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NONUNIFORM NONRESONANCE FOR NONLINEAR BOUNDARY VALUE PROBLEMS WITH y' DEPENDENCE

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ABSTRACT. A new nonuniform nonresonant result at the first eigenvalue is presented for the boundary value problem $y'' + q\,f(t, y, y') = 0$ a.e. on $[0, 1]$, $y(0) = y(1) = 0$.

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1. INTRODUCTION

This paper presents an existence result of 'nonuniform nonresonant' type for the boundary value problem

$$(1.1)\qquad \begin{cases} y'' + q\,f(t, y, y') = 0 \quad \text{a.e. on } [0, 1] \\ y(0) = y(1) = 0. \end{cases}$$

The result we present is new and improves results 'at the first eigenvalue' in [1–5]. To obtain our result we use a well known technique initiated by Mawhin and Ward [3] in the early 1980's. It is worth remarking here that we could consider Sturm–Liouville, Neumann and Periodic boundary data in (1.1); however since the arguments are essentially the same we will restrict our discussion to Dirichlet boundary data.

For the convenience of the reader we now recall some well known results from the literature [4, 5]. Consider the problem

$$\begin{cases} L\,y = \lambda\,y \quad \text{a.e. on } [0, 1] \\ y(0) = y(1) = 0, \end{cases}$$

where $L\,y = -(1/q)\,y''$ and $q \in L^1[0, 1]$ with $q > 0$ a.e. on $[0, 1]$. For notational purposes let $L^p_w[0, 1]$ ($p \geq 1$ and w a weight function) denote the space of functions

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y such that $\int_0^1 w(t)\,|y(t)|^p\,dt < \infty$; also for $u, v \in L^2_w[0,1]$ we define $\langle u, v\rangle = \int_0^1 w(t)\,u(t)\,\overline{v(t)}\,dt$. Let

$$D(L) = \left\{ y \in C^1[0,1] : \ y' \in AC[0,1] \ \text{ with } \ \frac{1}{q}\,y'' \in L^2_q[0,1] \ \text{ and } \ y(0) = y(1) = 0 \right\}.$$

Then L has a countable number [5] of real eigenvalues λ_i with corresponding eigenfunctions $\psi_i \in D(L)$. The eigenfunctions may be chosen so that they form an orthonormal set in $L^2_q[0,1]$ and we may arrange the eigenvalues so that $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ (note $\lambda_0 > 0$ and $\psi_0 > 0$ on $(0,1)$). In addition the set of eigenfunctions form a basis for $L^2_q[0,1]$. The results in Section 2 rely on the following well known Rayleigh–Ritz inequality [4, 5].

THEOREM 1.1. Suppose $q \in L^1[0,1]$ with $q > 0$ a.e. on $(0,1)$. Let λ_0 be the first eigenvalue of

$$\begin{cases} y'' + \lambda\, q\, y = 0 \quad \text{a.e. on } \ [0,1] \\ y(0) = y(1) = 0. \end{cases}$$

Then

$$\lambda_0 \int_0^1 q(t)\,|v(t)|^2\,dt \le \int_0^1 |v'(t)|^2\,dt$$

for all functions $v \in AC[0,1]$ with $v' \in L^2[0,1]$ and $v(0) = v(1) = 0$.

For notational purposes in this paper, for appropriate functions y we let

$$\|y\| = \left(\int_0^1 |y(t)|^2\,dt \right)^{1/2} \quad \text{and} \quad |y|_\infty = \sup_{[0,1]} |y(t)|.$$

2. EXISTENCE THEORY

In this section we present a nonuniform nonresonant result at the first eigenvalue for the Dirichlet boundary value problem

$$\begin{cases} y'' + q\, f(t, y, y') = 0 \quad \text{a.e. on } \ [0,1] \\ y(0) = y(1) = 0, \end{cases} \tag{2.1}$$

with

$$q \in L^1[0,1] \ \text{ and } \ q > 0 \ \text{ a.e. on } \ [0,1] \tag{2.2}$$

and

$$q\, f : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \ \text{ a } \ L^1 - \text{Carathéodory function [5]} \tag{2.3}$$

holding.

THEOREM 2.1. Suppose (2.2) and (2.3) hold and in addition assume f has the decomposition

$$f(t, u, w) = g(t, u, w) + h(t, u, w)$$

with the following holding:

$$(2.4)\qquad \begin{cases} \exists\, a,\, b \in C[0,1], \quad \text{with } \ a(t) \geq 0,\, b(t) \geq 0 \ \text{ on } \ [0,1] \ \text{ and} \\ b\sqrt{q} \in C[0,1], \quad \text{with } \ u\, g(t,u,w) \leq a(t)\, u^2 + b(t)\, |u|\, |w| \\ \text{for } \ t \in [0,1], \ u \in \mathbb{R} \ \text{ and } \ w \in \mathbb{R} \end{cases}$$

$$(2.5)\qquad \begin{cases} \text{either (i) } \ a(t) < |a|_\infty \ \text{ on a subset of } \ [0,1] \ \text{ of positive} \\ \text{measure or (ii) } \ b(t) < |b|_\infty \ \text{ on a subset of } \ [0,1] \ \text{ of} \\ \text{positive measure with in addition } \ q \in C[0,1] \end{cases}$$

$$(2.6)\qquad \begin{cases} \dfrac{|a|_\infty}{\lambda_0} + \dfrac{|b\sqrt{q}|_\infty}{\sqrt{\lambda_0}} \leq 1 \ \text{ if (i) occurs in (2.5), or} \\[2ex] \dfrac{|a|_\infty}{\lambda_0} + \dfrac{|b|_\infty\, |\sqrt{q}|_\infty}{\sqrt{\lambda_0}} \leq 1 \ \text{ if (ii) occurs in (2.5)} \end{cases}$$

$$(2.7)\qquad \begin{cases} \exists\ \phi_i,\, i = 1,2,3,4, \quad \text{with } \ \phi_i \geq 0 \ \text{ a.e. on } \ [0,1], \\ \text{and } \ \exists\ \alpha,\, \beta,\, \gamma,\, \theta \text{ such that } \ 0 \leq \gamma < 1,\, 0 \leq \theta < 1, \\ \alpha \geq 0,\, \beta \geq 0 \ \text{ and } \ \alpha + \beta < 1, \quad \text{with} \\ |h(t,u,w)| \leq \phi_1(t) + \phi_2(t)\, |u|^\gamma + \phi_3(t)\, |w|^\theta + \phi_4(t)\, |u|^\alpha\, |w|^\beta \\ \text{for a.e. } \ t \in [0,1], \ u \in \mathbb{R} \ \text{ and } \ w \in \mathbb{R} \end{cases}$$

$$(2.8)\qquad \phi_1 \in L^1_q[0,1], \ \phi_2 \in L^1_q[0,1], \ q\,\phi_3 \in L^{2/(2-\theta)}[0,1] \ \text{ and } \ q\,\phi_4 \in L^{2/(2-\beta)}[0,1]$$

$$(2.9)\qquad \begin{cases} \text{for any } \ M_0 > 0, \ \exists\ \phi_i \text{ (which may depend on } M_0),\ i = 5,6,7, \\ \text{with } \ \phi_i \geq 0 \ \text{ a.e. on } \ [0,1], \ \text{ and } \ \exists\ \delta, \ 0 \leq \delta < 2 \ \text{ with} \\ |g(t,u,w)| \leq \phi_5(t) + \phi_6(t)\, |w|^\delta + \phi_7(t)\, w^2 \ \text{ for a.e. } \ t \in [0,1], \\ w \in \mathbb{R} \ \text{ and } \ u \in [-M_0, M_0] \end{cases}$$

and

$$(2.10)\qquad \phi_5 \in L^1_q[0,1], \ q\,\phi_7 \in C[0,1] \ \text{ and } \ q\,\phi_6 \in L^{2/(2-\delta)}[0,1].$$

Then (2.1) has a solution in $C^1[0,1]$ (in fact in $W^{2,1}[0,1]$).

REMARK 2.1. Theorem 2.1 extends and improves some results in [1–3].

PROOF. Let y be any solution to

$$(2.11)_\lambda\qquad \begin{cases} y'' + \lambda\, q\, f(t,y,y') = 0 \quad \text{a.e. on } \ [0,1] \\ y(0) = y(1) = 0, \end{cases}$$

for $0 < \lambda < 1$. To show (2.1) has a solution it suffices to show [4] that there exists M, independent of λ, with

$$(2.12)\qquad \max\{|y|_\infty\,,\, |y'|_\infty\} \,\leq\, M.$$

Multiply $(2.11)_\lambda$ by $-y$ and integrate from 0 to 1 to obtain

(2.13)
$$\begin{aligned}\|y'\|^2 \le & \int_0^1 a(t)q(t)y^2(t)dt + \int_0^1 b(t)q(t)|y(t)||y'(t)|dt + \int_0^1 \phi_1(t)q(t)|y(t)|dt \\ & + \int_0^1 \phi_2(t)q(t)|y(t)|^{\gamma+1}dt + \int_0^1 \phi_3(t)q(t)|y(t)||y'(t)|^\theta dt \\ & + \int_0^1 \phi_4(t)q(t)|y(t)|^{\alpha+1}|y'(t)|^\beta dt.\end{aligned}$$

Case (A). Suppose $a(t) < |a|_\infty$ on a subset of $[0,1]$ of positive measure.

Let
$$R(y) = \frac{|a|_\infty}{\lambda_0}\int_0^1 [y'(t)]^2\,dt - \int_0^1 a(t)\,q(t)\,y^2(t)\,dt.$$
We claim that there exists $\epsilon > 0$ with
$$R(y) \ge \epsilon\left(\int_0^1 q(t)\,y^2(t)\,dt + \|y'\|^2\right). \tag{2.14}$$
Notice first that
$$R(y) \ge \frac{|a|_\infty}{\lambda_0}\int_0^1 [y'(t)]^2\,dt - |a|_\infty\int_0^1 q(t)\,y^2(t)\,dt = \sum_{i=0}^\infty\left(\frac{|a|_\infty}{\lambda_0}\lambda_i - |a|_\infty\right)c_i^2 \ge 0$$
since $\lambda_i \ge \lambda_0$ for $i \in \{0,1,2,\cdots\}$; here
$$y = \sum_{i=0}^\infty c_i\,\psi_i \quad\text{and}\quad c_i = \langle y,\psi_i\rangle$$
(of course $R(y) \ge 0$ also follows immediately from Theorem 1.1). Also if $R(y) = 0$ then $c_i = 0$ for $i \in \{1,2,\cdots\}$. Thus $y = c_0\,\psi_0$ and so
$$\begin{aligned}0 = R(y) &= \frac{|a|_\infty}{\lambda_0}c_0^2\int_0^1 [\psi_0'(t)]^2\,dt - c_0^2\int_0^1 a(t)\,q(t)\,[\psi_0(t)]^2\,dt \\ &= c_0^2\int_0^1 [|a|_\infty - a(t)]\,q(t)\,[\psi_0(t)]^2\,dt.\end{aligned}$$
Note $a(t) < |a|_\infty$ on a subset of $[0,1]$ of positive measure and $\psi_0 > 0$ on $(0,1)$ implies $c_0 = 0$ and so $y \equiv 0$.

If (2.14) is not true then there exists a sequence $\{y_n\}$ with
$$\int_0^1 q(t)\,[y_n(t)]^2\,dt + \|y_n'\|^2 = 1 \tag{2.15}$$
and
$$R(y_n) \to 0 \text{ as } n \to \infty. \tag{2.16}$$
A standard argument [5, Chapter 11] guarantees that there is a subsequence S of integers with
$$y_n \to y \text{ in } C[0,1] \text{ and } y_n' \rightharpoonup y' \text{ in } L^2[0,1] \tag{2.17}$$

as $n \to \infty$ in S; here $\rightharpoonup$ denotes weak convergence. Also

$$\int_0^1 [y'(t)]^2\,dt \le \liminf \int_0^1 [y_n'(t)]^2\,dt. \tag{2.18}$$

Now (2.16), (2.17) and (2.18) immediately implies $R(y) \le 0$. Thus $R(y) = 0$, so $y = 0$. Hence

$$\int_0^1 q(t)[y_n(t)]^2 dt + \|y_n'\|^2 = \frac{\lambda_0}{|a|_\infty} R(y_n) + \int_0^1 \left(1 + \frac{\lambda_0}{|a|_\infty} a(t)\right) q(t)[y_n(t)]^2 dt \;\to\; 0$$

as $n \to \infty$ in S,

which is impossible.

Consequently (2.14) is true, so in particular

$$\int_0^1 a(t)\,q(t)\,[y(t)]^2\,dt \le \left(\frac{|a|_\infty}{\lambda_0} - \epsilon\right) \int_0^1 [y'(t)]^2\,dt. \tag{2.19}$$

Put (2.19) into (2.13) to obtain

$$\begin{aligned} \|y'\|^2 \;\le\; & \left(\frac{|a|_\infty}{\lambda_0} - \epsilon\right) \|y'\|^2 + |b\sqrt{q}|_\infty \left(\int_0^1 q(t)\,[y(t)]^2\,dt\right)^{1/2} \|y'\| \\ & + |y|_\infty \int_0^1 \phi_1(t)\,q(t)\,dt + |y|_\infty^{\gamma+1} \int_0^1 \phi_2(t)\,q(t)\,dt \\ & + |y|_\infty\,\|y'\|^\theta \left(\int_0^1 [\phi_3(t)\,q(t)]^{2/(2-\theta)}\,dt\right)^{(2-\theta)/2} \\ & + |y|_\infty^{\alpha+1} \int_0^1 \phi_4(t)\,q(t)\,|y'(t)|^\beta\,dt. \end{aligned}$$

This together with Theorem 1.1 yields

$$\begin{aligned} \|y'\|^2 \;\le & \left(\frac{|a|_\infty}{\lambda_0} - \epsilon\right) \|y'\|^2 + \frac{|b\sqrt{q}|_\infty}{\sqrt{\lambda_0}} \|y'\|^2 \\ & + |y|_\infty \int_0^1 \phi_1(t)\,q(t)\,dt + |y|_\infty^{\gamma+1} \int_0^1 \phi_2(t)\,q(t)\,dt \\ & + |y|_\infty \|y'\|^\theta \left(\int_0^1 [\phi_3(t) q(t)]^{2/(2-\theta)} dt\right)^{(2-\theta)/2} \\ & + |y|_\infty^{\alpha+1} \|y'\|^\beta \left(\int_0^1 [\phi_4(t) q(t)]^{2/(2-\beta)} dt\right)^{(2-\beta)/2}. \end{aligned}$$

Now $y(0)=y(1)=0$ immediately implies $|y|_\infty \le (1/\sqrt{2})\,\|y'\|$ and this together with (2.6) yields

$$\begin{aligned}
\epsilon\,\|y'\|^2 \;&\le\; \frac{1}{\sqrt{2}}\,\|y'\|\int_0^1 \phi_1(t)\,q(t)\,dt + \left(\frac{1}{\sqrt{2}}\right)^{\gamma+1}\|y'\|^{\gamma+1}\int_0^1 \phi_2(t)\,q(t)\,dt \\
&\quad + \frac{1}{\sqrt{2}}\,\|y'\|^{\theta+1}\left(\int_0^1 [\phi_3(t)\,q(t)]^{2/(2-\theta)}\,dt\right)^{(2-\theta)/2} \\
&\quad + \left(\frac{1}{\sqrt{2}}\right)^{\alpha+1}\|y'\|^{\alpha+1+\beta}\left(\int_0^1 [\phi_4(t)\,q(t)]^{2/(2-\beta)}\,dt\right)^{(2-\beta)/2}.
\end{aligned}$$

Since $\gamma<1$, $\theta<1$ and $\alpha+\beta<1$, there exists K_0 (independent of λ) with

$$\|y'\| \le K_0 \tag{2.20}$$

holding.

Case (B). Suppose $b(t)<|b|_\infty$ on a subset of $[0,1]$ of positive measure and $q\in C[0,1]$.

The argument in Case (A) guarantees that there exists $\epsilon>0$ with

$$\frac{|b|_\infty^2}{\lambda_0}\int_0^1 [y'(t)]^2\,dt - \int_0^1 b^2(t)\,q(t)\,y^2(t)\,dt \ge \epsilon\left(\int_0^1 q(t)\,y^2(t)\,dt + \|y'\|^2\right)$$

and so in particular

$$\int_0^1 b^2(t)\,q(t)\,[y(t)]^2\,dt \le \left(\frac{|b|_\infty^2}{\lambda_0}-\epsilon\right)\int_0^1 [y'(t)]^2\,dt. \tag{2.21}$$

Put (2.21) into (2.13) to obtain

$$\begin{aligned}
\|y'\|^2 \;&\le\; |a|_\infty\int_0^1 q(t)\,[y(t)]^2\,dt + |\sqrt{q}|_\infty\left(\int_0^1 b^2(t)\,q(t)\,[y(t)]^2\,dt\right)^{1/2}\|y'\| \\
&\quad + |y|_\infty\int_0^1 \phi_1(t)\,q(t)\,dt + |y|_\infty^{\gamma+1}\int_0^1 \phi_2(t)\,q(t)\,dt \\
&\quad + |y|_\infty\,\|y'\|^\theta\left(\int_0^1 [\phi_3(t)\,q(t)]^{2/(2-\theta)}\,dt\right)^{(2-\theta)/2} + |y|_\infty^{\alpha+1}\int_0^1 \phi_4(t)\,q(t)\,|y'(t)|^\beta\,dt \\
&\le\; \frac{|a|_\infty}{\lambda_0}\,\|y'\|^2 + |\sqrt{q}|_\infty\left(\frac{|b|_\infty^2}{\lambda_0}-\epsilon\right)^{1/2}\|y'\|^2 \\
&\quad + \frac{1}{\sqrt{2}}\,\|y'\|\int_0^1 \phi_1(t)\,q(t)\,dt + \left(\frac{1}{\sqrt{2}}\right)^{\gamma+1}\|y'\|^{\gamma+1}\int_0^1 \phi_2(t)\,q(t)\,dt \\
&\quad + \frac{1}{\sqrt{2}}\,\|y'\|^{\theta+1}\left(\int_0^1 [\phi_3(t)\,q(t)]^{2/(2-\theta)}\,dt\right)^{(2-\theta)/2} \\
&\quad + \left(\frac{1}{\sqrt{2}}\right)^{\alpha+1}\|y'\|^{\alpha+1+\beta}\left(\int_0^1 [\phi_4(t)\,q(t)]^{2/(2-\beta)}\,dt\right)^{(2-\beta)/2};
\end{aligned}$$

here we also used Theorem 1.1. Now there exists a $\rho > 0$ with

$$\left(\frac{|b|_\infty^2}{\lambda_0} - \epsilon\right)^{1/2} \leq \frac{|b|_\infty}{\sqrt{\lambda_0}} - \rho,$$

and this together with (2.6) yields

$$\begin{aligned} \rho\,|\sqrt{q}|_\infty\,\|y'\|^2 \;\leq\; & \frac{1}{\sqrt{2}}\,\|y'\| \int_0^1 \phi_1(t)\,q(t)\,dt + \left(\frac{1}{\sqrt{2}}\right)^{\gamma+1} \|y'\|^{\gamma+1} \int_0^1 \phi_2(t)\,q(t)\,dt \\ & + \frac{1}{\sqrt{2}}\,\|y'\|^{\theta+1} \left(\int_0^1 [\phi_3(t)\,q(t)]^{2/(2-\theta)}\,dt\right)^{(2-\theta)/2} \\ & + \left(\frac{1}{\sqrt{2}}\right)^{\alpha+1} \|y'\|^{\alpha+1+\beta} \left(\int_0^1 [\phi_4(t)\,q(t)]^{2/(2-\beta)}\,dt\right)^{(2-\beta)/2}. \end{aligned}$$

Thus there exists K_0 (independent of λ) with (2.20) holding.

In both cases (2.20) holds, and now since $|y|_\infty \leq (1/\sqrt{2})\,\|y'\|$ we have

$$|y|_\infty \;\leq\; \frac{1}{\sqrt{2}}\,K_0 \equiv M_0 \tag{2.22}$$

for any solution y to $(2.11)_\lambda$. Also (2.7), (2.9), (2.21) and (2.22) implies

$$\begin{aligned} \int_0^1 |y''(t)|dt \;\leq\; & \int_0^1 \phi_5(t)q(t)dt + \int_0^1 \phi_6(t)q(t)|y'(t)|^\delta dt \\ & + \int_0^1 \phi_7(t)q(t)|y'(t)|^2 dt + \int_0^1 \phi_1(t)q(t)dt \\ & + M_0^\gamma \int_0^1 \phi_2(t)\,q(t)\,dt + \int_0^1 \phi_3(t)\,q(t)\,|y'(t)|^\theta\,dt \\ & + M_0^\alpha \textstyle\int_0^1 \phi_4(t)\,q(t)\,|y'(t)|^\beta\,dt \\ \leq\; & \int_0^1 \phi_5(t)\,q(t)\,dt + K_0^\delta \left(\int_0^1 [\phi_6(t)\,q(t)]^{2/(2-\delta)}\,dt\right)^{(2-\delta)/2} + |\phi_7\,q|_\infty\,K_0^2 \\ & + \int_0^1 \phi_1(t)\,q(t)\,dt + M_0^\gamma \int_0^1 \phi_2(t)\,q(t)\,dt \\ & + K_0^\theta \left(\int_0^1 [\phi_3(t)\,q(t)]^{2/(2-\theta)}\,dt\right)^{(2-\theta)/2} \\ & + M_0^\alpha\,K_0^\beta \left(\int_0^1 [\phi_4(t)\,q(t)]^{2/(2-\beta)}\,dt\right)^{(2-\beta)/2} \\ \equiv\; & M_1. \end{aligned}$$

Finally notice since $y(0) = y(1) = 0$ that

$$|y'|_\infty \leq \int_0^1 |y''(t)|\,dt \leq M_1,$$

so (2.12) follows with $M = \max\{M_0, M_1\}$. ∎

REMARK 2.2. From the proof we see that

$$a(t)\,u^2 + b(t)\,|u|\,|w|$$

in (2.4) can be replaced by

$$[a(t)+\eta]\,u^2 + [b(t)+\eta]\,|u|\,|w|,$$

provided $\eta > 0$ is chosen sufficiently small.

REMARK 2.3. In (2.4) notice $b(t)\,|u|\,|w|$ could be replaced by $b(t)\,|u|^{\xi}\,|w|^{2-\xi}$, $1 < \xi < 2$ (of course (2.6) has to be adjusted also).

REMARK 2.4. In the uniform nonresonance case (i.e. $a(t) = a_0$ and $b(t) = b_0$ in (2.4)) then Theorem 2.1 guarantees a solution provided (2.6) is replaced by

$$\frac{a_0}{\lambda_0} + \frac{b_0\,|\sqrt{q}|_\infty}{\sqrt{\lambda_0}} \;<\; 1.$$

In this case we do not need to consider Cases (A) and (B) and there is no need to discuss $R(y)$ since (2.13) reduces to

$$\begin{aligned} \|y'\|^2 \;\le\; & \left(\frac{a_0}{\lambda_0} + \frac{b_0\,|\sqrt{q}|_\infty}{\sqrt{\lambda_0}}\right)\|y'\|^2 + \int_0^1 \phi_1(t)\,q(t)\,|y(t)|\,dt \\ & + \int_0^1 \phi_2(t)\,q(t)\,|y(t)|^{\gamma+1}\,dt + \int_0^1 \phi_3(t)\,q(t)\,|y(t)|\,|y'(t)|^{\theta}\,dt \\ & + \int_0^1 \phi_4(t)\,q(t)\,|y(t)|^{\alpha+1}\,|y'(t)|^{\beta}\,dt. \end{aligned}$$

REMARK 2.5. It is easy to replace (2.6) with a quadratic form condition as described in [1, Theorem 2.1]. The details are left to the reader.

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