb{R}^n$. Then for*

$$\beta_n := -U^T(b_n)\,\eta(b_n) + X^T(b_n)\,U(b)\,d$$

we have

$$\lim_{n\to\infty} F\big(t,\beta_n,\eta(t)\big)\,\big|_a^{b_n} = d^T X^T(b)\,U(b)\,d - \eta^T(a)\,W(a)\,\eta(a),$$

where $F(t,\alpha,\eta)$ is defined by (4.6) *with matrices $W(t)$ and $W^*(t)$ as in* (4.5).

5. NONNEGATIVITY OF $\mathcal{J}$

The following theorem is one of the main results of this paper.

Theorem 5.1 (Characterization of $\mathcal{J} \geq 0$)**.** *Let* (1.5) *and the strengthened Legendre condition* (1.6) *hold. Then the following conditions are equivalent.*

(i) $\mathcal{J} \geq 0$ *over* $\mathcal{M}_a \eta(a) = 0$ *and* $\eta(b) = 0$.

(ii) *The interval* (a, b) *contains no points conjugate to* a*, and* b *is not strictly conjugate to* a *if* b *is left-scattered, i.e. the Jacobi condition holds.*

(iii) *The natural conjoined basis* X *of* (J) *has* $X(t)$ *invertible for all* $t \in (a, b)$*, and satisfies*

$$X^T(a)\, S(a)\, X^\sigma(a) > 0 \quad \text{on } \operatorname{Ker} \mathcal{M}_a, \quad \text{if } a \text{ is right-scattered,} \tag{5.1}$$

$$X^T(t)\, S(t)\, X^\sigma(t) > 0 \quad \text{for all } t \in (a, \rho(b)), \tag{5.2}$$

$$[X^\rho(b)]^T S^\rho(b)\, X(b) \geq 0 \quad \text{if } b \text{ is left-scattered.} \tag{5.3}$$

(iv) *There exists a symmetric solution* $W(t)$ *on* (a, b) *of the explicit Riccati equation* (R)*,* $t \in (a, \rho(b))$*, satisfying*

$$R(a) + \mu(a)\, W(a) > 0 \quad \text{on } \operatorname{Ker} \mathcal{M}_a, \quad \text{if } a \text{ is right-scattered,} \tag{5.4}$$

$$R[W](a)\, (I - \mathcal{M}_a) = 0 \quad \text{and} \quad W(a) = \hat{\Gamma}_a \quad \text{if } a \text{ is right-scattered,} \tag{5.5}$$

$$\lim_{t \to a^+} W(t)\, X(t) = \hat{\Gamma}_a + \mathcal{M}_a \quad \text{if } a \text{ is right-dense,} \tag{5.6}$$

$$\mu(a)\, \mathcal{D}(a)\, \mathcal{M}_a = 0, \tag{5.7}$$

$$R(t) + \mu(t)\, W(t) > 0 \quad \text{for all } t \in (a, \rho(b)), \tag{5.8}$$

$$R^\rho(b) + \mu(\rho(b))\, W^\rho(b) \geq 0 \quad \text{if } b \text{ is left-scattered,} \tag{5.9}$$

where the matrix $\mathcal{D}(a)$ *is defined by* (4.1) *and* X *is the natural conjoined basis of* (J).

Remark 5.2. From [19], the nonnegativity of the second variation at the optimal solution $\hat{y}$ of the nonlinear problem (P) is a necessary condition for optimality. Thus, under (1.5) and (1.6), Theorem 5.1 provides equivalent second order necessary optimality conditions.

When specialized to the zero initial endpoint, Theorem 5.1 yields the following result. Note that in this case $\sigma(a)$ cannot be conjugate to a if a is right-scattered (i.e. if $\sigma(a)$ is left-scattered).

Corollary 5.3 (Characterization of $\mathcal{J} \geq 0$, zero endpoints)**.** *Let* (1.5) *and the strengthened Legendre condition* (1.6) *hold, and* $\mathcal{M}_a = I$*. Then the following conditions are equivalent.*

(i) $\mathcal{J} \geq 0$ *over* $\eta(a) = 0$ *and* $\eta(b) = 0$.

(ii) *The interval (a, b) contains no points conjugate a, and b is not strictly conjugate to a if b is left-scattered.*

(iii) *The principal solution $\hat{X}$ of* (J) *has $\hat{X}(t)$ invertible for all $t \in (a, b)$, and satisfies conditions* (5.2), (5.3).

(iv) *There exists a symmetric solution $W(t)$ on (a, b) of the explicit Riccati equation* (R), *$t \in (a, \rho(b))$, satisfying conditions* (5.8), (5.9), *and*

$$W(a) = 0 \quad \text{if } a \text{ is right-scattered}, \tag{5.10}$$

$$\lim_{t \to a^+} W(t)\, \hat{X}(t) = I \quad \text{if } a \text{ is right-dense}, \tag{5.11}$$

$$\mu(a)\, \mathcal{D}(a) = 0. \tag{5.12}$$

For the case of the free left endpoint we have that $X(a) = I$ is invertible, so that the solution $W(t)$ of the Riccati equation always exists at $t = a$, i.e. $W(t)$ is always defined on $[a, b)$.

Corollary 5.4 (Characterization of $\mathcal{J} \geq 0$, free left endpoint)**.** *Let* (1.5) *and the strengthened Legendre condition* (1.6) *hold, and $\mathcal{M}_a = 0$. Then the following conditions are equivalent.*

(i) *$\mathcal{J} \geq 0$ over $\eta(b) = 0$.*

(ii) *The interval (a, b) contains no points conjugate a, and b is not strictly conjugate to a if b is left-scattered.*

(iii) *The solution $\tilde{X}$ of* (J) *given by $\tilde{X}(a) = I$, $\tilde{U}(a) = \hat{\Gamma}_a$ is invertible for all $t \in [a, b)$,*

$$\tilde{X}^T(t)\, S(t)\, \tilde{X}^\sigma(t) > 0 \quad \text{for all } t \in [a, \rho(b)),$$

and satisfies condition (5.3).

(iv) *There exists a symmetric solution $W(t)$ on $[a, b)$ of the explicit Riccati equation* (R), *$t \in [a, \rho(b))$, satisfying $W(a) = \hat{\Gamma}_a$,*

$$R(t) + \mu(t)\, W(t) > 0 \quad \text{for all } t \in [a, \rho(b)),$$

and condition (5.9) *holds.*

Proof of Theorem 5.1. The proof will consist of the following implications

$$\begin{tikzcd}
\text{(i)} \arrow[r, Rightarrow, "\text{L.5.5}"] & \text{(ii)} \arrow[d, Rightarrow, "\text{L.5.6}"] & \text{(iv)} \arrow[l, Rightarrow, "\text{L.5.8}"'] \\
 & \text{(iii)} \arrow[ul, Rightarrow, "\text{L.5.9}"] \arrow[ur, Rightarrow, "\text{L.5.7}"'] &
\end{tikzcd}$$

with the aid of Lemmas 5.5-5.9 proven below. ∎

Assume (1.5) and (1.6). The rest of this section is devoted to the proof of Theorem 5.1. In order to shorten our calculations, we abbreviate the integrand of the functional $\mathcal{I}$ by $\Omega(t)$, i.e.

$$\Omega(t) := \big\{(\eta^\sigma)^T C\, \eta^\sigma + q^T B\, q\big\}(t), \quad \tilde{\Omega}(t) := \big\{(\tilde{\eta}^\sigma)^T C\, \tilde{\eta}^\sigma + \tilde{q}^T B\, \tilde{q}\big\}(t).$$

Lemma 5.5. *In Theorem 5.1, (i) implies (ii).*

Proof. Assume $\mathcal{J} \geq 0$. Let η be a nontrivial solution of (J) satisfying (3.5). Then, by Lemma 3.5, $(\eta, q) = (X\alpha, U\alpha)$ on $[a, b]$ for some vector $\alpha \neq 0$, where X is the natural conjoined basis of (J). If condition (ii) of Theorem 5.1 does not hold, then there exist $\alpha \neq 0$ and $c \in (a, b)$ such that for $(\eta, q) := (X\alpha, U\alpha)$ one of the following Cases I-III holds true:

- Case I (c is left-dense point and $\eta(c) = 0$). Then $X(c)\,\alpha = 0$. Define $(\tilde{\eta}, \tilde{q})$ by (3.7). Then Lemma 3.7 yields $\mathcal{J}(\tilde{\eta}) = \eta^T(c)\, q(c) = 0$. Since we assume $\mathcal{J} \geq 0$, it follows that $\tilde{\eta}$ is optimal for $\mathcal{J}$ and hence, by [19, Theorem 1], $\tilde{\eta}$ solves the Jacobi equation (J) on $[a, \rho(b)]$. But since $\tilde{\eta}(t) \equiv 0$ on $[c, b]$ (note that $c < b$), i.e. $\tilde{\eta}(t) = 0$ at least at two consecutive points, it results that $\tilde{\eta}(t) \equiv 0$ on $[a, b]$. Hence, $\eta(t) = \tilde{\eta}(t) = 0$ on $[a, c)$, which implies that $\eta^\Delta(a) = 0$ (use $a < c$). Consequently, $q(a) = 0$. Therefore, we have

$$X(a)\,\alpha = \eta(a) = 0, \quad U(a)\,\alpha = q(a) = 0.$$

This in turn yields $\alpha = 0$, which is a contradiction.

- Case II (c is left-scattered such that $[\eta^\rho(c)]^T S^\rho(c)\, \eta(c) \leq 0$ and $\eta^\rho(c) \neq 0$). Define $(\tilde{\eta}, \tilde{q})$ by (3.8). Then Lemma 3.7 yields $\mathcal{J}(\tilde{\eta}) = \{\frac{1}{\mu}\, \eta^T S\, \eta^\sigma\}^\rho(c) \leq 0$. As we assume $\mathcal{J} \geq 0$, it follows that $\mathcal{J}(\tilde{\eta}) = 0$, i.e. $\tilde{\eta}$ is optimal for $\mathcal{J}$. Thus, $\tilde{\eta}$ must solve the Jacobi equation (J). But since $\tilde{\eta} \equiv 0$ on $[c, b]$, i.e. $\tilde{\eta}(t) = 0$ at least at two consecutive points (note $c < b$), we get $\tilde{\eta}(t) \equiv 0$ on $[a, b]$. This yelds that $\eta^\rho(c) = \tilde{\eta}^\rho(c) = 0$, which contradicts $\eta^\rho(c) \neq 0$.

- Case III (the point b is left-scattered, $\eta^\rho(b) \neq 0$, and $[\eta^\rho(b)]^T S^\rho(b)\, \eta(b) < 0$). Define $(\tilde{\eta}, \tilde{q})$ by (3.8) with $c = b$. Then Lemma 3.7 (with $c = b$) yields $\mathcal{J}(\tilde{\eta}) = \{\frac{1}{\mu}\, \eta^T S\, \eta^\sigma\}^\rho(b) < 0$. This however contradicts the assumption $\mathcal{J} \geq 0$. ☐

Lemma 5.6. *In Theorem 5.1, (ii) implies (iii).*

Proof. Suppose that the condition (ii) of Theorem 5.1 holds. Let X be the natural conjoined basis of (J). If the conditions in (iii) of Theorem 5.1 are not satisfied, then one of the following Cases I-IV holds true:

- Case I (a is right scattered and $d^T X^T(a)\, S(a)\, X^\sigma(a)\, d \leq 0$ for some nonzero vector $d \in \operatorname{Ker} \mathcal{M}_a$). We set $\eta(t) := X(t)\, d$ and $q(t) := U(t)\, d$ on $[a, b]$. Then (η, q) solves (H), satisfies (3.5), and $\eta(a) = d \neq 0$. Moreover, since $\eta^T(a)\, S(a)\, \eta^\sigma(a) = d^T X^T(a)\, S(a)\, X^\sigma(a)\, d \leq 0$, it follows that $\sigma(a) \in (a, b)$ is conjugate to a. This is a contradiction.

• Case II (there exists a point $c \in (a,b)$ such that $X(c)$ is not invertible). Then $X(c)\,d = 0$ for some nonzero vector $d \in \mathbb{R}^n$. Set $\eta(t) := X(t)\,d$ and $q(t) := U(t)\,d$ on $[a,b]$. Then (η, q) is a nontrivial solution of (H) satisfying (3.5) such that $\eta(c) = 0$.

- ▷ Suppose that c is left-dense. Then $c \in (a,b)$ is a point conjugate to a, which is a contradiction.
- ▷ Suppose that c is left-scattered. Depending on the value of $\eta^\rho(c)$, we have the following subcases:
  - – Subcase IIa ($\eta^\rho(c) = 0$). Since also $\eta(c) = 0$, it follows that $\eta(t) \equiv 0$ on $[a,b]$, and hence, $q(t) \equiv 0$ on $[a,b]$ and thus there is a contradiction.
  - – Subcase IIb ($\eta^\rho(c) \neq 0$). Then $[\eta^\rho(c)]^T S^\rho(c)\,\eta(c) = 0$. As $c < b$, this means that $c \in (a,b)$ is conjugate to a, which is a contradiction.

• Case III (there exists a point $c \in (a, \rho(b))$ such that $X^T(c)\,S(c)\,X^\sigma(c) \not> 0$). Since the strengthened Legendre condition (1.6) holds, such a point c must be right-scattered and $\sigma(c) < b$. Then there exists a nonzero vector $d \in \mathbb{R}^n$ such that $d^T X^T(c)\,S(c)\,X^\sigma(c)\,d \leq 0$. Set $\eta(t) := X(t)\,d$ and $q(t) := U(t)\,d$ on $[a,b]$. Then (η, q) is a nontrivial solution of (H) satisfying (3.5) and, for $\bar{c} := \sigma(c)$ we have

$$[\eta^\rho(\bar{c})]^T S^\rho(\bar{c})\,\eta(\bar{c}) = \left\{\eta^T S\,\eta^\sigma\right\}^\rho(\bar{c}) = \eta^T(c)\,S(c)\,\eta^\sigma(c) \leq 0. \tag{5.13}$$

Depending on the value of $\eta(c)$ we have the following:

- ▷ If $\eta(c) \neq 0$, i.e. $\eta^\rho(\bar{c}) \neq 0$, then, by using $\bar{c} < b$, equation (5.13) means that the left-scattered point $\sigma(c) = \bar{c} \subseteq (a,b)$ and is conjugate to a, which is a contradiction.
- ▷ If $\eta(c) = 0$, then we will distinguish further subcases (note that we use $a < c$):
  - – Subcase IIIa (c is left-dense). Then $c \in (a,b)$ is conjugate to a, which is a contradiction.
  - – Subcase IIIb (c is left-scattered). Dividing into another subsubcases we have:
    - ∗ Subsubcase IIIb1 ($\eta^\rho(c) = 0$). This condition together with $\eta(c) = 0$ implies that $\eta(t) \equiv 0$ on $[a,b]$, which is a contradiction.
    - ∗ Subsubcase IIIb2 ($\eta^\rho(c) \neq 0$). Then we have $[\eta^\rho(c)]^T S^\rho(c)\,\eta(c) = 0$, so that $c \in (a,b)$ is conjugate to a, which is a contradiction.

• Case IV ($[X^\rho(b)]^T S^\rho(b)\,X(b) \not\geq 0$ if b is left-scattered). Then there exists a nonzero vector $d \in \mathbb{R}^n$ such that $d^T [X^\rho(b)]^T S^\rho(b)\,X(b)\,d < 0$. Set $\eta(t) := X(t)\,d$ and $q(t) := U(t)\,d$ on $[a,b]$. Then η is a nontrivial solution of (J) satisfying (3.5) such that $[\eta^\rho(b)]^T S^\rho(b)\,\eta(b) < 0$ and $\eta^\rho(b) \neq 0$. Hence, b is strictly conjugate to a, which is a contradiction. □

Lemma 5.7. *In Theorem 5.1, (iii) implies (iv).*

Proof. Let X be the natural conjoined basis of (J) and define the function W by (4.2). Then if a is right-scattered, set $W(a) := \hat{\Gamma}_a$. By Remark 4.2, $W(t) = U(t)\,X^{-1}(t)$ on

(a,b), $W(t)$ is symmetric on $[a,b)$, it solves (R) on $(a,\rho(b))$ and satisfies both (5.5) and (5.6).

For (5.4), let a be right-scattered and $0\neq\alpha\in\operatorname{Ker}\mathcal{M}_a$. Then $\alpha=X(a)\,\alpha$ and

$$\begin{aligned}\alpha^T\big\{R(a)+\mu(a)\,W(a)\big\}\,\alpha&=\alpha^TX^T(a)\,\big\{R(a)+\mu(a)\,\hat{\Gamma}_a\big\}\,X(a)\,\alpha\\&=\alpha^TX^T(a)\,S(a)\,X^\sigma(a)\,\alpha>0,\end{aligned}$$

by (5.1) and (3.4). Hence, (5.4) holds. In this case, $\{0\}=\operatorname{Ker}X^\sigma(a)\subseteq\operatorname{Ker}X(a)$, and thus, by [4, Lemma 2], the symmetric matrix $\mathcal{D}(a)$ equals to the matrix $D(a):=X(a)\,[X^\sigma(a)]^{-1}S^{-1}(a)$, so that

$$\mu(a)\,\mathcal{D}(a)\,\mathcal{M}_a=\mu(a)\,D^T(a)\,\mathcal{M}_a=\mu(a)\,S^{T-1}(a)\,[X^\sigma(a)]^{T-1}(I-\mathcal{M}_a)\,\mathcal{M}_a=0.$$

Thus, (5.7) holds. Next, using (2.1) and that $WX=UX^\dagger X$ it follows that for $t\in[a,\rho(b)]$ we have (the argument (t) is omitted in the next computations)

$$X^TSX^\sigma=X^TS(\tilde{A}X+\mu\tilde{A}BU)=X^T(R+\mu W)X. \tag{5.14}$$

Hence, for $t\in(a,\rho(b))$, where $X(t)$ is invertible, we get from (5.14) that $X^TSX^\sigma>0$ if and only if $R+\mu W>0$, i.e. (5.8) holds. For $t=\rho(b)$ we obtain from (5.14) that $X^TSX^\sigma\geq 0$ if and only if (5.9) holds. □

Lemma 5.8. *In Theorem 5.1, (iv) implies (ii).*

Proof. Let η be a solution of (J) satisfying (3.5) and $\eta(t)\not\equiv 0$. By Lemma 3.5, $\eta(t)=X(t)\,\alpha$ and $q(t)=U(t)\,\alpha$ on $[a,b]$, where $\alpha\in\mathbb{R}^n$ is a nonzero vector and where X is the natural conjoined basis of (J). Let $W(t)$ be the solution of (R) from the condition (iv) of Theorem 5.1. If the condition (ii) of Theorem 5.1 is not satisfied, there exist $\alpha\neq 0$ and $c\in(a,b)$ such that for $(\eta,q):=(X\alpha,U\alpha)$ one of the following Cases I-III holds true:

- Case I (c is left-dense point and $\eta(c)=0$). Define $(\tilde{\eta},\tilde{q})$ by (3.7). Then by Lemma 3.7, $\mathcal{J}(\tilde{\eta})=\mathcal{I}(\tilde{\eta},\tilde{q})=\eta^T(c)\,q(c)=0$. On the other hand, we can calculate the value of $\mathcal{J}(\tilde{\eta})$ by using the Picone identity:

▷ Subcase I-A (a is right-dense). Let $a_n\searrow a$ be a right-sequence for a. Then the Picone identity (Proposition 4.1) on $[a_n,c]$, $\eta(c)=0$, and Remarks 2.1, 2.3 yield

$$\int_{a_n}^c\tilde{\Omega}(t)\,\Delta t=\eta^T(c)\,W(c)\,\eta(c)-\eta^T(a_n)\,W(a_n)\,\eta(a_n)+\int_{a_n}^c\big\{z^T\mathcal{D}z\big\}(t)\,\Delta t,$$

where $z = \tilde{q} - W\tilde{\eta}$ and $\mathcal{D} = (R+\mu W)^{-1}$ on $[a_n, c]$. Hence, by using $\eta(t) = X(t)\alpha$, $X(a) = I - \mathcal{M}_a$, and (5.6), we get

$$\begin{aligned}
\mathcal{J}(\tilde{\eta}) = \mathcal{I}(\tilde{\eta}, \tilde{q}) &= \tilde{\eta}^T(a)\,\hat{\Gamma}_a\,\tilde{\eta}(a) + \lim_{n\to\infty} \left\{ \int_{a_n}^{c} + \int_{c}^{b} \right\} \tilde{\Omega}(t)\,\Delta t \\
&= \eta^T(a)\,\hat{\Gamma}_a\,\eta(a) - \lim_{n\to\infty} \left\{ \eta^T(a_n)\,W(a_n)\,\eta(a_n) + \int_{a_n}^{c} \{z^T \mathcal{D} z\}(t)\,\Delta t \right\} \\
&= \lim_{n\to\infty} \left\{ \eta^T(a_n)\,[\mathcal{M}_a + \hat{\Gamma}_a - W(a_n)\,X(a_n)]\,\alpha + \int_{a_n}^{c} \{z^T \mathcal{D} z\}(t)\,\Delta t \right\} \\
&= \int_a^c \{z^T \mathcal{D} z\}(t)\,\Delta t.
\end{aligned}$$

Consequently, using (5.8), we obtain $\mathcal{D}(t)\,z(t) = 0$ on (a, c).

▷ Subcase I-B (a is right-scattered). Then the Picone identity (Proposition 4.3 with $\alpha = 0$) on $[a, \sigma(a)]$ and (Proposition 4.1) on $[\sigma(a), c]$, (5.5), (5.7), (5.8), and Remarks 2.1, 2.3 yield

$$\begin{aligned}
\mathcal{J}(\tilde{\eta}) = \mathcal{I}(\tilde{\eta}, \tilde{q}) &= \tilde{\eta}^T(a)\,\hat{\Gamma}_a\,\tilde{\eta}(a) + \left\{ \int_a^{\sigma(a)} + \int_{\sigma(a)}^{c} + \int_c^b \right\} \tilde{\Omega}(t)\,\Delta t \\
&= \eta^T(a)\,[\hat{\Gamma}_a - W(a)]\,\eta(a) + \eta^T(c)\,W(c)\,\eta(c) + \left\{ \int_a^{\sigma(a)} + \int_{\sigma(a)}^{c} \right\} \{z^T \mathcal{D} z\}(t)\,\Delta t \\
&= \mu(a)\,z^T(a)\,\mathcal{D}(a)\,z(a) + \int_{\sigma(a)}^{c} \{z^T \mathcal{D} z\}(t)\,\Delta t
\end{aligned}$$

where $z = \tilde{q} - W\tilde{\eta}$ on $[a, c]$ and $\mathcal{D} = (R + \mu W)^{-1} > 0$ on $[\sigma(a), c)$. By using (5.5) and (3.5) we get $\mu(a)\,z(a) = \mu(a)\,\mathcal{M}_a \gamma_a$. Together with (5.7) we obtain

$$\mu(a)\,z^T(a)\,\mathcal{D}(a)\,z(a) = z^T(a)\,\{\mu(a)\,\mathcal{D}(a)\,\mathcal{M}_a\}\gamma_a = 0, \tag{5.15}$$

which implies that $\mathcal{J}(\tilde{\eta}) \geq \int_a^c \{z^T \mathcal{D} z\}(t)\,\Delta t$. Hence, $\mathcal{D}(t)\,z(t) = 0$ on $[\sigma(a), c)$.

Thus, in both Subcases I-A & I-B we have $\mathcal{D}(t)\,z(t) = 0$ on (a, c), i.e. $z(t) = 0$ on (a, c). This implies that $\tilde{q}(t) = W(t)\,\tilde{\eta}(t)$ on (a, c), i.e. we have from the definition of $(\tilde{\eta}, \tilde{q})$ that $U(t)\,\alpha = W(t)\,X(t)\,\alpha$ on (a, c). As the point c is left-dense, take now a left-sequence $c_n \nearrow c$. Since X, U, and W are continuous at $t = c$, and since $\eta(c) = 0$, we get

$$U(c)\,\alpha = \lim_{n\to\infty} U(c_n)\,\alpha = \lim_{n\to\infty} W(c_n)\,X(c_n)\,\alpha = W(c)\,X(c)\,\alpha = W(c)\,\eta(c) = 0.$$

Thus, we have $U(c)\,\alpha = 0$, which together with $X(c)\,\alpha = 0$ yields $\alpha = 0$. This contradicts $\alpha \neq 0$.

• Case II (c is left-scattered such that $[\eta^\rho(c)]^T S^\rho(c)\,\eta(c) \leq 0$ and $\eta^\rho(c) \neq 0$). Define $(\tilde{\eta}, \tilde{q})$ by (3.8). Then by Lemma 3.7, $\mathcal{J}(\tilde{\eta}) = \mathcal{I}(\tilde{\eta}, \tilde{q}) = \{\frac{1}{\mu}\,\eta^T S\,\eta^\sigma\}^\rho(c) \leq 0$. On the other hand, by using the Picone identity, we have the following:

▷ Subcase II-A (a is right-dense). Let $a_n \searrow a$ be a right-sequence for a. Then the Picone identity (Proposition 4.1) on $[a_n, \rho(c)]$, (5.8), and Remarks 2.1, 2.3 yield

$$\begin{aligned}\int_{a_n}^{\rho(c)} \tilde{\Omega}(t)\,\Delta t &= \eta^T(t)\,W(t)\,\eta(t)\,\big|_{a_n}^{\rho(c)} + \int_{a_n}^{\rho(c)} \big\{z^T \mathcal{D} z\big\}(t)\,\Delta t \\ &\geq [\eta^\rho(c)]^T W^\rho(c)\,\eta^\rho(c) - \eta^T(a_n)\,W(a_n)\,\eta(a_n),\end{aligned}$$

where $z = \tilde{q} - W\tilde{\eta}$ and $\mathcal{D} = (R + \mu W)^{-1} > 0$ on $[a_n, \rho(c)]$. Hence, by using $\eta(t) = X(t)\alpha$, $\tilde{\eta}(c) = 0$, (3.5) and (5.6), we get

$$\begin{aligned}\mathcal{J}(\tilde{\eta}) = \mathcal{I}(\tilde{\eta}, \tilde{q}) &= \tilde{\eta}^T(a)\,\hat{\Gamma}_a\,\tilde{\eta}(a) + \lim_{n\to\infty} \left\{ \int_{a_n}^{\rho(c)} + \int_{\rho(c)}^{c} + \int_{c}^{b} \right\} \tilde{\Omega}(t)\,\Delta t \\ &\geq \eta^T(a)\,\hat{\Gamma}_a\,\eta(a) - \lim_{n\to\infty} \eta^T(a_n)\,W(a_n)\,\eta(a_n) \\ &\qquad + \big\{\eta^T W\,\eta + (\mu\,\tilde{\eta}^\sigma)^T C\,\tilde{\eta}^\sigma + \mu\,\tilde{q}^T B\,\tilde{q}\big\}^\rho(c) \\ &= \left\{\tfrac{1}{\mu}\,\eta^T(R + \mu W)\,\eta\right\}^\rho(c) > 0.\end{aligned}$$

Thus, we showed $\mathcal{J}(\tilde{\eta}) > 0$.

▷ Subcase II-B (a is right-scattered). We need to distinguish further subsubcases:

– Subsubcase II-B1 ($c > \sigma(a)$). Then the Picone identity (Proposition 4.3 with $\alpha = 0$) on $[a, \sigma(a)]$ and (Proposition 4.1) on $[\sigma(a), \rho(c)]$, (5.15), (5.8), and Remarks 2.1, 2.3 yield

$$\begin{aligned}\int_{a}^{\rho(c)} \tilde{\Omega}(t)\,\Delta t &= \eta^T(t)\,W(t)\,\eta(t)\,\big|_{a}^{\rho(c)} + \left\{ \int_{a}^{\sigma(a)} + \int_{\sigma(a)}^{\rho(c)} \right\} \big\{z^T \mathcal{D} z\big\}(t)\,\Delta t \\ &\geq \mu(a)\,z^T(a)\,\mathcal{D}(a)\,z(a) - \eta^T(a)\,W(a)\,\eta(a) + [\eta^\rho(c)]^T W^\rho(c)\,\eta^\rho(c) \\ &= -\eta^T(a)\,W(a)\,\eta(a) + [\eta^\rho(c)]^T W^\rho(c)\,\eta^\rho(c),\end{aligned}$$

where $z = \tilde{q} - W\tilde{\eta}$ on $[a, \rho(c)]$ and $\mathcal{D} = (R + \mu W)^{-1} > 0$ on $[\sigma(a), \rho(c)]$. Hence, by using (5.5), $\tilde{\eta}(c) = 0$, and (5.8) at $t = \rho(c)$ we get

$$\begin{aligned}\mathcal{J}(\tilde{\eta}) = \mathcal{I}(\tilde{\eta}, \tilde{q}) &= \tilde{\eta}^T(a)\,\hat{\Gamma}_a\,\tilde{\eta}(a) + \left\{ \int_{a}^{\rho(c)} + \int_{\rho(c)}^{c} + \int_{c}^{b} \right\} \tilde{\Omega}(t)\,\Delta t \\ &\geq \eta^T(a)\,[\hat{\Gamma}_a - W(a)]\,\eta(a) + \big\{\eta^T W\,\eta + \mu\,(\tilde{\eta}^\sigma)^T C\,\tilde{\eta}^\sigma + \mu\,\tilde{q}^T B\,\tilde{q}\big\}^\rho(c) \\ &= \left\{\tfrac{1}{\mu}\,\eta^T(R + \mu W)\,\eta\right\}^\rho(c) > 0.\end{aligned}$$

Thus, we showed $\mathcal{J}(\tilde{\eta}) > 0$.

– Subsubcase II-B2 ($c = \sigma(a)$). This yields $\eta(a) = \eta^\rho(c) \neq 0$. By using $\tilde\eta^\sigma(a) = 0$, $W(a) = \hat\Gamma_a$, and (5.4),

$$\begin{aligned}\mathcal{J}(\tilde\eta) = \mathcal{I}(\tilde\eta, \tilde q) &= \tilde\eta^T(a)\,\hat\Gamma_a\,\tilde\eta(a) + \left\{\int_a^{\sigma(a)} + \int_{\sigma(a)}^b\right\}\tilde\Omega(t)\,\Delta t \\ &= \left\{\eta^T\hat\Gamma_a\,\eta + \mu\,(\tilde\eta^\sigma)^T C\,\tilde\eta^\sigma + \mu\,\tilde q^T B\,\tilde q\right\}(a) \\ &= \tfrac{1}{\mu(a)}\,\eta^T(a)\,[R(a) + \mu(a)\,W(a)]\,\eta(a) > 0.\end{aligned}$$

Thus, $\mathcal{J}(\tilde\eta) > 0$.

Hence, in both Subcases II-A & II-B(1,2) we showed that $0 < \mathcal{J}(\tilde\eta) \le 0$, which is a contradiction.

• Case III (b is left-scattered and $[\eta^\rho(b)]^T S^\rho(b)\,\eta(b) < 0$). Define $(\tilde\eta, \tilde q)$ by (3.8) with $c = b$. Then by Lemma 3.7, $\mathcal{J}(\tilde\eta) = \mathcal{I}(\tilde\eta, \tilde q) = \{\frac{1}{\mu}\,\eta^T S\,\eta^\sigma\}^\rho(b) < 0$. On the other hand, the Picone identity yields, by the Subcases II-A & II-B1 with $c = b$, that $\mathcal{J}(\tilde\eta) \ge \{\frac{1}{\mu}\,\eta^T(R + \mu W)\,\eta\}^\rho(b) \ge 0$, by (5.9). Hence, $0 \le \mathcal{J}(\tilde\eta) < 0$, which is a contradiction. The proof of this lemma is therefore complete. □

Lemma 5.9. *In Theorem 5.1, (iii) implies (i).*

Proof. Let X be the natural conjoined basis of (J). Set $W(t) := U(t)X^{-1}(t)$ for $t \in (a, b)$ and $W(a) := \hat\Gamma_a$ if a is right-scattered. Then the proof of Lemma 5.7 shows that $W(t)$ satisfies condition (iv) of Theorem 5.1. Let (η, q) be admissible with $\mathcal{M}_a\eta(a) = 0$ and $\eta(b) = 0$.

• Case I (a is right-dense and b is left-dense). Let $a_n \searrow a$ be a right-sequence for a and $b_n \nearrow b$ a left-sequence for b. Then we can pick a point $c \in (a, b)$ such that $a_n < c < b_n$ for n large enough. Then

$$\begin{aligned}\mathcal{J}(\eta) = \mathcal{I}(\eta, q) &= \eta^T(a)\,\hat\Gamma_a\,\eta(a) + \lim_{n\to\infty}\left\{\int_{a_n}^c + \int_c^{b_n}\right\}\Omega(t)\,\Delta t \\ &= \eta^T(a)\,\hat\Gamma_a\,\eta(a) + \underbrace{\lim_{n\to\infty}\int_{a_n}^c \Omega(t)\,\Delta t}_{\text{L1}} + \underbrace{\lim_{n\to\infty}\int_c^{b_n} \Omega(t)\,\Delta t}_{\text{L2}}.\end{aligned}$$

We know that $X(t)$ is invertible on (a, b), in particular, $X(c)$ is invertible. To calculate L1, we first apply the Picone identity (Proposition 4.3) on $[a_n, c]$ with $\alpha = \alpha_n$,

$$\alpha_n := -U^T(a_n)\,\eta(a_n) + X^T(a_n)\,U(a)\,\eta(a). \tag{5.16}$$

Note that $\alpha_n + U^T(a_n)\,\eta(a_n) = X^T(a_n)\,U(a)\,\eta(a) \in \operatorname{Im} X^T(a_n)$ holds. Hence, we get

$$\int_{a_n}^c \Omega(t)\,\Delta t = F\big(t, \alpha_n, \eta(t)\big)\,\Big|_{a_n}^c + \int_{a_n}^c \left\{z^T\mathcal{D}z\right\}(t)\,\Delta t,$$

where $z = q - W\eta - X^{T-1}\alpha_n$ and $\mathcal{D} = (R + \mu W)^{-1} > 0$ on $[a_n, c]$. Thus, by Proposition 4.4 on $[a, c]$ with $d = \eta(a)$, with α_n as in (5.16), and by Remarks 2.1, 2.3

we have

$$\begin{aligned}\mathrm{L1} &= \lim_{n\to\infty}\left\{F\big(t,\alpha_n,\eta(t)\big)\,\big|_{a_n}^{c} + \int_{a_n}^{c}\big\{z^T\mathcal{D}z\big\}(t)\,\Delta t\right\}\\ &\geq \eta^T(c)\,W(c)\,\eta(c) - \eta^T(a)\,X^T(a)\,U(a)\,\eta(a).\end{aligned}$$

Similarly, for L2, the application of the extended Picone identity (Proposition 4.3) on $[c,b_n]$ with $\alpha := \beta_n$,

$$\beta_n := -U^T(b_n)\,\eta(b_n), \tag{5.17}$$

yields (note $\beta_n \in \operatorname{Im} X^T(c)$ since $X(c)$ is invertible)

$$\int_c^{b_n}\Omega(t)\,\Delta t = F\big(t,\beta_n,\eta(t)\big)\,\big|_c^{b_n} + \int_c^{b_n}\big\{z^T\mathcal{D}z\big\}(t)\,\Delta t,$$

where $z = q - W\eta - X^{T-1}\beta_n$ and $\mathcal{D} = (R+\mu W)^{-1} > 0$ on $[c,b_n]$. Thus, by Proposition 4.5 on $[c,b]$ with $d=0$, i.e. $\eta(b) = 0 = X(b)\,d$, with β_n as in (5.17), and by Remarks 2.1, 2.3 we have

$$\begin{aligned}\mathrm{L2} &= \lim_{n\to\infty}\left\{F\big(t,\beta_n,\eta(t)\big)\,\big|_c^{b_n} + \int_c^{b_n}\big\{z^T\mathcal{D}z\big\}(t)\,\Delta t\right\}\\ &\geq d^T X^T(b)\,U(b)\,d - \eta^T(c)\,W(c)\,\eta(c) = -\eta^T(c)\,W(c)\,\eta(c).\end{aligned}$$

Hence,

$$\mathcal{J}(\eta) = \eta^T(a)\,\hat\Gamma_a\,\eta(a) + \mathrm{L1} + \mathrm{L2} \geq \eta^T(a)\,\hat\Gamma_a\,\eta(a) - \eta^T(a)\,X^T(a)\,U(a)\,\eta(a) = 0,$$

since $X^T(a)\,U(a) = \hat\Gamma_a$. Thus, $\mathcal{J}(\eta)\geq 0$. This ends Case I.

• Case II (a right-scattered and b left-dense). Let $b_n \nearrow$ be a left sequence for b and without loss of generality assume that $\sigma(a) < b_n$ for n large enough. Then

$$\begin{aligned}\mathcal{J}(\eta) = \mathcal{I}(\eta,q) &= \eta^T(a)\,\hat\Gamma_a\,\eta(a) + \lim_{n\to\infty}\left\{\int_a^{\sigma(a)} + \int_{\sigma(a)}^{b_n}\right\}\Omega(t)\,\Delta t\\ &= \eta^T(a)\,\hat\Gamma_a\,\eta(a) + \underbrace{\int_a^{\sigma(a)}\Omega(t)\,\Delta t}_{\mathrm{L3}} + \underbrace{\lim_{n\to\infty}\int_{\sigma(a)}^{b_n}\Omega(t)\,\Delta t}_{\mathrm{L4}}.\end{aligned}$$

For L3, note $\eta(a)\in\operatorname{Im} X(a)$ holds and the Picone identity (Proposition 4.3 with $\alpha = 0$) on $[a,\sigma(a)]$ yields

$$\mathrm{L3} = \eta^T(t)W(t)\eta(t)\,\big|_a^{\sigma(a)} + \int_a^{\sigma(a)}\big\{z^T\mathcal{D}z\big\}(t)\,\Delta t = \big\{(\eta^\sigma)^T W^\sigma\eta^\sigma - \eta^T W\,\eta + \mu\, z^T\mathcal{D}\,z\big\}(a),$$

where $z(a) := q(a) - W(a)\,\eta(a)$. Now use (5.15) to obtain $\mu(a)\,z^T(a)\,\mathcal{D}(a)\,z(a) = 0$. For L4, use first the extended Picone identity (Proposition 4.3) on $[\sigma(a),b_n]$ with $\alpha = \beta_n$ given by (5.17) (note $\beta_n \in \operatorname{Im}[X^\sigma(a)]^T$, since $X^\sigma(a)$ is invertible),

$$\int_{\sigma(a)}^{b_n}\Omega(t)\,\Delta t = F\big(t,\beta_n,\eta(t)\big)\,\big|_{\sigma(a)}^{b_n} + \int_{\sigma(a)}^{b_n}\big\{z^T\mathcal{D}z\big\}(t)\,\Delta t,$$

where $z = q - W\eta - X^{T-1}\beta_n$ and $\mathcal{D} = (R + \mu W)^{-1} > 0$ on $[\sigma(a), b_n]$. Next, apply Proposition 4.5 on $[\sigma(a), b]$ with $\alpha = \beta_n$ and $d = 0$ and Remarks 2.1, 2.3 to get

$$\begin{aligned} \mathrm{L4} &= \lim_{n\to\infty} \left\{ F\big(t, \beta_n, \eta(t)\big)\,\Big|_{\sigma(a)}^{b_n} + \int_{\sigma(a)}^{b_n} \big\{z^T \mathcal{D} z\big\}(t)\,\Delta t \right\} \\ &\geq d^T X^T(b)\, U(b)\, d - [\eta^\sigma(a)]^T W^\sigma(a)\, \eta^\sigma(a) = -[\eta^\sigma(a)]^T W^\sigma(a)\, \eta^\sigma(a). \end{aligned}$$

Thus, in view of (5.15),

$$\begin{aligned} \mathcal{J}(\eta) &= \eta^T(a)\, \hat{\Gamma}_a\, \eta(a) + \mathrm{L3} + \mathrm{L4} \\ &\geq \Big\{ \eta^T (\hat{\Gamma}_a - W)\, \eta + (\eta^\sigma)^T W^\sigma \eta^\sigma - (\eta^\sigma)^T W^\sigma \eta^\sigma \Big\}(a) = 0. \end{aligned}$$

Hence, we showed that $\mathcal{J}(\eta) \geq 0$. This ends Case II.

- Case III (a right-dense and b left-scattered). Let $a_n \searrow a$ be a right-sequence for a and without loss of generality assume that $a_n < \rho(b)$ for n large enough. Then

$$\begin{aligned} \mathcal{J}(\eta) = \mathcal{I}(\eta, q) &= \eta^T(a)\, \hat{\Gamma}_a\, \eta(a) + \lim_{n\to\infty} \left\{ \int_{a_n}^{\rho(b)} + \int_{\rho(b)}^{b} \right\} \Omega(t)\,\Delta t \\ &= \eta^T(a)\, \hat{\Gamma}_a\, \eta(a) + \underbrace{\lim_{n\to\infty} \int_{a_n}^{\rho(b)} \Omega(t)\,\Delta t}_{\mathrm{L5}} + \underbrace{\int_{\rho(b)}^{b} \Omega(t)\,\Delta t}_{\mathrm{L6}}. \end{aligned}$$

For L5, apply first the extended Picone identity (Proposition 4.3) on $[a_n, \rho(b)]$ with $\alpha = \alpha_n$ given by (5.16) (note $\alpha_n \in \operatorname{Im} X^T(a_n)$, since $X(a_n)$ is invertible),

$$\int_{a_n}^{\rho(b)} \Omega(t)\,\Delta t = F\big(t, \alpha_n, \eta(t)\big)\,\Big|_{a_n}^{\rho(b)} + \big\{z^T \mathcal{D} z\big\}(t)\,\Delta t,$$

where $z = q - W\eta - X^{T-1}\alpha_n$ on $[a_n, \rho(b)]$ and $\mathcal{D} = (R + \mu W)^{-1} > 0$ on $[a_n, \rho(b))$. Next, apply Proposition 4.4 on $[a, \rho(b)]$ with $d = \eta(a)$ and α_n as in (5.16) and Remarks 2.1, 2.3 to get

$$\begin{aligned} \mathrm{L5} &= \lim_{n\to\infty} \left\{ F\big(t, \alpha_n, \eta(t)\big)\,\Big|_{a_n}^{\rho(b)} + \int_{a_n}^{\rho(b)} \big\{z^T \mathcal{D} z\big\}(t)\,\Delta t \right\} \\ &\geq [\eta^\rho(b)]^T W^\rho(b)\, \eta^\rho(b) - \eta^T(a)\, X^T(a)\, U(a)\, \eta(a). \end{aligned}$$

For L6, observe that since $\eta(b) = 0$, the equation of motion at $t = \rho(b)$ yields $\eta^\rho(b) = -\{\mu\, B\, q\}^\rho(b)$, and thus we obtain

$$\mathrm{L6} = \big\{\mu\, (\eta^\sigma)^T C\, \eta^\sigma + \mu\, q^T B\, q\big\}^\rho(b) = \Big\{\tfrac{1}{\mu}\, \eta^T R\, \eta\Big\}^\rho(b).$$

Therefore, from (5.9) it follows that

$$\begin{aligned} \mathcal{J}(\eta) &= \eta^T(a)\, \hat{\Gamma}_a\, \eta(a) + \mathrm{L5} + \mathrm{L6} \\ &\geq \Big\{\eta^T (\hat{\Gamma}_a - X^T U)\, \eta\Big\}(a) + \Big\{\eta^T W\, \eta + \tfrac{1}{\mu}\, \eta^T R\, \eta\Big\}^\rho(b) = \Big\{\tfrac{1}{\mu}\, \eta^T (R + \mu W)\, \eta\Big\}^\rho(b) \geq 0. \end{aligned}$$

Hence, we again have $\mathcal{J}(\eta) \geq 0$ and this ends Case III.

- Case IV (a right-scattered and b left-scattered). Then

$$\mathcal{J}(\eta) = \mathcal{I}(\eta, q) = \eta^T(a)\,\hat{\Gamma}_a\,\eta(a) + \underbrace{\int_a^{\rho(b)} \Omega(t)\,\Delta t}_{\text{L7}} + \underbrace{\int_{\rho(b)}^{b} \Omega(t)\,\Delta t}_{\text{L6}}.$$

For L7, note that $\eta(a) \in \operatorname{Im} X(a)$ and apply the Picone identity (Proposition 4.3 with $\alpha = 0$) on $[a, \rho(b)]$,

$$\int_a^{\rho(b)} \Omega(t)\,\Delta t = \eta^T(t)\,W(t)\,\eta(t)\,\Big|_a^{\rho(b)} + \int_a^{\rho(b)} \big\{z^T \mathcal{D} z\big\}(t)\,\Delta t,$$

where $z = q - W\eta$ on $[a, \rho(b)]$ and $\mathcal{D} = (R + \mu W)^{-1} > 0$ on $[\sigma(a), \rho(b))$. Hence, by using (5.15), (5.9), and Remarks 2.1, 2.3 we get

$$\begin{aligned}
\mathcal{J}(\eta) &= \eta^T(a)\,[\hat{\Gamma}_a - W(a)]\,\eta(a) + \Big\{\eta^T W\,\eta + \tfrac{1}{\mu}\,\eta^T R\eta\Big\}^{\rho}(b) \\
&\qquad\qquad + \left\{\int_a^{\sigma(a)} + \int_{\sigma(a)}^{\rho(b)}\right\} \big\{z^T \mathcal{D} z\big\}(t)\,\Delta t, \\
&\geq \Big\{\tfrac{1}{\mu}\,\eta^T (R + \mu W)\,\eta\Big\}^{\rho}(b) + \big\{\mu z^T \mathcal{D} z\big\}(a) \geq 0.
\end{aligned}$$

Thus, we proved $\mathcal{J}(\eta) \geq 0$, which ends Case IV. The proof of this lemma is now complete. □

Remark 5.10. In the proof of Theorem 5.1 we closed the loop(s) by showing "(iv) ⇒ (ii)" and "(iii) ⇒ (i)". For the case of both zero endpoints, as in Corollary 5.3, the proof can be much simpler by directly showing "(iv) ⇒ (i)". The method is the same as the one of Lemma 5.9, but a different conjoined basis $\bar{X}$ is used instead of the natural one. This is proven below. The proof below also shows that, in the case of zero endpoints, the limit condition (5.11) is not needed in the statement of Corollary 5.3.

Proof of "(iv) ⇒ (i)" in Corollary 5.3. Let $W(t)$ be the solution of (R) from condition (iv) of Corollary 5.3. First we construct an auxiliary conjoined basis $(\bar{X}, \bar{U})$ of (H) with $\bar{X}(t)$ invertible on (a, b). Let $c \in (a, b)$ be fixed and define X to be the solution of the system

$$X^{\Delta} = \tilde{A}(t)\,[A(t) + B(t)\,W(t)]\,X, \quad X(c) = I, \quad t \in (a, b). \tag{5.18}$$

Note that equation (5.18) has a unique solution, since the coefficient matrix is rd-continuous and regressive, i.e. $I + \mu\tilde{A}(A + BW) = S^{-1}(R + \mu W)$ is invertible. Then $X(t)$ is invertible for all $t \in (a, b)$. Set $U(t) := W(t)\,X(t)$ on (a, b). Then (X, U) solves (H) on (a, b). Now let $(\bar{X}, \bar{U})$ be the solution of (H) given by the initial conditions $\bar{X}(c) = X(c) = I$ and $\bar{U}(c) = U(c) = W(c)$. This conjoined basis is now defined on $[a, b]$ and, by the uniqueness of solutions of (H), $(\bar{X}, \bar{U})$ coincides with (X, U) on (a, b), i.e. $\bar{X}(t)$ is invertible for all $t \in (a, b)$.

Now we can proceed in proving that $\mathcal{J} \geq 0$ over $\eta(a) = 0 = \eta(b)$ by the same way as in the proof of Lemma 5.9. More precisely, in each of the Cases I-IV we will use the conjoined basis $\bar{X}$ instead of the natural conjoined basis. Furthermore, whenever we needed in the proof of Lemma 5.9 that $\eta(a)$ lies in the image of the natural conjoined basis, we use now $\eta(a) = 0 \in \operatorname{Im} \bar{X}(a)$. □

6. POSITIVITY OF $\mathcal{J}$

By strengthening of the conditions of Theorem 5.1 at $t = \rho(b)$ we obtain the characterization of the positivity of $\mathcal{J}$. For the $\mathrm{C}^1_{\mathrm{rd}}$ solutions, the equivalence of $\mathcal{J} > 0$ with the (natural) conjoined basis condition can be obtained from the known results in [1, 13, 14, 15, 16].

Theorem 6.1 (Characterization of $\mathcal{J} > 0$)**.** *Let* (1.5) *and the strengthened Legendre condition* (1.6) *hold. The following conditions are equivalent.*

(i) $\mathcal{J} > 0$ *over* $\mathcal{M}_a \eta(a) = 0$, $\eta(b) = 0$, *and* $\eta \not\equiv 0$.

(ii) *The interval* $(a, b]$ *contains no points conjugate to* a, *i.e. the strengthened Jacobi condition holds.*

(iii) *The natural conjoined basis* X *of* (J) *has* $X(t)$ *invertible for all* $t \in (a, b]$, *satisfies condition* (5.1), *and*

$$X^T(t)\, S(t)\, X^\sigma(t) > 0 \quad \textit{for all } t \in (a, \rho(b)]. \tag{6.1}$$

(iv) *There exists a symmetric solution* $W(t)$ *on* $(a, b]$ *of the explicit Riccati equation* (R), $t \in (a, \rho(b)]$, *satisfying conditions* (5.4), (5.5), (5.6), (5.7), *and*

$$R(t) + \mu(t)\, W(t) > 0 \quad \textit{for all } t \in (a, \rho(b)]. \tag{6.2}$$

Proof. The proof follows the same implications as the proof of Theorem 5.1. More precisely, in the proof of "(i) ⇒ (ii)" (as in Lemma 5.5) we only have Case I with $c \in (a, b]$. In the proof of "(ii) ⇒ (iii)" (as in Lemma 5.6) we have Case I, Case II with $c \in (a, b]$, and Case III with $c \in (a, \rho(b)]$. In the proof of "(iii) ⇒ (iv)" (as in Lemma 5.7) we have $W(t) := U(t)\, X^{-1}(t)$ on $(a, b]$. In the proof of "(iv) ⇒ (ii)" (as in Lemma 5.8) we have Case I with $c \in (a, b]$, and Case II with $c \in [\sigma(a), \rho(b)]$. In the proof of "(iii) ⇒ (i)" (as in Lemma 5.9) we distinguish only two cases (a right-dense or right-scattered) and apply the Picone identity on $[a_n, b]$ if a is right-dense, or on $[a, b]$ if a is right-scattered. This way we obtain $\mathcal{J} \geq 0$. In order to prove $\mathcal{J} > 0$, we must show that $\mathcal{J}(\eta) = 0$ implies $\eta(t) \equiv 0$ on $[a, b]$. If $\mathcal{J}(\eta) = 0$ for some admissible η with $\mathcal{M}_a \eta(a) = 0$ and $\eta(b) = 0$, then the proof of Lemma 5.9 yields

$$\mathcal{D}(t)\, z(t) = 0 \quad \begin{cases} \text{on } [a_n, \rho(b)] \text{ if } a \text{ is right-dense,} \\ \text{on } [a, \rho(b)] \text{ if } a \text{ is right-scattered,} \end{cases} \tag{6.3}$$

where $z(t) := q(t) - W(t)\,\eta(t)$ on $[a,b]$, $W(t) := U(t)\,X^{-1}(t)$ on $(a,b]$, and $W(a) := \hat{\Gamma}_a$ if a is right-scattered. Note that this $W(t)$ satisfies condition (iv) of this theorem, by the earlier implication "(iii) $\Rightarrow$ (iv)". Since, by (6.2), $\mathcal{D} = (R+\mu W)^{-1}$ is invertible on $(a,\rho(b)]$, it follows from (6.3) that in both cases we have $q(t) = W(t)\,\eta(t)$ on $(a,\rho(b)]$. Hence, the definition of q in (3.1) yields

$$\eta^{\Delta}(t) = S^{-1}(t)\,[W(t) - Q^T(t)]\,\eta(t) \quad \text{for all } t \in (a,\rho(b)]. \tag{6.4}$$

Since $I + \mu S^{-1}(W - Q^T) = S^{-1}(R + \mu W)$ is invertible, the coefficient matrix in (6.4) is regressive and rd-continuous. Thus, the initial value problem (6.4) with $\eta(b) = 0$ possesses only the trivial solution, namely $\eta(t) \equiv 0$ on $(a,b]$.

Now, if a is right-dense, then the continuity of $\eta(t)$ at a yields $\eta(a) = 0$. If a is right-scattered, the condition $\mathcal{D}(a)\,z(a) = 0$ in (6.3) implies $q(a) - W(a)\,\eta(a) = \mu(a)\,\mathcal{M}_a\gamma_a$ for some $\gamma_a \in \mathbb{R}^n$. Multiplying the latter equation by $\mu(a)$ and using (3.1) at $t = a$, we get

$$\{S\eta^{\sigma} - (R + \mu W)\,\eta\}(a) = \mu^2(a)\,\mathcal{M}_a\gamma_a.$$

Now use the fact that $\eta^{\sigma}(a) = 0$ and then multiply the above equation by $\eta^T(a)$ from the left to obtain $\{\eta^T(R + \mu W)\,\eta\}(a) = 0$. But then (5.4) implies $\eta(a) = 0$.

Hence, $\eta(t) \equiv 0$ on $[a,b]$ and, therefore, $\mathcal{J} > 0$. The proof is complete. □

For completeness and comparison, we also state below the special cases of Theorem 6.1 for the zero endpoints and for the free left endpoint. For the $\mathrm{C}^1_{\mathrm{rd}}$ solutions, the equivalence (i) $\Leftrightarrow$ (iii) in Corollary 6.2 is [1, Result 3].

Corollary 6.2 (Characterization of $\mathcal{J} > 0$, zero endpoints)**.** *Let* (1.5) *and the strengthened Legendre condition* (1.6) *hold, and* $\mathcal{M}_a = I$. *Then the following conditions are equivalent.*

(i) $\mathcal{J} > 0$ *over* $\eta(a) = 0$, $\eta(b) = 0$, *and* $\eta \not\equiv 0$.
(ii) *The interval* $(a,b]$ *contains no points conjugate to* a.
(iii) *The principal solution* $\hat{X}$ *of* (J) *has* $\hat{X}(t)$ *invertible for all* $t \in (a,b]$ *and satisfies condition* (6.1).
(iv) *There exists a symmetric solution* $W(t)$ *on* $(a,b]$ *of the explicit Riccati equation* (R), $t \in (a,\rho(b)]$, *satisfying conditions* (5.10), (5.11), (5.12), *and* (6.2).

Remark 6.3. The proof of Corollary 6.2 can be simpler than that of Theorem 6.1 by showing directly "(iv) $\Rightarrow$ (i)" as in Remark 5.10.

Corollary 6.4 (Characterization of $\mathcal{J} > 0$, free left endpoint)**.** *Let* (1.5) *and the strengthened Legendre condition* (1.6) *hold, and* $\mathcal{M}_a = 0$. *Then the following conditions are equivalent.*

(i) $\mathcal{J} > 0$ *over* $\eta(b) = 0$ *and* $\eta \not\equiv 0$.

(ii) *The interval* $(a, b]$ *contains no points conjugate to* a.

(iii) *The solution* $\tilde{X}$ *of* (J) *given by* $\tilde{X}(a) = I$, $\tilde{U}(a) = \hat{\Gamma}_a$ *has* $\tilde{X}(t)$ *invertible for all* $t \in [a, b]$ *and satisfies*

$$\tilde{X}^T(t)\, S(t)\, \tilde{X}^\sigma(t) > 0 \quad \textit{for all } t \in [a, \rho(b)].$$

(iv) *There exists a symmetric solution* $W(t)$ *on* $[a, b]$ *of the explicit Riccati equation* (R), $t \in [a, \rho(b)]$, *satisfying* $W(a) = \hat{\Gamma}_a$ *and*

$$R(t) + \mu(t)\, W(t) > 0 \quad \textit{for all } t \in [a, \rho(b)]. \tag{6.5}$$

Remark 6.5. For the zero endpoints and $\mathrm{C}^1_{\mathrm{rd}}$ solutions, [15, Theorem 1] or [7, Theorem 10.52] yields a characterization of the positivity of $\mathcal{J} > 0$ via a solution $W(t)$ of the Riccati equation (R) on the *closed* interval $[a, \rho(b)]$, like in condition (iv) of Corollary 6.4. The "dense-normality" assumption in [15, Theorem 1] holds in the present setting trivially, since $R(t)$ is assumed to be invertible. Hence, each of the conditions (i)-(iv) of Corollary 6.2 is also equivalent to the following.

(v) *There exists a solution* X *of* (J) *such that* $X(t)$ *invertible for all* $t \in [a, b]$ *and satisfying*

$$X^T(t)\, S(t)\, X^\sigma(t) > 0 \quad \textit{for all } t \in [a, \rho(b)].$$

(vi) *There exists a symmetric solution* $W(t)$ *on* $[a, b]$ *of the explicit Riccati equation* (R), $t \in [a, \rho(b)]$, *satisfying condition* (6.5).

In the following example we show how the results of this paper can be applied to an *impulsive* control dynamical system.

Example 6.6. Consider the following *time-dependent impulsive* control dynamical system on the connected interval $[0, \frac{3\pi}{4}]$,

$$\begin{cases} \text{for } t \neq \frac{\pi}{4}, & \dot{x}(t) = u(t), \quad \dot{y}(t) = -x^2(t) + u^2(t), \\ \text{for } t = \frac{\pi}{4}, & \delta x(t) = \frac{\pi}{4}\, u(t), \\ & \delta y(t) = \left(\frac{\pi}{4}\right)^2 u^2(t) + \frac{\pi}{2}\,(1-\sqrt{2})\, x(t)\, u(t) + 2\,(1-\sqrt{2})\, x^2(t), \end{cases} \tag{6.6}$$

with $x(0) = 0 = y(0)$ and $x\left(\frac{3\pi}{4}\right) = 0$. Here, for a function z, $\delta z(t)$ stands for $z(t^+) - z(t)$. For more details about such impulsive systems, see e.g. [2, 9, 10, 11].

The optimal control problem is

$$\text{minimize} \;\; y(\tfrac{3\pi}{4}) \text{ subject to } (x, y, u) \text{ satisfying (6.6)}. \tag{IC}$$

The state functions x and y are real valued and are usually *discontinuous* at the resetting time $t = \frac{\pi}{4}$, but they are supposed to be left continuous there. The control function u is piecewise continuous.

We shall see that the impulsive optimal control problem (IC) can be reformulated as minimizing a quadratic functional of the form $\mathcal{J}$, for a specific choice of the time scale $\mathbb{T}$ and the functions P, Q, and R. In fact, for given functions (x, y, u) set

$$\big(\eta(t),\, \xi(t),\, w(t)\big) := \begin{cases} \big(x(t),\, y(t),\, u(t)\big), & \text{for } t \in [0, \frac{\pi}{4}], \\ \big(x(\frac{\pi}{4}^+),\, y(\frac{\pi}{4}^+),\, u(\frac{\pi}{4}^+)\big), & \text{for } t = \frac{\pi}{2}, \\ \big(x(t-\frac{\pi}{4}),\, y(t-\frac{\pi}{4}),\, u(t-\frac{\pi}{4})\big), & \text{for } t \in (\frac{\pi}{2}, \pi], \end{cases}$$

Consider now the time scale $\mathbb{T} := [0, \frac{\pi}{4}] \cup [\frac{\pi}{2}, \pi]$. From the above definition it follows that on this time scale, the functions η and ξ are in $\mathrm{C}^1_{\mathrm{prd}}$ and the function w is in $\mathrm{C}_{\mathrm{prd}}$. Furthermore, one can easily show that: (x, y, u) satisfies the impulsive system (6.6) with the associated boundary conditions is *equivalent* to saying that the above defined triplet (η, ξ, w) satisfies on $\mathbb{T}$

$$\text{(6.7)} \quad \begin{cases} \text{for } t \neq \frac{\pi}{4}, & \eta^\Delta(t) = w(t), \quad \xi^\Delta(t) = -\eta^2(t) + w^2(t), \\ \text{for } t = \frac{\pi}{4}, & \eta^\Delta(t) = w(t), \\ & \xi^\Delta(t) = \frac{\pi}{4}\, w^2(t) + 2\,(1-\sqrt{2})\,\eta(t)\,w(t) + \frac{8}{\pi}\,(1-\sqrt{2})\,\eta^2(t), \end{cases}$$

with $\eta(0) = 0 = \xi(0)$ and $\eta(\pi) = 0$. Recall that here $\eta^\Delta(t) = \dot{\eta}(t)$ for $t \neq \frac{\pi}{4}$ and $\eta^\Delta(\frac{\pi}{4}) = [\,\eta(\frac{\pi}{2}) - \eta(\frac{\pi}{4})\,] \,/\, \frac{\pi}{4}$, similarily is the case for $\xi^\Delta(t)$ when $t \neq \frac{\pi}{4}$ and for $\xi^\Delta(\frac{\pi}{4})$.

The objective function $y(\frac{3\pi}{4})$ for (IC) is $\xi(\pi)$ which can be calculated by integrating the system (6.7) as

$$\begin{aligned} \xi(\pi) = {} & \int_0^{\pi/4} \big\{w^2(t) - \eta^2(t)\big\}\, dt + \int_{\pi/2}^{\pi} \big\{w^2(t) - \eta^2(t)\big\}\, dt \\ & + \tfrac{\pi}{4} \left\{ \tfrac{8}{\pi}\,(1-\sqrt{2})\,\big(\eta^\sigma(\tfrac{\pi}{4})\big)^2 - 2\,(1-\sqrt{2})\,\eta^\sigma(\tfrac{\pi}{4})\,\eta^\Delta(\tfrac{\pi}{4}) + \tfrac{\pi}{4}\,\big(\eta^\Delta(\tfrac{\pi}{4})\big)^2 \right\}. \end{aligned}$$

Thus, the problem (IC) is equivalent to

$$\text{(TSC)} \quad \text{minimize } \mathcal{J}(\eta) := \int_0^\pi \Big\{ P(t)\,\big(\eta^\sigma(t)\big)^2 + 2\,Q(t)\,\eta^\sigma(t)\,\eta^\Delta(t) + R(t)\,\big(\eta^\Delta(t)\big)^2 \Big\}\,\Delta t$$

subject to $\eta \in \mathrm{C}^1_{\mathrm{prd}}$ satisfying the boundary conditions $\eta(0) = 0 = \eta(\pi)$, where the time scale is $\mathbb{T} := [0, \frac{\pi}{4}] \cup [\frac{\pi}{2}, \pi]$ and the coefficients are defined by $P(t) := -1$, $Q(t) := 0$, $R(t) := 1$ for $t \neq \frac{\pi}{4}$ while $P(\frac{\pi}{4}) := \frac{8}{\pi}\,(1-\sqrt{2})$, $Q(\frac{\pi}{4}) := \sqrt{2} - 1$, and $R(\frac{\pi}{4}) := \frac{\pi}{4}$. It follows that $P(\cdot)$, $Q(\cdot)$, $R(\cdot)$ are rd-continuous, $R(t)$ and $S(t) = R(t) + \mu(t)\,Q(t)$ are invertible for all $t \in \mathbb{T}$ so that (1.5) is satisfied, and our strengthened Legendre condition (1.6) holds. The principal solution $\hat{X}(t) = \sin t$ on $\mathbb{T}$ of the associated Jacobi equation (J) is nonzero for all $t \in (0, \pi) \cap \mathbb{T}$ and satisfies condition (iii) of Corollary 5.3. Hence, by this corollary, $\mathcal{J} \geq 0$. Note that the solution $W(t)$ of the Riccati equation satisfying condition (iv) of Corollary 5.3 is $W(t) = \cot t$ for $t \in (0, \pi) \cap \mathbb{T}$. Obviously, $\mathcal{J}(\hat{X}) = 0$.

By the equivalence between the two problems (TSC) and (IC) it turns out that the impulsive optimal control problem (IC) has a nonnegative objective function with a minimum value equal to 0. In fact, set

$$\big(\,\hat{x}(t),\hat{y}(t)\,\big) := \begin{cases} \big(\,\sin t,\ \frac{1}{2}\,\sin 2t\,\big), & \text{for } t\in[0,\frac{\pi}{4}],\\ \big(\,\sin(t+\frac{\pi}{4}),\ \frac{1}{2}\,\sin(2t+\frac{\pi}{2})\,\big), & \text{for } t\in(\frac{\pi}{4},\frac{3\pi}{4}],\end{cases}$$

and

$$\hat{u}(t) := \begin{cases} \cos t, & \text{for } t\in[0,\frac{\pi}{4}),\\ \frac{2}{\pi}\,(2-\sqrt{2}), & \text{for } t=\frac{\pi}{4},\\ \cos(t+\frac{\pi}{4}), & \text{for } t\in(\frac{\pi}{4},\frac{3\pi}{4}].\end{cases}$$

Then, the triplet $(\hat{x},\hat{y},\hat{u})$ is admissible for the problem (IC) and satisfies $\hat{y}(\frac{3\pi}{4})=0$, hence it is optimal.

Example 6.7. Consider the same data as in the problem (TSC) with the exception of the time scale which is now $\mathbb{T}_b=[0,\frac{\pi}{4}]\cup[\frac{\pi}{2},b]$ with $\frac{\pi}{2}<b<\pi$. We have, by Corollary 6.2, that $\mathcal{J}>0$ over the zero endpoints $\eta(0)=0=\eta(b)$. On the other hand, for $b>\pi$ we get by Corollary 5.3 that $\mathcal{J}\ngeq 0$.

Remark 6.8. In [19] it is shown that the *coercivity* of the second variation of (P) over (1.1) is a second order sufficient condition for the strict weak local optimality in (P). For the discrete-time case and for the continuous-time case over absolutely continuous admissible functions η it is known in [26] that the positivity and coercivity of $\mathcal{J}(\eta)$ are equivalent. This result is still open for the general time scale setting, and so does the conclusion regarding the sufficiency for the problem (P) of the conditions involved in Theorem 6.1 and the subsequent corollaries.

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