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STABILITY IN NEUTRAL NONLINEAR DYNAMIC EQUATIONS ON A TIME SCALE WITH FUNCTIONAL DELAY

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ABSTRACT. Let $\mathbb{T}$ be a time scale that is unbounded above and below and such that $0 \in \mathbb{T}$. Let $\tau : \mathbb{T} \to \mathbb{T}$ be such that $\tau(\mathbb{T})$ is a time scale. We use fixed point theorems to obtain stability results about the zero solution of the nonlinear neutral dynamic equation with functional delay

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + c(t)x^{\tilde{\Delta}}(\tau(t)) + q\big(x(t), x(\tau(t))\big), t \in \mathbb{T},$$

where f^{Δ} is the Δ-derivative on $\mathbb{T}$ and $f^{\tilde{\Delta}}$ is the Δ-derivative on $\tau(\mathbb{T})$.

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1. INTRODUCTION

In this paper we consider the neutral nonlinear dynamic equation on a time scale,

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + c(t)x^{\tilde{\Delta}}(\tau(t)) + q\big(x(t), x(\tau(t))\big),\ t \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T}$ is unbounded above and below. Throughout this paper we assume that $0 \in \mathbb{T}$ for convenience. We also assume that $a, b : \mathbb{T} \to \mathbb{R}$ are continuous and that $c : \mathbb{T} \to \mathbb{R}$ is continuously delta-differentiable. In order for the function $x(\tau(t))$ to be well-defined and differentiable over $\mathbb{T}$, we assume that $\tau : \mathbb{T} \to \mathbb{T}$ is an increasing mapping such that $\tau(\mathbb{T})$ is closed.

For the reader who is unfamiliar with the theory of dynamic equations on time scales, we include some basic definitions and theorems in the appendix. For more information, we refer the reader to the texts [1] and [2]. Throughout this paper, intervals subscripted with a $\mathbb{T}$ represent real intervals intersected with $\mathbb{T}$. For example, for $a, b \in \mathbb{T}$, $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \le t \le b\}$.

In the case $\mathbb{T} = \mathbb{R}$, it has been shown that neutral differential equations have many applications. For example, these equations arise in the study of two or more

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simple oscillatory systems with some interconnections between them [4] and [7], and in modelling physical problems such as vibration of masses attached to an elastic bar [7].

For $\mathbb{T} = \mathbb{R}$, in [6] the second author studied the stability of the zero solution. Recently, in [5] the authors considered (2.1) and (2.12) when the delay is constant and showed the existence of a periodic solution by appealing to Krasnosel'skii fixed point theorem. Also, the uniqueness of the periodic solution was deduced by using the contraction mapping principle.

2. STABILITY VIA FIXED POINT THEORY

We begin by considering the dynamic equation

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + b(t)x(\tau(t)) + c(t)x^{\tilde{\Delta}}(\tau(t)),\ t \in \mathbb{T}. \tag{2.1}$$

In addition to the conditions on τ mentioned in Section 1, we need that

$$\tau^{\Delta}(t) \neq 0 \tag{2.2}$$

for all $t \in \mathbb{T}$. Furthermore, the exponential function $e_a(t,0)$ must satisfy

$$e_{\ominus a}(t,0) \to 0 \quad \text{as } t \to \infty, \tag{2.3}$$

as well as the initial value problem $y^{\Delta}(t) = ay(t), y(0) = 1$. As such, we require that $a(t) \geq 0$ for all $t \in \mathbb{T}$. Since $a(t) \geq 0$ for all $t \in \mathbb{T}$, then $1 + \mu(t)a(t) \geq 1 > 0$ for all t and so $a \in \mathcal{R}^+$.

Lemma 2.1. *Suppose* (2.2) *holds. Then x is a solution of equation* (2.1) *if and only if*

$$\begin{aligned} x(t) = &\left(x(0) - \frac{c(0)}{\tau^{\Delta}(0)}x(\tau(0))\right) e_{\ominus a}(t,0) + \frac{c(t)}{\tau^{\Delta}(t)}x\big(\tau(t)\big) \\ &- \int_0^t \big[r(s)x^{\sigma}(\tau(s)) - b(s)x(\tau(s))\big] e_{\ominus a}(t,s)\,\Delta s \end{aligned} \tag{2.4}$$

where

$$r(s) = \frac{\big(c^{\Delta}(s) + c^{\sigma}(s)a(s)\big)\big(\tau^{\Delta}(s)\big) + \tau^{\Delta\Delta}(s)c(s)}{\big(\tau^{\Delta}(s)\big)\big(\tau^{\Delta}(\sigma(s))\big)}. \tag{2.5}$$

Proof. We begin by rewriting (2.1) as

$$x^{\Delta}(t) + a(t)x^{\sigma}(t) = b(t)x(\tau(t)) + c(t)x^{\tilde{\Delta}}(\tau(t)).$$

Multiply both sides of the above equation by $e_a(t,0)$ and integrate from 0 to t to obtain

$$\int_0^t \big[e_a(s,0)x(s)\big]^{\Delta}\,\Delta s = \int_0^t \big[b(s)x(\tau(s)) + c(s)x^{\tilde{\Delta}}(\tau(s))\big] e_a(s,0)\,\Delta s.$$

As a consequence, we arrive at

$$e_a(t,0)x(t) - x(0) = \int_0^t \big[b(s)x(\tau(s)) + c(s)x^{\tilde{\Delta}}(\tau(s))\big]e_a(s,0)\,\Delta s.$$

Add $x(0)$ to both sides and multiply both sides by $e_{\ominus a}(t,0)$ to conclude

$$x(t) = x(0)e_{\ominus a}(t,0) + \int_0^t \big[b(s)x(\tau(s)) + c(s)x^{\tilde{\Delta}}(\tau(s))\big]e_{\ominus a}(t,s)\,\Delta s. \tag{2.6}$$

Here we have used Lemma 3.8 to simplify the exponential. We want to pull the factor $x^{\Delta}(\tau(s))$ from under the integral in (2.6). Clearly,

$$\int_0^t c(s)x^{\tilde{\Delta}}(\tau(s))e_{\ominus a}(t,s)\,\Delta s = \int_0^t x^{\tilde{\Delta}}(\tau(s))\big(\tau^{\Delta}(s)\big)\cdot\frac{c(s)}{\tau^{\Delta}(s)}e_{\ominus a}(t,s)\,\Delta s.$$

Using the integration by parts formula,

$$\int_0^t f^{\Delta}(s)h(s)\,\Delta s = (fh)(t) - (fh)(0) - \int_0^t h^{\Delta}(s)f^{\sigma}(s)\,\Delta s,$$

and Theorems 3.6 and 3.7 we obtain,

$$\begin{aligned}&\int_0^t c(s)x^{\tilde{\Delta}}(\tau(s))e_{\ominus a}(t,s)\,\Delta s = \\ &\qquad \frac{c(t)}{\tau^{\Delta}(t)}x(\tau(t)) - \frac{c(0)}{\tau^{\Delta}(0)}x(\tau(0))e_{\ominus a}(t,0) - \int_0^t r(s)x^{\sigma}(\tau(s))\,\Delta s,\end{aligned} \tag{2.7}$$

where $r(s)$ is given by (2.5). Finally, substituting the right hand side of (2.7) into (2.6) completes the proof. $\square$

Let $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ be a given bounded Δ-differentiable initial function. We say $x := x(\cdot, 0, \psi)$ is a solution of (2.1) if $x(t) = \psi(t)$ for $t \le 0$ and satisfies (2.1) for $t \ge 0$.

We say the zero solution of (1.1) is stable at t_0 if for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\big[\psi : (-\infty, t_0]_{\mathbb{T}} \to \mathbb{R}$ with $\|\psi\| < \delta\big]$ implies $|x(t, t_0, \psi)| < \varepsilon$.

Let $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$ be the space of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ and define the set S by

$$S = \{\varphi \in C_{rd} : \varphi(t) = \psi(t) \text{ if } t \le 0,\ \varphi(t) \to 0 \text{ as } t \to \infty, \text{ and } \varphi \text{ is bounded}\}.$$

Then $\Big(S, ||\cdot||\Big)$ is a complete metric space where $||\cdot||$ is the supremum norm.

For the next theorem we assume there is an $\alpha > 0$ such that

$$\left|\frac{c(t)}{\tau^{\tilde{\Delta}}(t)}\right| + \int_0^t |r(s) - b(s)|e_{\ominus a}(t,s)\,\Delta s \le \alpha < 1,\ t \ge 0, \tag{2.8}$$

and

$$\tau(t) \to \infty \text{ as } t \to \infty. \tag{2.9}$$

Theorem 2.2. *If* (2.2), (2.3), (2.8) *and* (2.9) *hold, then every solution* $x(\cdot, 0, \psi)$ *of* (2.1) *with small continuous initial function* ψ, *is bounded and goes to zero as* $t \to \infty$. *Moreover, the zero solution is stable at* $t_0 = 0$.

Proof. Define the mapping $P : S \to S$ by

$$\big(P\varphi\big)(t) = \psi(t), \text{ if } t \leq 0$$

and

$$\begin{aligned}\big(P\varphi\big)(t) = &\left(\varphi(0) - \frac{c(0)}{\tau^{\tilde{\Delta}}(0)}\varphi(\tau(0))\right) e_{\ominus a}(t,0) + \frac{c(t)}{\tau^{\tilde{\Delta}}(t)}\varphi\big(\tau(t)\big) \\ &- \int_0^t \big[r(s)\varphi^\sigma(\tau(s)) - b(s)\varphi(\tau(s))\big] e_{\ominus a}(t,s)\,\Delta s,\ t \geq 0.\end{aligned}$$

It is clear that for $\varphi \in S$, $P\varphi$ is continuous. Let $\varphi \in S$ with $||\varphi|| \leq K$, for some positive constant K. Let ψ be a small given continuous initial function with $\|\psi\| < \delta, \delta > 0$. Then using (2.8) in the definition of $(P\varphi)(t)$, we have

$$\begin{aligned}\|P\varphi\| &\leq \left|1 - \frac{c(0)}{\tau^{\tilde{\Delta}}(0)}\right|\delta + \left|\frac{c(t)}{\tau^{\tilde{\Delta}}(t)}\right| K + \int_0^t |r(s) - b(s)| e_{\ominus a}(t,s)\,\Delta s\, K \\ &\leq \left|1 - \frac{c(0)}{\tau^{\tilde{\Delta}}(0)}\right|\delta + \alpha K, \qquad (2.10)\end{aligned}$$

which implies that, $\|P\varphi\| \leq K$, for the right δ. Thus, (2.10) implies that $P\varphi$ is bounded. Next we show that $\big(P\varphi\big)(t) \to 0$ as $t \to \infty$. The first term on the right side of $P\varphi$ tends to zero, by condition (2.3). Also, the second term on the right side tends to zero, because of (2.9) and the fact that $\varphi \in S$. It is left to show that the integral term goes to zero as $t \to \infty$.

Let $\varepsilon > 0$ be given and $\varphi \in S$ with $||\varphi|| \leq K$, $K > 0$. Then, there exists a $t_1 > 0$ so that for $t > t_1$, $|\varphi(\tau(t))| < \varepsilon$. Due to condition (2.3), there exists a $t_2 > t_1$ such that for $t > t_2$ implies that $e_{\ominus a}(t, t_1) < \frac{\varepsilon}{\alpha K}$. For $t > t_2$, we have

$$\begin{aligned}&\Big|\int_0^t [r(s)\varphi^\sigma(\tau(s)) - b(s)\varphi(\tau(s))] e_{\ominus a}(t,s)\,\Delta s\Big| \\ &\quad\leq K\int_0^{t_1} |r(s) - b(s)| e_{\ominus a}(t,s)\,\Delta s + \varepsilon\int_{t_1}^t |r(s) - b(s)| e_{\ominus a}(t,s)\,\Delta s \\ &\quad\leq K e_{\ominus a}(t,t_1)\int_0^{t_1} |r(s) - b(s)| e_{\ominus a}(t_1,s)\,\Delta s + \alpha\varepsilon \\ &\quad\leq \alpha K e_{\ominus a}(t,t_1) + \alpha\varepsilon \\ &\quad\leq \varepsilon + \alpha\varepsilon.\end{aligned}$$

Hence, $\big(P\varphi\big)(t) \to 0$ as $t \to \infty$. It is left to show that $P\varphi$ is a contraction under the supremum norm.

Let $\zeta, \eta \in S$. Then

$$\begin{aligned}\left|(P\zeta)(t) - (P\eta)(t)\right| &\leq \left\{\left|\frac{c(t)}{\tau^{\tilde{\Delta}}(t)}\right| + \int_0^t |r(s) - b(s)| e_{\ominus a}(t,s)\,\Delta s\right\} \|\zeta - \eta\| \\ &\leq \alpha \|\zeta - \eta\|.\end{aligned}$$

Thus, by the contraction mapping principle, P has a unique fixed point in S which solves (2.1), bounded and tends to zero as t tends to infinity. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing K by ε. This completes the proof. □

Example 2.3. Let

$$\begin{aligned}\mathbb{T} &= (-\infty, -1] \cup \left\{\left(1/2\right)^{\mathbb{Z}} - 1\right\} \\ &= (-\infty, -1] \cup \left\{\ldots, \left(1 - 2^n\right)/2^n, \ldots, -3/4, -1/2, 0, 1, 3, \cdots 2^n - 1, \ldots\right\}.\end{aligned}$$

Then for any small continuous initial function, $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$, every solution, $x(\cdot, 0, \psi)$ of the linear neutral differential equation,

$$x^{\Delta}(t) = -3x^{\sigma}(t) + c_0 x^{\tilde{\Delta}}\left(\frac{t}{2} - \frac{1}{2}\right), \tag{2.11}$$

where c_0 is a constant, is bounded and goes to 0 as $t \to \infty$

Proof. In (2.11), we have $\tau(t) = \frac{t}{2} - \frac{1}{2}$. Let $t \in (1/2)^{\mathbb{Z}} - 1$. Then there exists an $n \in \mathbb{Z}$ such that $t = (1/2)^n - 1$. Hence,

$$\begin{aligned}\tau(t) &= \frac{1}{2}\left(\left(\frac{1}{2}\right)^n - 1\right) - \frac{1}{2} \\ &= \left(\frac{1}{2}\right)^{n+1} - \frac{1}{2} - \frac{1}{2} \\ &= \left(\frac{1}{2}\right)^{n+1} - 1 \in \mathbb{T}.\end{aligned}$$

So, $\tau : \mathbb{T} \to \mathbb{T}$. Furthermore $\tau(\mathbb{T})$ is a time scale. Also, $\tau(t) = \frac{1}{2}t - \frac{1}{2} \to \infty$ as $t \to \infty$ and $[\tau(t)]^{\tilde{\Delta}} = (t/2 - 1/2)^{\tilde{\Delta}} = 1/2$. Consequently, conditions (2.2) and (2.9) are satisfied. Since $1 + 3\mu(t) > 0$ for all $t \in \mathbb{T}$, then $3 \in \mathcal{R}^+$ and condition (2.3) is satisfied as well.

One may easily arrive at $r(s) = 6c_0$. Also

$$\begin{aligned}\left|\frac{c(t)}{\tau^{\tilde{\Delta}}(t)}\right| + \int_0^t |r(s) - b(s)| e_{\ominus a}(t,s)\,\Delta s &\leq 2c_0 + 6c_0 \int_0^t e_{\ominus 3}(t,s)\,\Delta s \\ &\leq 2c_0 + 2c_0 - 2c_0 e_{\ominus 3}(t, 0) \\ &\leq 4c_0.\end{aligned}$$

Hence, (2.8) is satisfied for $|c_0| \leq \frac{\alpha}{4}, \alpha \in (0,1)$. Let ψ be a given initial function that is continuous with $|\psi(t)| \leq \delta$ for all $t \in \mathbb{T}$ and define

$$S = \{\varphi \in C_{rd} : \varphi(t) = \psi(t) \text{ if } t \leq 0,\ \varphi(t) \to 0 \text{ as } \ t \to \infty, \text{ and } \varphi \text{ is bounded}\}.$$

Define

$$\big(P\varphi\big)(t) = \psi(t) \text{ if } t \leq 0$$

and

$$\begin{aligned} \big(P\varphi\big)(t) \ &= \ \big(\psi(0) - 2c_0\psi(-1/2)\big)e_{\ominus 3}(t,0) + 2c_0\varphi\left(\frac{1}{2}t - \frac{1}{2}\right) \\ &\quad - \int_0^t 6c_0\varphi^\sigma\left(\frac{s}{2} - \frac{1}{2}\right)e_{\ominus 3}(t,s)\,\Delta s,\ t \geq 0. \end{aligned}$$

Then, for $\varphi \in S$ with $\|\varphi\| \leq K$, we have $\|P\varphi\| \leq |1-2c_0|\delta + 4|c_0|K \leq |1-2c_o|\delta + K\alpha$. This implies that $\|P\varphi\| \leq K$, for $K \geq \frac{(1-\alpha)}{|1-2c_0|\delta}$. To see that P defines a contraction mapping, we let $\zeta, \eta \in S$. Then

$$\begin{aligned} \Big|(P\zeta)(t) - (P\eta)(t)\Big| \ &\leq \ 2|c_0|\ \|\zeta - \eta\| + 2|c_0|\|\zeta - \eta\| \\ &\leq \ \alpha||\zeta - \eta||. \end{aligned}$$

Hence, by Theorem 2.2, every solution $x(\cdot, 0, \psi)$ of (2.11) with small continuous initial function $\psi : (-\infty,\ 0]_{\mathbb{T}} \to \mathbb{R}$, is in S, bounded and goes to zero as $t \to \infty$ and the proof is complete. $\square$

Next we turn our attention to the nonlinear neutral differential equation with unbounded delay

$$x^\Delta(t) = -a(t)x^\sigma(t) + c(t)x^{\tilde{\Delta}}(\tau(t)) + q\big(x(t), x(\tau(t))\big),\ t \in \mathbb{T}. \tag{2.12}$$

where a, c, and τ are defined as before. Here, we assume $q(0,0) = 0$ and is locally Lipschitz continuous in x and y. That is, there is a $K > 0$ so that if $|x|, |y|, |z|$ and $|w| \leq K$ then

$$|q(x,y) - q(z,w)| \leq L|x - z| + E|y - w| \tag{2.13}$$

for some positive constants L and E.

Note that

$$\begin{aligned} |q(x,y)| \ &= \ |q(x,y) - q(0,0) + q(0,0)| \\ &\leq \ |q(x,y) - q(0,0)| + |q(0,0)| \\ &\leq \ L|x| + E|y|. \end{aligned}$$

Let

$$S = \{\varphi \in C_{rd} : \|\varphi\| \leq K, \varphi(t) = \psi(t) \text{ if } t \leq 0, \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

Define a map $P : S \to S$ by

$$\big(P\varphi\big)(t) = \psi(t), \text{ if } t \leq 0$$

and for $t \geq 0$,

$$\begin{aligned}\big(P\varphi\big)(t) &= \left(\varphi(0) - \frac{c(0)}{\tau^{\Delta}(0)}\varphi(\tau(0))\right) e_{\ominus a}(t,0) + \frac{c(t)}{\tau^{\Delta}(t)}\varphi\big(\tau(t)\big) \\ &\quad + \int_0^t \Big[-r(s)\varphi^{\sigma}(\tau(s)) + q\big(\varphi(s), \varphi(\tau(s))\big)\Big] e_{\ominus a}(t,s)\,\Delta s.\end{aligned}$$

It is clear that for $\varphi \in S$, $P\varphi$ is continuous. If P has a fixed point, say φ, then φ is a solution of (2.12). In order for P to be a contraction, we assume that, there is an $\alpha > 0$ such that

$$\left|\frac{c(t)}{\tau^{\tilde{\Delta}}(t)}\right| + \int_0^t (|r(s)| + L + E) e_{\ominus a}(t,s)\,\Delta s \leq \alpha < 1, \;\; t \geq 0. \tag{2.14}$$

Theorem 2.4. *If* (2.2), (2.3), (2.13) *and* (2.14) *hold, then every solution* $x(\cdot, 0, \psi)$ *of* (2.12) *with small continuous initial function* $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$, *goes to zero as* $t \to \infty$. *Moreover, the zero solution is stable at* $t_0 = 0$.

Proof. Let $\varphi \in S$ and take t_1 and t_2 be as in the proof of Theorem 2.2. Then for $t > t_2$,

$$\begin{aligned}&\left|\int_0^t q\big(\varphi(s), \varphi(\tau(s))\big) e_{\ominus a}(t,s)\,\Delta s\right| \\ &\qquad \leq K(L+E)\int_0^{t_1} e_{\ominus a}(t,s)\,\Delta s + \varepsilon(L+E)\int_{t_1}^t e_{\ominus a}(t,s)\,\Delta s \\ &\qquad \leq K(L+E) e_{\ominus a}(t,t_1) \int_0^{t_1} e_{\ominus a}(t_1,s)\,\Delta s + \varepsilon\,\alpha \\ &\qquad \leq \alpha K e_{\ominus a}(t,t_1) + \varepsilon\,\alpha \\ &\qquad \leq \varepsilon + \varepsilon\,\alpha.\end{aligned}$$

This along with the proof of Theorem 2.2, shows that $\big(P\varphi\big)(t) \to 0$ as $t \to \infty$. Let ψ be a small given continuous initial function with $\|\psi\| < \delta, \delta > 0$. Then, by using (2.14) we arrive at

$$\begin{aligned}\|P\varphi\| &\leq \left|1 - \frac{c(0)}{\tau^{\tilde{\Delta}}(0)}\right|\delta + \left|\frac{c(t)}{\tau^{\tilde{\Delta}}(t)}\right| K \\ &\quad + \int_0^t (|r(s)| + L + E) e_{\ominus a}(t,s)\,\Delta s K \\ &\leq \left|1 - \frac{c(0)}{\tau^{\tilde{\Delta}}(0)}\right|\delta + \alpha K, \qquad (2.15)\end{aligned}$$

which implies that, $\|P\varphi\| \leq K$, for the right choice of δ and α. It is left to show that $P\varphi$ is a contraction. Let $\zeta, \eta \in S$. Then

$$\begin{aligned}
&\Big|(P\zeta)(t) - (P\eta)(t)\Big| \\
&\qquad \leq \Big\{ \Big|\frac{c(t)}{\tau^{\tilde{\Delta}}(t)}\Big| \;||\zeta - \eta|| \\
&\qquad\quad + \int_0^t \big|r(s)\big| \, |\zeta^\sigma(\tau(s)) - \eta^\sigma(\tau(s))| e_{\ominus a}(t,s)\,\Delta s \\
&\qquad\quad + \int_0^t |q\big(\zeta(s), \zeta(\tau(s))\big) - q\big(\eta(s), \eta(\tau(s))\big)| e_{\ominus a}(t,s)\,\Delta s \Big\} \\
&\qquad \leq \Big\{ \Big|\frac{c(t)}{\tau^{\tilde{\Delta}}(t)}\Big| + \int_0^t (|r(s)| + L + E) e_{\ominus a}(t,s)\,\Delta s \Big\} \|\zeta - \eta\| \\
&\qquad \leq \alpha||\zeta - \eta||.
\end{aligned}$$

Thus, by the contraction mapping principle, P has a unique fixed point in S which solves (2.12) and tends to zero as t tends to infinity. This completes the proof. □

Appendix - Time Scales

In his 1988 PhD dissertation, Stefan Hilger [3] introduced the theory of analysis on time scales with the intention of unifying and extending the continuous and discrete calculi. Since then many authors have engaged in studying dynamic equations on time scales. See for example the references in the monographs [1] and [2]. The following definitions and lemmas can be found in aforementioned texts.

A *time scale* $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the *forward jump operator*, σ, and the *backward jump operator*, ρ, respectively, are defined as $\sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\}$ and $\rho(r) = \sup\{\tau \in \mathbb{T} \mid \tau < r\}$. These operators allow elements in the time scale to be classified as follows. We say t is *right scattered* if $\sigma(t) > t$ and *right dense* if $\sigma(t) = t$. We say t is *left scattered* if $\rho(t) < t$ and *left dense* if $\sigma(t) = t$. The *graininess function*, $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset \equiv \sup \mathbb{T}$ and $\sup \emptyset \equiv \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum, M, we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$. Otherwise, we define $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum, m, we define $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}_\kappa = \mathbb{T}$.

Let $t \in \mathbb{T}^\kappa$ and let $f : \mathbb{T} \to \mathbb{R}$. The *delta derivative* of $f(t)$, denoted $f^\Delta(t)$, is defined to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood, U, of t such that

$$\Big|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]\Big| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^\Delta(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is *right dense continuous*, (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

We are now ready to state some properties of the delta-derivative of f. Note $f(\sigma(t)) = f^\sigma(t)$.

Theorem 3.5. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$ and let α be a scalar.*

(i) $\big(f + g\big)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$.
(ii) $(\alpha f)^\Delta(t) = \alpha f^\Delta(t)$.
(iii) *The product rules.*

$$\begin{aligned} \big(fg\big)^\Delta(t) &= f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t). \\ \big(fg\big)^\Delta(t) &= f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t). \end{aligned}$$

(iv) *If $g(t)g^\sigma(t) \neq 0$ then,*

$$\left(\frac{f}{g}\right) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}.$$

The next theorem is the chain rule on time scales.

Theorem 3.6. *Assume $\nu : \mathbb{T} \to \mathbb{T}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $w : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^\Delta(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^\kappa$, then*

$$(w \circ \nu)^\Delta = (w^{\tilde{\Delta}} \circ \nu)\nu^\Delta.$$

We also need the substitution rule.

Theorem 3.7. *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^\Delta(t)\,\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\,\tilde{\Delta}s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all *positively regressive* functions, $\mathcal{R}^+$, is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$. The *exponential function* is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(\tau)}\mathrm{Log}\big(1 + \mu(\tau)p(\tau)\big)\,\Delta\tau\right). \tag{3.16}$$

If $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 3.8. *Let $p \in \mathcal{R}$. Then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(ii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;

(iv) $e_p(\cdot,s) = \frac{1}{e_p(s,\cdot)} = e_{\ominus p}(s,\cdot)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

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