Dynamic Systems and Applications 16 (2007) 595-609

ON THE EXISTENCE OF MULTIPLE SOLUTIONS TO BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

RAVI P. AGARWAL, HAROLD B. THOMPSON, AND CHRISTOPHER C. TISDELL

Department of Mathematical Sciences, Florida Institute of Technology,
Melbourne, FL 32901-6975 (agarwal@fit.edu)
Department of Mathematics, University of Queensland, Queensland 4072,
Australia (hbt@maths.uq.edu.au) (cct@maths.uq.edu.au)

ABSTRACT. We establish existence results for multiple solutions to boundary value problems for nonlinear, second order, ordinary differential equations subject to nonlinear boundary conditions involving two points. We apply our theory to a problem from chemical reactor theory. Our results are extended to systems of equations.

AMS (MOS) Subject Classification. 34B15

Keywords and Phrases. Multiple solutions, lower and upper solutions, Nagumo growth condition, degree theory, nonlinear boundary conditions, chemical reactor theory

1. INTRODUCTION

It is well–known that nonlinear equations, by their very nature, can admit more than one solution. Many physical problems have motivated research regarding the existence of multiple solutions to differential equations. For example, many significant results on multiple solutions to boundary value problems arising in chemical reactor theory appear in [5,6,9,19,22,25,27]. A more theoretical approach to non–uniqueness is seen in [11,17,38] with interesting existence results presented for multiple fixed points of operators and multiple solutions to differential and integral equations.

Motivated by the theory and applications in the above works we investigate the existence of multiple solutions to the boundary value problem

$$y'' = f(x, y, y'),\ 0 \le x \le 1, \tag{1}$$

$$G(y(0), y(1), y'(0), y'(1)) = (0, 0), \tag{2}$$

where f is continuous, real–valued, nonlinear, and G is continuous and possibly nonlinear.

Received July 24, 2005

By a solution to problem (1), (2) we mean a twice continuously differentiable function $y(x)$ satisfying (1), (2) for all $x \in [0, 1]$. We present some existence theorems for solutions to problem

This work was completed while the second author was a Visiting Professor at the School of Mathematics of the University of Minnesota.

(1) and (2). Moreover, we formulate conditions under which there exist at least three distinct solutions. A particular motivation for this work was the research concerning the existence of multiple solutions to (1), (2) conducted in [13] for the special case $G = (y(0), y(1)) = (0, 0)$. Our results are new when $G \neq (y(0), y(1))$. They apply to many different types of boundary conditions including those of Dirichlet, Neumann, periodic and Sturm–Liouville, and complement the results in [13].

As an application of our results we show that for a range of parameter values there exist three distinct solutions to a boundary value problem arising in chemical reactor theory which was studied by Cohen [9]. For this range of parameter values Cohen constructed singular perturbation expansions for three distinct solutions although he stopped short of showing that three distinct solutions exist.

The methods used in our work follow along similar lines to those in [13] and [33]. We assume that there exist two pairs of upper and lower solutions for problem (1) and (2) and that the right hand side of (1) satisfies the Nagumo growth condition with respect to y'. We use the upper and lower solutions to modify f and establish a priori bounds on solutions of the modified problem. Then, motivated by the remarkable applications in [5] and in [14], we employ degree theory. Using an existence theorem from [33] we see that (1), (2) has at least two distinct solutions. Here we assume that G satisfies a degree–based relationship with the lower and upper solutions and, to compute degree, we construct additional pairs of upper and lower solutions which also have a degree–based alliance with the boundary conditions.

By formulating a condition which incorporates the additivity property of the above degree–based connection between the boundary conditions and the two pairs of lower and upper solutions chosen, we show that a third distinct solution to problem (1), (2) exists.

For the application to chemical reactor theory, under the range of parameters mentioned above, we construct the pairs of lower and upper solutions which are compatible with the boundary conditions and apply our results to conclude that there are at least three solutions.

For further works on multiple solutions to differential equations we refer the reader to [1–3,7,11,13,18,21,23,26,29,31,37,39].

2. PRELIMINARY RESULTS

We denote the boundary of a set U by ∂U and the closure of U by $\bar{U}$. If U, $V \subseteq \mathbb{R}^n$, then we denote the space of m times continuously differentiable functions mapping from U to V by $C^m(U;V)$. If $V = \mathbb{R}$ then we simply write $C^m(U)$. Define $y(x) \not\leq z(x)$ on an interval I if there is at least one $x \in I$ such that $y(x) > z(x)$. If U is a bounded, open subset of $\mathbb{R}^n$, $p \in \mathbb{R}^n$, $f \in C(\bar{U};\mathbb{R}^n)$ and $p \notin f(\partial U)$ then we denote the Brouwer degree of f on U at p by $d(f,U,p)$.

The following lower and upper solutions are used to obtain a priori bounds on solutions to (1).

DEFINITION 1. We call α (β) a lower (upper) solution for (1) if α (β) $\in C^2([0,1])$

$$\alpha''(x) \geq f(x,\alpha(x),\alpha'(x)),\ (\beta''(x) \leq f(x,\beta(x),\beta(x))),$$

for all $x \in [0,1]$. We say α (β) is a strict lower (strict upper) solution for (1) if the above inequalities are strict. If $\alpha \leq \beta$ we shall refer to the pair as non–degenerate when $\Delta_\alpha^\beta = (\alpha(0),\beta(0)) \times (\alpha(1),\beta(1)) \neq \emptyset$, i.e., $\alpha(0) < \beta(0)$ and $\alpha(1) < \beta(1)$, and set $\alpha_m = \min\{\alpha(x) : x \in [0,1]\}$, and $\beta_M = \max\{\beta(x) : x \in [0,1]\}$.

We now give the Bernstein–Nagumo growth condition on f with respect to y' and state the associated Nagumo lemma which ensures a priori bounds on first derivatives of C^2 solutions to (1).

DEFINITION 2. Let $\alpha \leq \beta \in C^2([0,1])$. We say that $f \in C([0,1] \times \mathbb{R}^2)$ satisfies a Bernstein–Nagumo condition with respect to α and β with Nagumo function h if there exists $h \in C([0,\infty);\ (0,\infty))$, such that

$$|f(x,y,p)| \leq h(|p|),\ \text{for } (x,y,p) \in [0,1] \times [\alpha(x),\beta(x)] \times \mathbb{R} \tag{3}$$

and

$$\int_\mu^L \frac{sds}{h(s)} > \beta_M - \alpha_m, \tag{4}$$

where $\alpha_m = \min\{\alpha(x) : x \in [0,1]\}$, $\beta_M = \max\{\beta(x) : x \in [0,1]\}$, and $\mu = \max\{|\alpha(0) - \beta(1)|,\ |\beta(0) - \alpha(1)|\}$.

LEMMA [Nagumo, 24]. Let $\alpha \leq \beta \in C^2([0,1])$ and let $y \in C^2([0,1])$ satisfy

$$\alpha_m - \epsilon \leq y \leq \beta_M + \epsilon \tag{5}$$

and

$$|y''| \leq h(|y'|) + \epsilon \text{ on } [0,1]. \tag{6}$$

where $h \in C([0,\infty);(0,\infty))$, satisfies (4) and $\epsilon > 0$. Then, for $\epsilon > 0$ sufficiently small, there exists $L(\alpha,\beta,h) > 0$ such that

$$\int_\mu^L \frac{sd\,s}{h(s)+\epsilon} > \beta_M - \alpha_m + 2\epsilon, \tag{7}$$

and $|y'| < L$ on $[0,1]$. Moreover we may choose L such that $L > \max\{|\alpha'(x)|,\ |\beta'(x)| : x \in [0,1]\}$.

PROOF. See [12], [15] or [24]. ∎

Modification of f is common practice for existence proofs of boundary value problems and we will make the necessary modifications by using the following functions.

DEFINITION 3. If $\alpha \leq \beta$ are given, let $\pi : \mathbb{R} \to [\alpha, \beta]$ be (the retraction) defined by $\pi(y, \alpha, \beta) = \max\{\min\{\beta, y\}, \alpha\}$. For each $\varepsilon > 0$, let $K \in C(\mathbb{R} \times (0, \infty); [-1, 1])$ satisfy

(i). $K(\cdot, \varepsilon)$ is an odd function, (ii). $K(t, \varepsilon) = 0$, if and only if $t = 0$, (iii). $K(t, \varepsilon) = 1$, for all $t \geq \varepsilon$.

Let $T \in C(\mathbb{R} \times (0, \infty))$ be given by $T(y, \alpha, \beta, \varepsilon) = K(y - \pi(y, \alpha, \beta), \varepsilon)$. Let

$$\begin{aligned} k_\epsilon(x, y, y') &= (1 - |T(y(x), \alpha(x), \beta(x), \varepsilon)|)\, f(x, \pi(y, \alpha, \beta), \pi(y', -L, L)) \\ &\quad + T(y(x), \alpha(x), \beta(x), \varepsilon)\, (|f(x, \pi(y, \alpha, \beta), \pi(y', -L, L))| + \varepsilon)\,. \end{aligned}$$

Let $X = C^1([0,1]) \times \mathbb{R}^2$ with the usual product norm.

REMARK 1. Let $\alpha,\ \beta \in C^2([0,1])$ and $\alpha \leq \beta$ on $[0,1]$. If k_ϵ satisfies the Bernstien–Nagumo condition with respect to α and β with Nagumo function h then

$$|k_\epsilon(x, y, p)| \leq h(|p|) + \epsilon, \quad \text{for} \quad (x, y, p) \in [0,1] \times \mathbb{R}^2. \tag{8}$$

Thus if y satisfies (5) and

$$y'' = k_\epsilon(x, y, y') \tag{9}$$

and if $\epsilon > 0$ is sufficiently small that (7) holds then $|y'| \leq L$, by Lemma 1, where L is given by (4).

3. NONLINEAR BOUNDARY CONDITIONS AND DEGREE THEORY

We now introduce the concept of compatible boundary conditions originally due to Thompson [33]. Compatibility can be thought of as a degree–based relationship between the boundary conditions and the lower and upper solutions. The main advantage of using this method is its unification of the theory regarding a diverse range of boundary conditions.

DEFINITION 4. We call the vector field $\Psi = (\psi^0, \psi^1) \in C(\bar{\Delta}_\alpha^\beta; \mathbb{R}^2)$ strongly inwardly pointing on $\Delta_\alpha^\beta = (\alpha(0), \beta(0)) \times (\alpha(1), \beta(1))$ if for all $(C, D) \in \partial\Delta_\alpha^\beta$

$$\psi^0(\alpha(0), D) > \alpha'(0),\ \ \psi^0(\beta(0), D) < \beta'(0),$$

$$\psi^1(C, \alpha(1)) < \alpha'(1),\ \ \psi^1(C, \beta(1)) > \beta'(1).$$

DEFINITION 5. Let $G \in C(\bar{\Delta}_\alpha^\beta \times \mathbb{R}^2; \mathbb{R}^2)$. We say G is strongly compatible with the pair $\alpha,\ \beta$ if for all strongly inwardly pointing vector fields Ψ on Δ_α^β

$$\mathbf{G}(C,D) \neq (0,0) \text{ for all } (C,D) \in \partial\Delta_\alpha^\beta,\ d(\mathbf{G}, \Delta_\alpha^\beta, (0,0)) \neq 0, \tag{10}$$

where $\mathbf{G}(C,D) = G(C,D,\Psi(C,D))$ for all $(C,D) \in \bar{\Delta}_\alpha^\beta$.

DEFINITION 6. We say G is strongly semi–compatible with the pair $\alpha,\ \beta$ if G satisfies Definition 5 with (10) omitted.

We shall require the following simple property from degree theory for the proof of our main theorem.

LEMMA 2 [Additivity of Degree]. If $\Omega = \Omega^1 \cup \Omega^2 \cup \Omega^3$, where Ω^i are open, bounded sets and pairwise disjoint, then

$$d(f,\Omega,0) = d(f,\Omega^1,0) + d(f,\Omega^2,0) + d(f,\Omega^3,0), \tag{11}$$

provided the degree in (11) is defined.

PROOF. See [20]. ∎

4. EXISTENCE OF TWO SOLUTIONS

The results in this section will guarantee the existence of two solutions to (1), (2) and will simplify the proof of our main result. The existence of two solutions to (1), (2) follows routinely and is nothing new, the existence of a third solution is the real challenge.

The following lemma mirrors standard results in the literature concerning solutions to the modified differential equation.

LEMMA 3. Let $\alpha \leq \beta$ be non–degenerate lower and upper solutions for (1) and let f satisfy a Bernstein–Nagumo condition with respect to α and β with Nagumo function h. Let y be a solution of (9) with $(y(0), y(1)) \in \bar{\Delta}_\alpha^\beta$. Then for $\epsilon > 0$ sufficiently small $\alpha(x) \leq y \leq \beta(x)$ and $|y'(x)| < L$ on $[0,1]$, where L is given in (4).

PROOF. For completeness we provide a proof. Choose $\epsilon > 0$ sufficiently small that (7) holds. We argue by contradiction. Assume that $y(t) < \alpha(t)$ for some $t \in [0,1]$. From continuity we may assume that $\alpha - y$ attains its positive maximum at some $t \in [0,1]$. Thus $\alpha'(t) = y'(t)$ and $\alpha''(t) \leq y''(t)$. From our assumptions we see $t \in (0,1)$. Now since $\alpha(t) - y(t) > 0$ we have

$$T(y(t), \alpha(t), \beta(t), \varepsilon) = K(y(t) - \alpha(t), \varepsilon) < 0.$$

Moreover, since $L \geq \max_{x\in[0,1]}\{|\alpha'(x)|, |\beta'(x)|\}$, we have

$$\begin{aligned} y''(t) &= k(t, y(t), y'(t)) = k(t, y(t), \alpha'(t)), \\ &= (1 - |T(y(t), \alpha(t), \beta(t), \varepsilon)|) f(t, \pi(y, \alpha, \beta), \pi(\alpha', -L, L)) \\ &\quad + T(y(t), \alpha(t), \beta(t), \varepsilon)(|f(t, \pi(y, \alpha, \beta), \pi(\alpha', -L, L))| + \varepsilon), \\ &= (1 - |K(y(t) - \alpha(t), \varepsilon)|) f(t, \pi(y, \alpha, \beta), \alpha') \\ &\quad + K(y(t) - \alpha(t), \varepsilon)(|f(t, \pi(y, \alpha, \beta), \alpha')| + \varepsilon) \\ &< f(t, \pi(y, \alpha, \beta), \alpha') = f(t, \alpha, \alpha') \leq \alpha''(t) \end{aligned}$$

which is a contradiction and thus $\alpha \leq y$ on $[0,1]$. Similarly, $y \leq \beta$ on $[0,1]$. Since $|y''| \leq |k_\epsilon| \leq h(|y'|) + \epsilon$ it follows from Lemma 1 that $|y'| < L$. ∎

REMARK 2. Under the assumptions of Lemma 3 it follows that any solution to (9) is also a solution to (1).

LEMMA 4. If G is strongly compatible with α and β then $G(C, D, l, m) \neq (0,0)$ if $(C, D) \in \partial\Delta$, where $C = \alpha(0)$ and $l > \alpha'(0)$, or $C = \beta(0)$ and $l < \beta'(0)$, or $D = \alpha(1)$ and $m < \alpha'(1)$, or $D = \beta(1)$ and $m > \beta'(1)$.

PROOF. Consider the case $(C, D) \in \partial\Delta$ with $C = \beta(0)$, $l < \beta'(0)$ and $\alpha(1) \leq D \leq \beta(1)$. It is easy to construct a strongly inwardly pointing vector field $\Psi = (\psi^0, \psi^1)$ with $\psi^0(\beta(0), D) = l$ and $\psi^1(\beta(0), D) = m$. Thus $G(C, D, l, m) = \mathbf{G}(\beta(0), D) \neq (0,0)$. The other cases follow in a similar fashion. ∎

To simplify our existence proof we shall need the following result from [33].

THEOREM 1. Assume that there exist non–degenerate lower and upper solutions $\alpha \leq \beta$ for (1), and that f satisfies a Bernstein–Nagumo condition with respect to α and β with Nagumo function h. Assume that $G \in C(\bar{\mathbf{\Delta}}_\alpha^\beta \times \mathbb{R}^2; \mathbb{R}^2)$ is strongly compatible with the pair α, β. Then problem (1) and (2) has a solution y satisfying $\alpha \leq y \leq \beta$.

PROOF. We sketch the main details of the proof as we will require them in the proof of our main result. Let ϵ, L, and k_ϵ be as chosen in Lemma 3. Assume G is strongly compatible with α and β, let Ψ be any strongly inwardly pointing vector field on Δ_α^β and consider (9). By Lemma 3 any solution to the modified problem, (9), is also a solution to (1). Let $\alpha_m = \min_{[0,1]} \alpha(x)$ and $\beta_M = \max_{[0,1]} \beta(x)$. Then $\alpha_m - \varepsilon$ and $\beta_M + \varepsilon$ are strict lower and upper solutions for (9). Let Ω_ε be given by

$$\Omega_\varepsilon = \{y \in C^1([0,1]) : \alpha_m - \varepsilon < y < \beta_M + \varepsilon,\ |y'| < L + 1 \text{ on } [0,1]\},$$

and let $\Gamma_\epsilon = \Omega_\varepsilon \times \Delta_\alpha^\beta$. Define $\mathbf{K} : C^1([0,1]) \to C([0,1])$ by

$$\mathbf{K}(y)(x) = k_\varepsilon(x, y(x), y'(x)),\ 0 \leq x \leq 1.$$

Define $\mathbf{CK} : C^1([0,1]) \to C([0,1])$ by

$$(\mathbf{CK})(y)(x) = -\left(\int_x^1 x(1-s) k_\varepsilon(s, y(s), y'(s)) ds + \int_0^x s(1-x) k_\varepsilon(s, y(s), y'(s)) ds\right),$$

and let $w(C,D)(x) = C(1-x) + Dx$ for $C,\ D \in \mathbb{R}$ and $0 \le x \le 1$.

Firstly observe that $(y, C, D) \in \Gamma_\varepsilon$ is a solution to

$$\Phi(y, C, D) = ((\mathbf{I} - \mathbf{CK})(y) + w(C,D), G(C, D, y'(0), y'(1))) = 0,$$

if and only if $y(0) = C$, $y(1) = D$ and y is a solution to problem (9) and (2).

Secondly observe that $d(\Phi, \Gamma_\epsilon, 0) = d(\mathbf{G}, \Delta_\alpha^\beta, 0)$, by homotopy, where $\mathbf{G}$ is given by Definition 5. Since $d(\mathbf{G}, \Delta_\alpha^\beta, 0) \ne 0$ it follows that problem (9) and (2) has a solution, y, as required. This completes the proof of Theorem 1. ∎

REMARK 3. As well as the two observations above we also need to observe that there are no solutions $(y, C, D) \in \Gamma_\epsilon \setminus \bar{\Gamma}_\alpha^\beta$ to $\Phi(y, C, D) = 0$, where $\Omega_\alpha^\beta = \{y \in C^1([0,1]) : \alpha < y < \beta,\ |y'| < L+1 \text{ on } [0,1]\}$ and $\Gamma_\alpha^\beta = \Omega_\alpha^\beta \times \Delta_\alpha^\beta$.

In Assumption T below we list the conditions on lower and upper solutions for (1) that we require for our main existence theorem. Some of these conditions have been used in [13].

ASSUMPTION T: Assume there exists two pairs of nondegenerate lower and upper solutions α_1, β_1 and α_2, β_2 for (1) satisfying:

(a) $\alpha_1 \le \alpha_2 \le \beta_2$,

(b) $\alpha_1 \le \beta_1 \le \beta_2$,

(c) $\alpha_2 \not\le \beta_1$,

(d) if y is a solution of (1), (2) with $y \ge \alpha_2$ on $[0,1]$ then $y > \alpha_2$ on $(0,1)$,

(e) if y is a solution of (1), (2) with $y \le \beta_1$ on $[0,1]$ then $y < \beta_1$ on $(0,1)$,

(f) if y is a solution of (1), (2) with $\alpha_2 \le y \le \beta_2$ and $y(x_0) = \alpha_2(x_0),\ y'(x_0) = \alpha_2'(x_0)$ for some $x_0 \in \{0,1\}$, then $y = \alpha_2$,

(g) if y is a solution of (1), (2) with $\alpha_1 \le y \le \beta_1$ on $[0,1]$ and $y(x_0) = \beta_1(x_0),\ y'(x_0) = \beta_1'(x_0)$ for some $x_0 \in \{0,1\}$, then $y = \beta_1$.

The following remarks may be applied to our problem; see Henderson and Thompson [13].

REMARK 4. Conditions (d), (f), and (g) of Assumption T will be satisfied if, for example, either

(i). solutions of initial value problems for (1) are unique, or

(ii). α and β are strict lower and strict upper solutions, respectively.

5. **THE MAIN RESULT**

We now present the main result.

THEOREM 2. Let Assumption T hold, and let f satisfy a Bernstein–Nagumo condition with respect to α_1 and β_2 with Nagumo function h. Let $G \in C(\bar{\Delta}_{\alpha_1}^{\beta_2} \times \mathbb{R}^2; \mathbb{R}^2)$

be strongly compatible with the pair α_1, β_1 and the pair α_2, β_2 and let G be strongly semi–compatible with the pair α_1 and β_2. Assume that

$$d(\mathbf{G}, \Delta_{\alpha_1}^{\beta_1}, 0) + d(\mathbf{G}, \Delta_{\alpha_2}^{\beta_2}, 0) \neq d(\mathbf{G}, \Delta_{\alpha_1}^{\beta_2}, 0), \tag{12}$$

where in $d(\mathbf{G}, \Delta_{\alpha_i}^{\beta_j}, 0)$, $\mathbf{G}(C, D) = G(C, D, \Psi(C, D))$ for some strongly inwardly pointing vector field, $\Psi(C, D)$, on $\Delta_{\alpha_i}^{\beta_j}$, for $1 \leq i \leq j \leq 2$. Then problem (1), (2) has at least three solutions $y_1,\ y_2$ and y_3 satisfying $\alpha_1 \leq y_1 \leq \beta_1$, $\alpha_2 \leq y_2 \leq \beta_2$, and $y_3 \nleq \beta_1$ and $y_3 \ngeq \alpha_2$, respectively.

PROOF. For the pair α_1 and β_2 of lower and upper solutions for (1) let $\epsilon > 0$, L and k_ϵ be as chosen in Lemma 3. Without loss of generality we may assume that $L > \max\{|\alpha_1'(x)|,\ |\alpha_2'(x)|,\ |\beta_1'(x)|, |\beta_2'(x)| : x \in [0,1]\}$. Thus solutions, y, of problem (9) such that $(y(0), y(1)) \in \bar{\Delta}_{\alpha_1}^{\beta_2}$ satisfy $\alpha_1 \leq y \leq \beta_2$ and $|y'| < L$ on $[0,1]$, so that solutions of problem (9) and (2) are the required solutions of problem (1) and (2). Let

$$\begin{aligned}
\Omega_\epsilon &= \{y \in C^1([0,1]) : \alpha_\epsilon < y < \beta_\epsilon,\ |y'| < L+1 \text{ on } [0,1]\} \\
\Gamma_\epsilon &= \Omega_\epsilon \times \Delta_{\alpha_1}^{\beta_2}, \\
\Omega_{\alpha_i}^{\beta_j} &= \{y \in C^1([0,1]) : \alpha_i < y < \beta_j,\ |y'| < L+1 \text{ on } [0,1]\}, \text{ and} \\
\Gamma_{\alpha_i}^{\beta_j} &= \Omega_{\alpha_i}^{\beta_j} \times \Delta_{\alpha_i}^{\beta_j} \text{ for } i,\ j = 1, 2,\ i \leq j,
\end{aligned}$$

where $\alpha_\epsilon = \min\{\alpha(x) - \epsilon : x \in [0,1]\}$ and $\beta_\epsilon = \max\{\beta(x) + \epsilon : x \in [0,1]\}$. From Theorem 1 and the associated observations $(y, C, D) \in \Gamma_\epsilon$ is a solution of

$$\Phi(y, C, D) = ((\mathbf{I} - \mathbf{CK})(y) + w(C, D), G(C, D, y'(0), y'(1))) = 0, \tag{13}$$

if and only if $(y(0), y(1)) = (C, D) \in \bar{\Delta}_{\alpha_1}^{\beta_2}$, $\alpha_1 \leq y \leq \beta_2$, and y is a solution to problem (9) and (2).

Let

$$\Gamma_{\alpha_2} = \{(y, C, D) \in \Gamma_\epsilon : y > \alpha_2 \text{ on } (0,1)\} \tag{14}$$

$$\Gamma^{\beta_1} = \{(y, C, D) \in \Gamma_\epsilon : y < \beta_1 \text{ on } (0,1)\}, \tag{15}$$

and

$$\Gamma = \Gamma_\epsilon \setminus \{\bar{\Gamma}_{\alpha_2} \cup \bar{\Gamma}^{\beta_1}\}. \tag{16}$$

By Lemma 3, if $(y, C, D) \in \bar{\Gamma}_{\alpha_2}$ is a solution of (13) then $(y, C, D) \in \bar{\Gamma}_{\alpha_2}^{\beta_2}$, while if $(y, C, D) \in \bar{\Gamma}^{\beta_1}$ is a solution of (13) then $(y, C, D) \in \bar{\Gamma}_{\alpha_1}^{\beta_1}$. By strong compatibility of the boundary conditions with α_2 and β_2 there are no solutions $(y, C, D) \in \bar{\Gamma}_{\alpha_2}$ with $y(0) = \alpha_2(0)$ and $y'(0) > \alpha_2'(0)$ or with $y(1) = \alpha_2(1)$ and $y'(1) < \alpha_2'(1)$. By Assumption T part (f) there are no solutions with $y(0) = \alpha_2(0)$ and $y'(0) = \alpha_2'(0)$ or with $y(1) = \alpha_2(1)$ and $y'(1) = \alpha_2'(1)$. Moreover, by Assumption T part (d) there are no solutions $(y, C, D) \in \bar{\Gamma}_{\alpha_2}$ with $y \geq \alpha_2$ on $[0,1]$ and $y(x_0) = \alpha_2(x_0)$ for some

$x_0 \in (0,1)$. Moreover any solution $(y,C,D) \in \bar{\Gamma}_{\alpha_2}^{\beta_2}$ satisfies $|y'| < L$. Thus there are no solutions in $\partial\Gamma_{\alpha_2}$. Similarly there are no solutions $(y,C,D) \in \partial\Gamma^{\beta_1}$.

Applying Theorem 1 to the pairs $\alpha_1,\ \beta_1$ and $\alpha_2,\ \beta_2$ we see that there are solutions (y_1,C_1,D_1) and (y_2,C_2,D_2) of (13) in Γ^{β_1} and in Γ_{α_2}, respectively. Assume that there is no solution $(y,C,D) \in \Gamma$. Thus $d(\Phi,\Gamma,0) = 0$. It follows that

$$\begin{aligned} d(\mathbf{G},\Delta_{\alpha_1}^{\beta_2},0) \;=\; d(\Phi,\Gamma_\epsilon,0) &= d(\Phi,\Gamma^{\beta_1},0) + d(\Phi,\Gamma_{\alpha_2},0) + d(\Phi,\Gamma,0) \\ &= d(\mathbf{G},\Delta_{\alpha_1}^{\beta_1},0) + d(\mathbf{G},\Delta_{\alpha_2}^{\beta_2},0) \end{aligned}$$

where Γ_ϵ is given in (16) and we set $\mathbf{G}(C,D) = G(C,D,\Psi(C,D))$ in $d(\mathbf{G},\Delta_{\alpha_i}^{\beta_j},0)$ for a strongly inwardly pointing vector field Ψ on $\Delta_{\alpha_i}^{\beta_j}$ for $1 \le i \le j \le 2$. From this contradiction we conclude that there is a third solution $(y_3,C_3,D_3) \in \Gamma_{\alpha_1}^{\beta_2}$. Since $(y_3,C_3,D_3) \notin \bar{\Gamma}_{\alpha_2}$ and $(y_3,C_3,D_3) \notin \bar{\Gamma}^{\beta_1}$ it follows that y_3 is the required third solution to problem (1) and (2). ∎

6. SOME WELL–KNOWN BOUNDARY CONDITIONS

Note that Theorems 1 and 2 are based on performing a 'test' on the given boundary conditions with respect to the pairs of upper and lower solutions chosen. In the well–known cases, compatibility, the degree dependent relationship between the boundary conditions and upper and lower solutions is equivalent to the standard assumptions.

Consider (1) subject to any of the following boundary conditions:

$$\begin{aligned} &(i)\ G = (y(0)-A, y(1)-B) = (0,0); \\ &(ii)\ G = (y'(0)-A, y'(1)-B) = (0,0); \\ &(iii)\ G = (y(0)-y(1), y'(0)-y'(1)) = (0,0). \end{aligned}$$

Then the usual assumptions regarding upper and lower solutions are respectively:

$$\begin{aligned} &(i^*)\ \alpha(0) \le A \le \beta(0),\ \alpha(1) \le B \le \beta(1); \\ &(ii^*)\ \alpha'(0) \le A \le \beta'(0),\ \alpha'(1) \le B \le \beta'(1); \\ &(iii^*)\ \alpha(0) \le \beta(0),\ \alpha(1) \le \beta(1),\ \alpha'(0) \le \beta'(0),\ \alpha'(1) \le \beta'(1). \end{aligned}$$

LEMMA 5. Let $\alpha \le \beta$ be nondegenerate lower and upper solutions for (1). Then the boundary conditions (i) - (iii) are compatible if and only if the respective inequalities (i*) – (iii*) hold. Moreover, if strict inequalities hold in (i*) – (iii*) then the boundary conditions (i) – (iii) are strongly compatible.

PROOF. The proof for compatibility in case (iii) may be found in [33]. Suppose (i^*) holds with strict inequalities. We shall prove strong compatibility for G given by (i). Now $\mathbf{G}\,(C,D) = (C-A, D-B)$, and α and β are nondegenerate. This implies $\mathbf{G} = (\mathbf{G}^0, \mathbf{G}^1)$ satisfies

$$\mathbf{G}^0(\alpha(0),D) < 0,\ \ \mathbf{G}^1(C,\alpha(1)) < 0,$$

$$\mathbf{G}^0(\beta(0), D) > 0,\ \mathbf{G}^1(C, \beta(1)) > 0,$$

and hence $d(\mathbf{G}, \Delta_\alpha^\beta, (0,0)) = 1 \neq 0$. Thus G is strongly compatible with α and β. The other cases follow by arguments similar to those above. ∎

LEMMA 6. Condition (12) will be satisfied for the boundary conditions $G = (y(0), y(1)) = (0,0)$ if strict inequalities hold in (i^*) for the pairs α_1, β_1 and α_2, β_2 when $A = B = 0$.

PROOF. Assume that strict inequalities hold in (i^*) for the pairs α_1, β_1 and α_2, β_2 when $A = B = 0$. From the previous lemma we see that

$$d(\mathbf{G}, \Delta_{\alpha_1}^{\beta_1}, 0) = 1 = d(\mathbf{G}, \Delta_{\alpha_2}^{\beta_2}, 0) = d(\mathbf{G}, \Delta_{\alpha_1}^{\beta_2}, 0).$$

Thus (12) holds. ∎

REMARK 5. Theorem 2 applies not only to the above case but to a wide variety of boundary conditions including nonlinear variations. Thus our results are extensions of those in [13].

EXAMPLE. Consider the problem

$$y'' = -(y')^2 + (y-1)^2(y-12),\ 0 \leq x \leq 1, \tag{17}$$

$$G = (y'(0) - y(0), y'(\pi) + y(\pi)) = (0,0). \tag{18}$$

Solutions to initial value problems for (17) are unique. We may choose $\alpha_1 = -1$, $\beta_1 = 1$, $\alpha_2 = 2\sin x$ and $\beta_2 = 12$ as our two pairs of lower and upper solutions.

It is not hard to verify that the boundary conditions are strongly compatible with the lower and upper solutions chosen with the degree calculations following routinely. Note also that f satisfies the conditions of Lemma 1. Thus all the conditions of Theorem 2 are satisfied and the problem has at least three solutions y_1, y_2, and y_3 satisfying $\alpha_1 \leq y_1 \leq \beta_1$, $\alpha_2 \leq y_2 \leq \beta_2$, and $y_3 \not\leq \beta_1$ and $y_3 \not\geq \alpha_2$, respectively.

7. AN APPLICATION TO CHEMICAL REACTOR THEORY

Cohen [9] considered certain chemical reactions in tubular reactors which can be mathematically described by the boundary value problem

$$y'' = (y' - g(y))/c,\ 0 \leq x \leq 1, \tag{19}$$

$$y'(0) - ly(0) = 0 = y'(1), \tag{20}$$

where y is the temperature in the reactor,

$$g(s) = d(q-s)e^{-k/(1+s)},\ 0 \leq s \leq q, \tag{21}$$

is the rates of chemical production of the species in the reactor and c, l, d and q are known positive constants.

We consider (19) and (20) where g satisfies the following assumption.

ASSUMPTION g.

1. g is twice continuously differentiable;
2. $g(s) > 0$, for $s < q$, $g(s) < 0$, for $s > q$, and
3. $\int^q ds/g(s) = \infty$.

We note that g given by (21) satisfies Assumption g.

For problem (19) and (20) subject to Assumption g, Cohen established the following:

1. any solution y satisfies $0 < y < q$ and $y' \geq 0$ on $[0, 1]$;
2. there is a minimal solution y_m and a maximal solution y_M;
3. $y_m = y_M$ if

$$\frac{d}{ds}\left(\frac{g(s)}{s}\right) < 0 \text{ for } 0 < s < q \tag{22}$$

   and hence the solution is unique.

In the case g is given by (21) and $ls - g(s) = 0$ has two (three) solutions in $(0, q)$, Cohen used a 'formal' singular perturbation analysis of problem (19) and (20) to conclude that there are at least two (three) solutions for $c > 0$ sufficiently small. Based on this, on other observations, and on numerical evidence Cohen asserts that, for g given by (21), it appears to be necessary and sufficient that

$$\frac{d}{ds}[g(s)/s] > 0, \text{ for some } s, \ 0 < s < q, \tag{23}$$

for multiple solutions of problem (19) and (20) to exist.

If $y_m \neq y_M$ it follows from the uniqueness of initial value problems that $y_m < y_M$ on $[0, 1]$. In this case we have the following result.

LEMMA 7. Let g be twice continuously differentiable and satisfy $d/ds[g(s)/s] \leq 0$ for $0 < s \leq q$. If there exist minimal and maximal solutions $y_m < y_M$ of problem (19) and (20) then $\lambda y_m + (1 - \lambda) y_M$ is a solution for each λ with $0 \leq \lambda \leq 1$ and hence problem (19) and (20) has infinity many solutions.

PROOF. Now $y_m(0) \leq y_m \leq y_M \leq y_M(1)$ on $[0, 1]$. Also

$$0 = \left[e^{-x/c}(y'_M y_m - y'_m y_M)\right]_0^1 = \int_0^1 e^{-x/c} y_m y_M (g(y_m)/y_m - g(y_M)/y_M) dx \geq 0,$$

since $g(y_m)/y_m - g(y_M)/y_M \geq 0$. Thus $g(y_m)/y_m - g(y_M)/y_M = 0$ on $[0, 1]$ and $g(s)/s = k$ for $y_m(0) \leq s \leq y_M(1)$. Thus y_m and y_M satisfy (19) and the result follows. ∎

In view of this result to complete our analysis of the role played by the sign of $d/ds[g(s)/s]$ in the existence of multiple solutions of problem (19) and (20) we need only consider the case g satisfies (23). Let

$$L = \{l \in \mathbb{R} : \exists\ a,\ b,\ 0 < b < a < q, \text{ such that } g(b)/b < l < g(a)/a\}. \tag{24}$$

LEMMA 8. $L \neq \emptyset$ if and only if (23) holds.

We have the following result.

THEOREM 3. Let g satisfy Assumption g and let $L \neq \emptyset$ be given by (24). For l in L and $c > 0$ sufficiently small, there are at least three solutions of problem (19) and (20).

PROOF. Let $G = (g_0, g_1)$ where $g_0 = y'(0) - ly(0)$ and $g_1 = y'(1)$. It suffices to construct lower solutions α_1 and α_2 and upper solutions β_1 and β_2 satisfying the assumptions of Theorem 2. Let $\alpha_1(x) = -1 - lx/2$ and $\beta_2(x) = q + qlx/2$. Clearly α_1 is a lower solution and β_2 is an upper solution for (19) on $[0, 1]$.

Since $l \in L$ there exists a and b with $0 < b < a < q$ such that $g(b)/b < l < g(a)/a$. Choose $\varepsilon > 0$ such that

$$g(b) + \varepsilon < lb \text{ and } la < g(a) - \varepsilon. \tag{25}$$

Let $\alpha_2(0) = a > b = \beta_1(0)$, $\alpha_2'(x) = g(\alpha_2) - \gamma x - \varepsilon$, $\beta_2'(x) = g(\beta_2) + \varepsilon$, where $\gamma \geq 0$ is chosen below and $\varepsilon > 0$ satisfies further restrictions given below.

Since $\int^q ds/g(s) = \infty$ we may choose $\varepsilon > 0$ sufficiently small that

$$\int_b^q ds/(g(s) + \varepsilon) > 1. \tag{26}$$

Now $\beta_1' = g(\beta_1) + \varepsilon \geq 0$ and by (26) β_1 satisfies $\beta_1 < q$ on $[0, 1]$. For $\gamma = 0$ we have $\alpha_2' \geq 0$ and $\alpha_2 \leq m < q$ on $[0, 1]$. Since $\alpha_2(x, \gamma)$ and $\alpha_2'(x, \gamma)$ depend continuously on γ while $0 \leq \alpha_2 \leq q$ it follows that we may choose $\gamma > 0$ such that $a \leq \alpha_2 \leq m$ and $-\varepsilon < \alpha_2'(1) < 0$.

Now

$$(\beta_1' - g(\beta_1))/c = \varepsilon/c > \beta_1'',\ 0 \leq x \leq 1,$$

and

$$(\alpha_2' - g(\alpha_2))/c = -\varepsilon/c - \gamma x/c \leq \alpha_2'',\ 0 \leq x \leq 1,$$

for $0 < c < \delta$ for some $\delta > 0$. Thus α_2 is a strict lower solution and β_1 is a strict upper solution for (19) on $[0, 1]$.

Now $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$ on $[0, 1]$ and $\alpha_2(0) > \beta_1(0)$ so $\beta_1 \ngeq \alpha_2$ on $[0, 1]$. Since any solution y satisfies $0 \leq y \leq q$ it suffices to show that there are at least three solutions between α_1 and β_2.

Now $\alpha_1'(0) < 0 < \beta_1'(1)$ and $\alpha_2'(1) < 0 < \beta_2'(1)$ while:

if $y(0) = \alpha_1(0) = -1$ then $y'(0) = -l < -l/2 = \alpha_1'(0)$;
if $y(0) = \alpha_2(0) = a$ then $y'(0) = la < g(a) - \varepsilon = \alpha_2'(0)$;
if $y(0) = \beta_1(0) = b$ then $y'(0) = lb > g(b) + \varepsilon = \beta_1'(0)$;
if $y(0) = \beta_2(0) = q$ then $y'(0) = lq > lq/2 = \beta_2'(0)$.

Thus $G = (g_0, g_1)$ satisfies $d(\mathbf{G}, \triangle, 0) = 1$ for $\triangle = \triangle_{\alpha_1}^{\beta_2}$, $\triangle_{\alpha_1}^{\beta_1}$ and $\triangle_{\alpha_2}^{\beta_2}$. Hence by Theorem 2 there are three solutions as required.

8. CONCLUDING REMARKS

REMARK 6. There are other conditions which guarantee a priori bounds on y'. For example, the Bernstein–Nagumo condition can be replaced with (see [4], [16] and [28]): Let there exist $\alpha,\ \beta \in C^2([0,1])$ such that $\alpha \leq \beta$ and let there exist $\varphi \leq \psi \in C^1([0,1] \times \mathbb{R})$ such that

$$f(x, y, \varphi(x, y)) > \frac{\partial \varphi}{\partial x}(x, y(x)) + \frac{\partial \varphi}{\partial y}(x, y(x))\varphi(x, y(x));$$

$$f(x, y, \psi(x, y)) < \frac{\partial \psi}{\partial x}(x, y(x)) + \frac{\partial \psi}{\partial y}(x, y(x))\psi(x, y(x))$$

on $\bar{\omega} = \{(x, y) \in [0,1] \times \mathbb{R} : \alpha(x) \leq y \leq \beta(x),\ x \in [0,1]\}$. Then for any solution $y \in C^2([0,1])$ of (1) such that $\alpha(x) \leq y \leq \beta(x)$ on $[0,1]$ and

$$\varphi(0, y(0)) \leq y'(0) \leq \psi(0, y(0)) \tag{27}$$

we have $\varphi(x, y) \leq y' \leq \psi(x, y)$ on $[0,1]$. If we assume that f satisfies a Bernstein–Nagumo condition with respect to α and β and we strengthen (4) to

$$\int^{\infty} \frac{sds}{h(s)} = \infty,$$

then we can construct ϕ and ψ satisfying (27). On the other hand ϕ and ψ satisfying (27) exist when $f(x, y, p)$ grows sufficiently fast with respect to p that (4) is not satisfied and, in this direction the result is more general than Theorem 2. On the other hand Theorem 2 does not require the additional assumption (27); this condition must follow from the boundary conditions.

REMARK 7. We may broaden our existence results to the case when f is vector–valued and weakly coupled, i.e.,

$$f(x, y, y') = (f_1(x, y, y_1'), \ldots, f_n(x, y, y_n')).$$

Start by suitably extending the notion of upper and lower solutions and Nagumo conditions for weakly coupled f, as originally defined in [15]. Compatibility for weakly coupled systems follows in a similar fashion to the definitions in Section 3, with the inequalities between vectors holding component–wise. For full details see [34].

REMARK 8. Existence theorems for multiple solutions to weakly coupled systems of equations follow under similar conditions to those in Theorem 2, and the associated sharpened remarks from this section. For more detailed research on weakly coupled systems we refer the reader to [10], [15], [30] and [34].

REMARK 9. For simplicity we have restricted our attention to two–point boundary conditions only. There is suitable scope for generalization of our results to the three–(or more) point case, once the appropriate definition of compatibility is extended, as in [35].

REMARK 10. It should be noted that our theorems only provide lower bounds for the number of solutions.

REFERENCES

[1] R.P. Agarwal and D. O'Regan, Multiple solutions for the Dirichlet second–order boundary value problem of nonsingular type, *Comput. Math. Appl.* **37**(1999), 99–106.

[2] R.P. Agarwal and D. O'Regan, Multiple positive solutions for a coupled system of boundary value problems, *Dyn. Cont. Disc. Impuls. Sys.* **7**(2000), 97–106.

[3] R.P. Agarwal, D. O'Regan and V. Lakshmikantham, Multiple solutions for continuous and discrete boundary value problems, *Nonlinear Studies* **6**(1999), 151–159.

[4] K. Ako, Subfunctions for ordinary differential equations V, *Funkcial. Ekvac.* **12**(1969), 239–249.

[5] H. Amann, Existence of multiple solutions for nonlinear elliptic boundary value problems, *Indiana Univ. Math. J.* **21**(1972), 925–935.

[6] N.R. Amundson and D. Luss, Qualitative and quantitative observations on the tubular reactor, *Canad. J. Chem. Eng.* **46**(1968), 424–433.

[7] R. Avery, Existence of multiple positive solutions to a conjugate boundary value problem, *Math. Sci. Res. Hot–Line* **2**(1998), 1–6.

[8] S.R. Bernfeld and V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value Problems*, *Academic Press*, New York, 1974.

[9] D.S. Cohen, Multiple stable solutions of nonlinear boundary value problems, *SIAM J. Appl. Math.* **20**(1971), 1–13.

[10] J. Ehme and J. Henderson, Multipoint boundary value problems for weakly coupled equations, *Nonlinear Times and Digest* **1**(1994), 133–140.

[11] D.J. Guo and V. Lakshmikantham, Multiple solutions of two–point boundary value problems of ordinary differential equations in Banach spaces, *J. Math. Anal. Appl.* **129**(1988), 211–222.

[12] P. Hartman, *Ordinary Differential Equations*, *John Wiley*, New York, 1964.

[13] J. Henderson and H.B. Thompson, Existence of multiple solutions for second order boundary value problems, *J. Diff. Eqns.* **166**(2000), 443–454.

[14] R.E. Gaines and J.L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, *Lecture Notes in Mathematics* **568**, *Springer–Verlag*, New York, 1977.

[15] R. Kannan and V. Lakshmikantham, Periodic solutions of nonlinear boundary value problems, *Nonlinear Analysis* **6**(1982), 1–10.

[16] H.W. Knobloch, Comparison theorems for nonlinear second order differential equations, *J. Diff. Eqns.* **1**(1965), 1–26.

[17] M.A. Krasnosel'skii and V.Ja. Stecenko, Some nonlinear problems with many solutions, *Amer. Math. Soc. Transl. (2),* **54**(1966), 29–48.

[18] T. Laetsch, Existence and bounds for multiple solutions of nonlinear equations, *SIAM J. Appl. Math.* **18**(1970), 389–400.

[19] R.W. Leggett and L.R. Williams, Multiple positive fixed points on ordered Banach spaces, *Indiana Univ. Math. J.* **28**(1979), 673–688.

[20] N.G. Lloyd, *Degree Theory*, *Cambridge University Press*, Cambridge, 1978.

[21] S. Li and J. Su, Existence of multiple solutions of a two–point boundary value problem at resonance, *Topol. Meth. Nonl. Anal.* **10**(1997), 123–135.

[22] L. Markus and N.R. Amundson, Nonlinear boundary-value problems arising in chemical reactor theory, *J. Diff. Eqns.* **4**(1968), 102–113.

[23] J. Mawhin and M. Willem, Multiple solutions of the periodic boundary value problem for some forced pendulum–type equations, *J. Diff. Eqns.* **52**(1984), 264–287.

[24] M. Nagumo, Über die Differentialgleichung $y'' = f(x, y, y')$, *Proc. Phys–Math. Soc. Japan* **19** (1937), 861–866.

[25] S.V. Parter, Solutions of a differential equation arising in chemical reactor processes, *SIAM J. Appl. Math.* **26**(1974), 687–716.

[26] S.V. Parter and V. Pereyra, Nonlinear two–point boundary value problems with multiple solutions, *Nordisk Tidskr. Informationsbehandling* **11**(1971), 53–83.

[27] S.V. Parter, M.L. Stein and P.R. Stein, On the multiplicity of solutions of a differential equation arising in chemical reactor theory, *Stud. Appl. Math.* **54**(1975), 293–314.

[28] K. Schmitt, Periodic solutions of nonlinear second order differential equations, *Math. Zeitschr.* **98**(1967), 200–207.

[29] K. Schmitt, Applications of variational inequalities to ordinary and partial differential equations — multiple solutions of boundary value problems, *J. Diff. Eqns.* **17**(1975), 154–186.

[30] V.S.H. Rao and A. S. Vatsala, Existence of solutions for a system of second order nonlinear periodic boundary value problems, *Applicable Analysis* **17**(1983), 23–36.

[31] M. Struwe, Multiple solutions of anticoercive boundary value problems for a class of ordinary differential equations of second order, *J. Diff. Eqns.* **37**(1980), 285–295.

[32] H.B. Thompson, Minimal solutions for two point boundary value problems, *Rend. Circ. Mat. Palermo (2)* **XXXVII**(1988), 261–281.

[33] H.B. Thompson, Second order ordinary differential equations with fully nonlinear two point conditions, *Pacific J. Math.* **172**(1996), 255–277.

[34] H.B. Thompson and C.C. Tisdell, Nonlinear multipoint boundary value problems for weakly coupled systems, *Bull. Austral. Math. Soc.* **60**(1999), 45–54.

[35] H.B. Thompson and C.C. Tisdell, Three—point boundary value problems for second–order, ordinary, differential equations, *Mathl. Comput. Modelling*, to appear.

[36] L.R. Williams and R.W. Leggett, Multiple fixed point theorems for problems in chemical reactor theory, *J. Math. Anal. Appl.* **69**(1979), 180–193.

[37] P.J.Y. Wong and R.P. Agarwal, Multiple solutions of generalized multipoint conjugate boundary value problems, *Georgian Math. J.* **6**(1999), 567–590.

[38] S. Yong and S. Jingxian, Multiple positive fixed points of weakly inward mappings, *J. Math. Anal. Appl.* **148**(1990), 431–439.

[39] W. Zou, Multiple solutions results for two–point boundary value problem with resonance, *Dis. Contin. Dynam. Sys.* **4**(1998), 485–496.