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MULTIPLE POSITIVE SOLUTIONS OF STURM-LIOUVILLE PROBLEMS FOR SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is devoted to study the existence of multiple positive solutions for the second order Sturm-Liouville problems with impulse effects. The proof is based on the theory of fixed point index in cones.

Keywords: Boundary value problems; Impulse effects; Multiple positive solutions; Fixed point index in cones

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1. INTRODUCTION

This paper is devoted to study the existence of multiple positive solutions for the boundary value problem with impulse effects

$$\begin{cases} -Lu = g(x,u), \qquad x \in I', \\ -\Delta(pu')|_{x=x_k} = I_k(u(x_k)), \quad k = 1,2,\dots,m, \\ R_1(u) = \alpha_1 u(0) + \beta_1 u'(0) = 0, \\ R_2(u) = \alpha_2 u(1) + \beta_2 u'(1) = 0, \end{cases} \tag{1.1}$$

here $Lu = (p(x)u')' + q(x)u$ is Sturm-Liouville operator, $I = [0,1]$ $I' = I \backslash \{x_1, x_2, \dots, x_m\}$ and $0 < x_1 < x_2 < \cdots < x_m < 1$ are given, $\mathbb{R}^+ = [0,\infty)$, $g \in \mathbb{C}(I \times \mathbb{R}^+, \mathbb{R}^+)$, $I_k \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^+)$, $\Delta(pu')|_{x=x_k} = p(x_k)u'(x_k^+) - p(x_k)u'(x_k^-)$, $u'(x_k^+)$ (respectively $u'(x_k^-)$) denotes the right limit (respectively left limit) of $u'(x)$ at $x = x_k$.

Throughout this paper, we always suppose that

(S_1) $p(x) \in \mathbb{C}^1([0,1],\mathbb{R})$, $p(x) > 0$, $q(x) \in \mathbb{C}([0,1],\mathbb{R})$, $q(x) \le 0$, $\alpha_1, \alpha_2, \beta_2 \ge 0$, $\beta_1 \le 0$, $\alpha_1^2 + \beta_1^2 > 0$, $\alpha_2^2 + \beta_2^2 > 0$.

In recent years, second-order differential boundary value problems with impulses have been studied extensively in the literature (see for instance [1, 3, 6, 7, 8, 9, 10, 11]

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and their references). However, most papers are concerned with the case $p(x) = 1$ and $q(x) = 0$. In this paper, we will consider the case $p(x) \neq 1$ and $q(x) \neq 0$. Here we also mention that second order dynamic inclusions on time scales with impulses has been studied in [2].

The existence of positive solutions of problem (1.1) has been studied in [5]. By employing Krasnoselskii fixed point theorem on compression and expansion of cones, it was proved in [5] that problem (1.1) has at least one positive solution when $g(x,u)$ is either superlinear or sublinear in u. Our results in this paper improve those in [5]. The proof is based on fixed point index theory in cones [4].

To conclude the introduction, we introduce the following notation:

$$g_0 = \liminf_{u\to 0^+} \min_{x\in[0,1]} \frac{g(x,u)}{u}, \qquad I_0(k) = \liminf_{u\to 0^+} \frac{I_k(u)}{u},$$

$$g_\infty = \liminf_{u\to +\infty} \min_{x\in[0,1]} \frac{g(x,u)}{u}, \qquad I_\infty(k) = \liminf_{u\to +\infty} \frac{I_k(u)}{u};$$

$$g^\infty = \limsup_{u\to +\infty} \max_{x\in[0,1]} \frac{g(x,u)}{u}, \qquad I^\infty(k) = \limsup_{u\to +\infty} \frac{I_k(u)}{u},$$

$$g^0 = \limsup_{u\to 0^+} \max_{x\in[0,1]} \frac{g(x,u)}{u}, \qquad I^0(k) = \limsup_{u\to 0^+} \frac{I_k(u)}{u}.$$

Moreover, for the simplicity in the following discussion, we introduce the following hypotheses.

(H$_1$) $$g_0 + \frac{\sigma \sum_{k=1}^m I_0(k)\phi_1(x_k)}{\int_0^1 \phi_1(x)dx} > \lambda_1, \quad g_\infty + \frac{\sigma \sum_{k=1}^m I_\infty(k)\phi_1(x_k)}{\int_0^1 \phi_1(x)dx} > \lambda_1.$$

(H$_2$) $$g^0 + \frac{\sum_{k=1}^m I^0(k)\phi_1(x_k)}{\int_0^1 (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\phi_1(x)dx} < \lambda_1, \quad g^\infty + \frac{\sum_{k=1}^m I^\infty(k)\phi_1(x_k)}{\int_0^1 (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\phi_1(x)dx} < \lambda_1,$$

here $\sigma = \min_{x\in[x_1,x_m]} \min\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\}$ (see section 2), and $\phi_1(x)$ is the eigenfunction related to the smallest eigenvalue λ_1 of the eigenvalue problem $-L\phi = \lambda\phi,\ R_1(\phi) = R_2(\phi) = 0$.

(H$_3$) There is a $p > 0$ such that $0 \le u \le p$ and $0 \le x \le 1$ implies

$$g(x,u) \le \eta p,\ I_k(u) \le \eta_k p,$$

here $\eta,\ \eta_k \ge 0$ satisfy $\eta + \sum_{k=1}^m \eta_k > 0,\ \eta\int_0^1 G(y,y)dy + \sum_{k=1}^m G(x_k,x_k)\eta_k < 1$ and $G(x,y)$ is the Green's function of boundary value problem $-Lu = 0,\ R_1(u) = R_2(u) = 0$ (see section 2).

(H$_4$) There is a $p > 0$ such that $\sigma p \le u \le p$ implies

$$g(x,u) \ge \lambda p,\ 0 \le x \le 1, \quad I_k(u) \ge \lambda_k p,$$

here $\lambda,\ \lambda_k \geq 0$ satisfy $\lambda + \sum_{k=1}^{m} \lambda_k > 0$ and $\lambda \int_{x_1}^{x_m} G(\frac{1}{2}, y)dy + \sum_{k=1}^{m} \lambda_k G(\frac{1}{2}, x_k) > 1$.

2. PRELIMINARIES

In this paper, we shall consider the following space

$$\begin{aligned}\mathbb{PC}'(I,\mathbb{R}) = \{u \in \mathbb{C}(I,\mathbb{R}); u'|_{(x_k,x_{k+1})} \in \mathbb{C}(x_k,x_{k+1}),\\ u'(x_k^-) = u'(x_k), \quad \exists\ u'(x_k^+),\ k = 1,2,\cdots,m\}\end{aligned}$$

with the norm $\|u\|_{\mathbb{PC}'} = \max\{\|u\|,\ \|u'\|\}$, here $\|u\| = \sup_{x\in[0,1]} |u(x)|$, $\|u'\| = \sup_{x\in[0,1]} |u'(x)|$. Then $\mathbb{PC}'(I,\mathbb{R})$ is a Banach space.

Definition 2.1. A function $u \in \mathbb{PC}'(I,\mathbb{R}) \cap \mathbb{C}^2(I',\mathbb{R})$ is a solution of (1.1) if it satisfies the differential equation

$$Lu + g(x,u) = 0, \quad x \in I'$$

and the function u satisfies conditions $\Delta(pu')|_{x=x_k} = -I_k(u(x_k))$ and $R_1(u) = R_2(u) = 0$.

Let $Q = I \times I$ and $Q_1 = \{(x,y) \in Q | 0 \leq x \leq y \leq 1\},\ Q_2 = \{(x,y) \in Q | 0 \leq y \leq x \leq 1\}$. Let $G(x,y)$ is the Green's function of the boundary value problem

$$-Lu = 0,\ R_1(u) = R_2(u) = 0.$$

Following from [5], $G(x,y)$ can be written by

$$G(x,y) := \begin{cases} \dfrac{m(x)n(y)}{\omega},\ (x,y) \in Q_1, \\ \dfrac{m(y)n(x)}{\omega},\ (x,y) \in Q_2. \end{cases} \tag{2.1}$$

Lemma 2.2. [4] *Suppose that* (S_1) *holds, then the Green's function* $G(x,y)$, *defined by* (2.1), *possesses the following properties:*

(i) $m(x) \in \mathbb{C}^2(I,R)$ *is increasing and* $m(x) > 0,\ x \in (0,1]$.

(ii) $n(x) \in \mathbb{C}^2(I,R)$ *is decreasing and* $n(x) > 0,\ x \in [0,1)$.

(iii) $(Lm)(x) \equiv 0,\ m(0) = -\beta_1,\ m'(0) = \alpha_1$.

(iv) $(Ln)(x) \equiv 0,\ n(1) = \beta_2,\ n'(1) = -\alpha_2$.

(v) ω *is a positive constant. Moreover,* $p(x)(m'(x)n(x) - m(x)n'(x)) \equiv \omega$.

(vi) $G(x,y)$ *is continuous and symmetrical over* Q.

(vii) $G(x,y)$ *has continuously partial derivative over* $Q_1,\ Q_2$.

(viii) *For each fixed* $y \in I$, $G(x,y)$ *satisfies* $LG(x,y) = 0$ *for* $x \neq y,\ x \in I$. *Moreover,* $R_1(G) = R_2(G) = 0$ *for* $y \in (0,1)$.

(viiii) G'_x *has discontinuous point of the first kind at* $x = y$ *and*

$$G'_x(y+0,y) - G'_x(y-0,y) = -\frac{1}{p(y)}, \qquad y \in (0,1).$$

Consider the linear Sturm-Liouvile problem

$$-(Lu)(x) = \lambda u(x), \qquad R_1(u) = R_2(u) = 0.$$

By the Sturm-Liouvile theory of ordinary differential equations (see, for example, [4], [11]), we know that there exists an eigenfunction $\phi_1(x)$ with respect to the first eigenvalue $\lambda_1 > 0$ such that $\phi_1(x) > 0$ for $x \in (0,1)$.

Following from Lemma 2.2, it is easy to see that

$$\min\left\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\right\} \frac{m(y)n(y)}{\omega} \le G(x,y) \le G(y,y) = \frac{m(y)n(y)}{\omega}, \quad (x,y) \in [0,1]\times[0,1]. \tag{2.2}$$

Let E be a Banach space and $K \subset E$ be a closed convex cone in E. For $r > 0$, let $K_r = \{u \in K : ||u|| < r\}$ and $\partial K_r = \{u \in K : ||u|| = r\}$. The following three Lemmas are needed in our argument, which can be found in [4].

Lemma 2.3. *Let* $\Phi : K \to K$ *be a continuous and completely continuous mapping and* $\Phi u \neq u$ *for* $u \in \partial K_r$. *Then the following conclusions hold:*

(i) If $||u|| \le \|\Phi u\|$ for $u \in \partial K_r$, then $i(\Phi, K_r, K) = 0$;

(ii) If $\|u\| \ge \|\Phi u\|$ for $u \in \partial K_r$, then $i(\Phi, K_r, K) = 1$.

Lemma 2.4. *Let* $\Phi : K \to K$ *be a continuous and completely continuous mapping with* $\mu\Phi u \neq u$ *for every* $u \in \partial K_r$ *and* $0 < \mu \le 1$. *Then* $i(\Phi, K_r, K) = 1$.

Lemma 2.5. *Let* $\Phi : K \to K$ *be a continuous and completely continuous mapping. Suppose that the following two conditions are satisfied:*

(i) $\inf_{u\in\partial K_r} ||\Phi u|| > 0$; (ii) $\mu\Phi u \neq u$ for every $u \in \partial K_r$ and $\mu \ge 1$.

Then, $i(\Phi, K_r, K) = 0$.

In applications below, we take $E = \mathbb{C}(I, \mathbb{R})$ and define

$$K = \{u \in \mathbb{C}(I,\mathbb{R}) : u(x) \ge \min\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\}\|u\|, x \in I\}.$$

One may readily verify that K is a cone in E.

Define an operator $\Phi : K \to K$ by

$$(\Phi u)(x) = \int_0^1 G(x,y)g(y,u(y))dy + \sum_{k=1}^{m} G(x,x_k)I_k(u(x_k)), \ x \in I.$$

Lemma 2.6. $\Phi(K) \subset K$. *Moreover,* $\Phi : K \to K$ *is continuous and completely continuous.*

Proof It is easy to see that $\Phi : K \to K$ is continuous and completely continuous. Thus we only need to show $\Phi(K) \subset K$.

In fact, for $u \in K$, by using inequalities (2.2), we have that

$$\|\Phi u\| \le \int_0^1 G(y,y)g(y,u(y))ds + \sum_{k=1}^{m} G(x_k,x_k)I_k(u(x_k))$$

and

$$\begin{aligned}(\Phi u)(x) &\ge \min\left\{\frac{m(x)}{m(1)},\frac{n(x)}{n(0)}\right\}\int_0^1 G(y,y)g(y,u(y))dy \\ &\quad + \min\left\{\frac{m(x)}{m(1)},\frac{n(x)}{n(0)}\right\}\sum_{k=1}^{m} G(x_k,x_k)I_k(u(x_k)) \\ &\ge \min\{\frac{m(x)}{m(1)},\frac{n(x)}{n(0)}\}\|\Phi u\|,\ x\in[0,1].\end{aligned} \tag{2.3}$$

Thus, $\Phi(K) \subset K$. □

Lemma 2.7. *If u is a fixed point of the operator Φ, then u is a solution of problem* (1.1).

3. MAIN RESULTS

Lemma 3.1. *If* (H_3) *is satisfied, then* $i(\Phi, K_p\ , K) = 1$.

Proof Let $u \in K$ with $\|u\| = p$. It follows from (H_3) that

$$\begin{aligned}\|\Phi u\| &\le \int_0^1 G(y,y)g(y,u(y))dy + \sum_{k=1}^{m} G(x_k,x_k)I_k(u(x_k)) \\ &\le p[\eta\int_0^1 G(y,y)dy + \sum_{k=1}^{m} G(x_k,x_k)\eta_k] < p = \|u\|.\end{aligned}$$

Thus

$$\|\Phi u\| < \|u\|, \quad \forall\ u \in \partial K_p.$$

It is obvious that $\Phi u \ne u$ for $u \in \partial K_p$. Therefore, $i(\Phi, K_p, K) = 1$, here we use Lemma 2.3. □

Lemma 3.2. *If* (H_4) *is satisfied, then* $i(\Phi, K_p, K) = 0$.

Proof Let $u \in K$ with $\|u\| = p$, then

$$u(x) \ge \min\{\frac{m(x)}{m(1)},\frac{n(x)}{n(0)}\}\|u\| \ge \min_{x\in[x_1,x_m]}\min\{\frac{m(x)}{m(1)},\frac{n(x)}{n(0)}\}\|u\| = \sigma p,\ x\in[x_1,x_m].$$

It follows from (H_4) that

$$\begin{aligned}(\Phi u)(\tfrac{1}{2}) &\geq \textstyle\int_{x_1}^{x_m} G(\tfrac{1}{2}, y)g(y, u(y))dy + \sum\limits_{k=1}^{m} G(\tfrac{1}{2}, x_k)I_k(u(x_k)) \\ &\geq p[\lambda \textstyle\int_{x_1}^{x_m} G(\tfrac{1}{2}, y)dy + \sum\limits_{k=1}^{m} \lambda_k G(\tfrac{1}{2}, x_k)] \\ &> p = \|u\|.\end{aligned}$$

Therefore,

$$\|\Phi u\| > \|u\|, \quad \forall\ u \in \partial K_p.$$

Clearly $\Phi u \neq u$ for $u \in \partial K_p$. So, $i(\Phi, K_p, K) = 0$, here we use Lemma 2.3. □

Theorem 3.3. *Assume that* (H_1) *and* (H_3) *are satisfied. Then problem* (1.1) *has at least two positive solutions* u_1 *and* u_2 *with*

$$0 < ||u_1|| < p < ||u_2||.$$

Proof According to Lemma 3.1, we have that

$$i(\Phi, K_p, K) = 1. \tag{3.1}$$

Since (H_1) holds, then there exists $0 < \varepsilon < 1$ such that

$$(1-\varepsilon)[g_0 + \frac{\sigma \sum\limits_{k=1}^{m} I_0(k)\phi_1(x_k)}{\int_0^1 \phi_1(x)dx}] > \lambda_1,\ (1-\varepsilon)[g_\infty + \frac{\sigma \sum\limits_{k=1}^{m} I_\infty(k)\phi_1(x_k)}{\int_0^1 \phi_1(x)dx}] > \lambda_1. \tag{3.2}$$

By the definitions of $g_0,\ I_0$, one can find $0 < r_0 < p$ such that

$$g(x, u) \geq g_0(1-\varepsilon)u,\ I_k(u) \geq I_0(k)(1-\varepsilon)u,\ \forall\ x \in [0, 1],\ 0 < u < r_0.$$

Let $r \in (0, r_0)$, then for $u \in \partial K_r$, $x \in [x_1, x_m]$, we have

$$u(x) \geq \min_{x\in[x_1,x_m]} \min\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\}\|u\| = \sigma r.$$

Thus

$$\begin{aligned}(\Phi u)(\frac{1}{2}) =& \int_0^1 G(\frac{1}{2}, y)g(y, u(y))dy + \sum_{k=1}^{m} G(\frac{1}{2}, x_k)I_k(u(x_k)) \\ \geq& \int_{x_1}^{x_m} G(\frac{1}{2}, y)g(y, u(y))dy + \sum_{k=1}^{m} G(\frac{1}{2}, x_k)I_k(u(x_k)) \\ \geq& g_0(1-\varepsilon)\int_{x_1}^{x_m} G(\frac{1}{2}, y)u(y)dy + (1-\varepsilon)\sum_{k=1}^{m} G(\frac{1}{2}, x_k)I_0(k)u(x_k) \\ \geq& (1-\varepsilon)\sigma r[g_0 \int_{x_1}^{x_m} G(\frac{1}{2}, y)dy + \sum_{k=1}^{m} G(\frac{1}{2}, x_k)I_0(k)],\end{aligned}$$

from which we see that $\inf\limits_{u\in\partial K_r} ||\Phi u|| > 0$, namely, hypothesis (i) of Lemma 2.5 holds.

Next we show that $\mu\Phi u \neq u$ for any $u \in \partial K_r$ and $\mu \geq 1$.

If this is not true, then there exist $u_0 \in \partial K_r$ and $\mu_0 \geq 1$ such that $\mu_0 \Phi u_0 = u_0$. Note that $u_0(x)$ satisfies

$$(3.3)\qquad \begin{cases} Lu_0(x) + \mu_0 g(x, u_0(x)) = 0, \quad x \in I', \\ -\Delta(pu_0')|_{x=x_k} = \mu_0 I_k(u_0(x_k)), \quad k = 1, 2, \cdots, m, \\ \alpha_1 u_0(0) + \beta_1 u_0'(0) = 0 \\ \alpha_2 u_0(1) + \beta_2 u_0'(1) = 0. \end{cases}$$

Multiply equation (3.3) by $\phi_1(x)$ and integrate from 0 to 1, note that

$$\begin{aligned}
&\int_0^1 \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx = \int_0^{x_1} \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx \\
&\quad + \sum_{k=1}^{m-1} \int_{x_k}^{x_{k+1}} \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx \\
&\quad + \int_{x_m}^1 \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx \\
&= \phi_1(x_1)p(x_1)u_0'(x_1 - 0) - \phi_1(0)p(0)u_0'(0) - \int_0^{x_1} p(x)u_0'(x)\phi_1'(x)dx \\
&\quad + \int_0^{x_1} q(x)u_0(x)\phi_1(x)dx + \sum_{k=1}^{m-1}[\phi_1(x_{k+1})p(x_{k+1})u_0'(x_{k+1} - 0) \\
&\quad - \phi_1(x_k)p(x_k)u_0'(x_k + 0) - \int_{x_k}^{x_{k+1}} p(x)u_0'(x)\phi_1'(x)dx \\
&\quad + \int_{x_k}^{x_{k+1}} q(x)u_0(x)\phi_1(x)dx] + \phi_1(1)p(1)u_0'(1) - \phi_1(x_m)p(x_m)u_0'(x_m + 0) \\
&\quad - \int_{x_m}^1 p(x)u_0'(x)\phi_1'(x)dx + \int_{x_m}^1 q(x)u_0(x)\phi_1(x)dx \\
&= -\sum_{k=1}^{m} \Delta(p(x_k)u_0'(x_k))\phi_1(x_k) - \int_0^1 p(x)\phi_1'(x)u_0'(x)dx + \int_0^1 q(x)\phi_1(x)u_0(x)dx \\
&\quad + \phi_1(1)p(1)u_0'(1) - \phi_1(0)p(0)u_0'(0).
\end{aligned}$$

Also note that

$$\begin{aligned}
\int_0^1 p(x)\phi_1'(x)u_0'(x)dx &= \int_0^1 p(x)\phi_1'(x)du_0(x) \\
&= p(1)\phi_1'(1)u_0(1) - p(0)\phi_1'(0)u_0(0) - \int_0^1 u_0(x)(p(x)\phi_1'(x))'dx \\
&= p(1)\phi_1'(1)u_0(1) - p(0)\phi_1'(0)u_0(0) + \int_0^1 u_0(x)q(x)\phi_1(x)dx \\
&\quad + \lambda_1 \int_0^1 u_0(x)\phi_1(x)dx.
\end{aligned}$$

Thus, by the boundary conditions, we have

$$\begin{aligned}\int_0^1 \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx &= -\sum_{k=1}^m \Delta(p(x_k)u_0'(x_k))\phi_1(x_k)\\ &\quad - p(1)\phi_1'(1)u_0(1) + p(0)\phi_1'(0)u_0(0)\\ &\quad - \int_0^1 u_0(x)q(x)\phi_1(x)dx - \lambda_1\int_0^1 u_0(x)\phi_1(x)dx\\ &\quad + \int_0^1 q(x)\phi_1(x)u_0(x)dx + \phi_1(1)p(1)u_0'(1) - \phi_1(0)p(0)u_0'(0)\\ &= -\sum_{k=1}^m \Delta(p(x_k)u_0'(x_k))\phi_1(x_k) - \lambda_1\int_0^1 u_0(x)\phi_1(x)dx\\ &= \sum_{k=1}^m \mu_0 I_k(u_0(x_k))\phi_1(x_k) - \lambda_1\int_0^1 u_0(x)\phi_1(x)dx.\end{aligned}$$

So we obtain

$$\begin{aligned}\lambda_1\int_0^1 u_0(x)\phi_1(x)dx &= \mu_0\sum_{k=1}^m I_k(u_0(x_k))\phi_1(x_k) + \mu_0\int_0^1 \phi_1(x)g(x,u_0(x))dx\\ &\geq (1-\varepsilon)\sum_{k=1}^m I_0(k)\phi_1(x_k)u_0(x_k) + (1-\varepsilon)g_0\int_0^1 \phi_1(x)u_0(x)dx.\end{aligned}$$

Since $u_0(x) \geq \min\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\}||u_0|| \geq \min\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\}r$, we have $\int_0^1 \phi_1(x)u_0(x)dx > 0$, and so from the above inequality we see that $\lambda_1 \geq (1-\varepsilon)g_0$. If $\lambda_1 = (1-\varepsilon)g_0$, then $I_0(k) = 0, k = 1, 2, \ldots, m$. But from (3.2) we have $(1-\varepsilon)g_0 > \lambda_1$, which is a contradiction. So $\lambda_1 > (1-\varepsilon)g_0$. Thus

$$\begin{aligned}[\lambda_1 - (1-\varepsilon)g_0]\int_0^1 u_0(x)\phi_1(x)dx &\geq (1-\varepsilon)\sum_{k=1}^m I_0(k)\phi_1(x_k)u(x_k)\\ &\geq (1-\varepsilon)\sigma r\sum_{k=1}^m I_0(k)\phi_1(x_k).\end{aligned}$$

Since$\int_0^1 u_0(x)\phi_1(x)dx \leq r\int_0^1 \phi_1(x)dx$, we have

$$[\lambda_1 - (1-\varepsilon)g_0]\int_0^1 \phi_1(x)dx \geq (1-\varepsilon)\sigma\sum_{k=1}^m I_0(k)\phi_1(x_k),$$

which contradicts (3.2) again. Hence Φ satisfies the hypotheses of Lemma 2.5 in K_r. Thus

$$i(\Phi, K_r, K) = 0. \tag{3.4}$$

On the other hand, from (H_1), there exists $H > p$ such that

$$g(x,u) \geq g_\infty(1-\varepsilon)u,\ I_k(u) \geq I_\infty(k)(1-\varepsilon)u, \forall\ x \in [0,1],\ u \geq H. \tag{3.5}$$

Let $C = \max\limits_{0\le u\le H}\max\limits_{0\le x\le 1}|g(x,u)-g_\infty(1-\varepsilon)u| + \sum\limits_{k=1}^{m}\max\limits_{0\le u\le H}|I_k(u)-I_\infty(k)(1-\varepsilon)u|$. It is clear that

$$g(x,u)\ge g_\infty(1-\varepsilon)u - C,\quad I_k(u)\ge I_\infty(k)(1-\varepsilon)u - C,\ \forall\, x\in[0,1],\ u\ge 0. \tag{3.6}$$

Choose $R > R_0 := \max\{\frac{H}{\sigma}, p\}$ and let $u\in\partial K_R$. Since $u(x)\ge\sigma||u|| = \sigma R > H$ for $x\in[x_1,x_m]$, from (3.5) we see that

$$g(x,u(x))\ge g_\infty(1-\varepsilon)u(x)\ge\sigma g_\infty(1-\varepsilon)R,\ \forall\, x\in[x_1,x_m].$$

$$I_k(u(x_k)\ge\sigma I_\infty(k)(1-\varepsilon)R.$$

Essentially the same reasoning as above yields $\inf\limits_{u\in\partial K_R}||\Phi u|| > 0$. Next we show that if R is large enough, then $\mu\Phi u\ne u$ for any $u\in\partial K_R$ and $\mu\ge 1$. In fact, if there exist $u_0\in\partial K_R$ and $\mu_0\ge 1$ such that $\mu_0\Phi u_0 = u_0$, then $u_0(x)$ satisfies equation (3.3).

Multiply equation (3.3) by $\phi_1(x)$ and integrate from 0 to 1, using integration by parts in the left side to obtain

$$\begin{aligned}\lambda_1\int_0^1 u_0(x)\phi_1(x)dx &= \mu_0\sum_{k=1}^{m} I_k(u_0(x_k))\phi_1(x_k) + \mu_0\int_0^1 g(x,u_0(x))\phi_1(x)dx\\ &\ge (1-\varepsilon)\sum_{k=1}^{m} I_\infty(k)\phi_1(x_k)u_0(x_k) + (1-\varepsilon)g_\infty\int_0^1 u_0(x)\phi_1(x)dx\\ &-\quad C(\sum_{k=1}^{m}\phi_1(x_k) + \int_0^1\phi_1(x)dx).\end{aligned}$$

If $g_\infty\le\lambda_1$, then we have

$$\begin{aligned}[\lambda_1-(1-\varepsilon)g_\infty]\int_0^1 u_0(x)\phi_1(x)dx + C\left(\sum_{k=1}^{m}\phi_1(x_k)+\int_0^1\phi_1(x)dx\right)&\\ \ge (1-\varepsilon)\sum_{k=1}^{m} I_\infty(k)\phi_1(x_k)u_0(x_k),&\end{aligned}$$

thus

$$\begin{aligned}\|u_0\|[\lambda_1-(1-\varepsilon)g_\infty]\int_0^1\phi_1(x)dx + C\left(\sum_{k=1}^{m}\phi_1(x_k)+\int_0^1\phi_1(x)dx\right)&\\ \ge (1-\varepsilon)\sigma\|u_0\|\sum_{k=1}^{m} I_\infty(k)\phi_1(x_k)&\end{aligned}$$

and

$$\|u_0\|\le\frac{C(\sum\limits_{k=1}^{m}\phi_1(x_k)+\int_0^1\phi_1(x)dx)}{(1-\varepsilon)\sigma\sum\limits_{k=1}^{m} I_\infty(k)\phi_1(x_k) - [\lambda_1-(1-\varepsilon)g_\infty]\int_0^1\phi_1(x)dx} =: \bar{R}. \tag{3.7$_a$}$$

If $g_\infty > \lambda_1$, we can choose $\varepsilon > 0$ such that $(1-\varepsilon)g_\infty > \lambda_1$, then we have

$$\begin{aligned} C\left(\sum_{k=1}^{m}\phi_1(x_k) + \int_0^1 \phi_1(x)dx\right) &\geq [(1-\varepsilon)g_\infty - \lambda_1]\int_0^1 \phi_1(x)u_0(x)dx \\ &\geq [(1-\varepsilon)g_\infty - \lambda_1]\|u_0\|\int_0^1 \left(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)}\right)\phi_1(x)dx. \end{aligned}$$

Thus

$$(3.7_b) \qquad \|u_0\| \leq \frac{C\left(\sum\limits_{k=1}^{m}\phi_1(x_k) + \int_0^1 \phi_1(x)dx\right)}{[(1-\varepsilon)g_\infty - \lambda_1]\int_0^1 (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\phi_1(x)dx} =: \bar{R}.$$

Let $R > \max\{p, \bar{R}\}$, then for any $u \in \partial K_R$ and $\mu \geq 1$, we have $\mu\Phi u \neq u$. Hence hypothesis (ii) of Lemma 2.5 is satisfied and

$$(3.8) \qquad i(\Phi, K_R\ , K) = 0.$$

In view of (3.1), (3.4) and (3.8), we obtain

$$i(\Phi, K_R \setminus \bar{K}_p, K) = -1,\ i(\Phi, K_p \setminus \bar{K}_r, K) = 1.$$

Then Φ has fixed points u_1 and u_2 in $K_p \setminus \bar{K}_r$ and $K_R \setminus \bar{K}_p$, respectively, which means $u_1(x)$ and $u_2(x)$ are positive solution of the problem (1.1) and $0 < \|u_1\| < p < \|u_2\|$. □

Corollary 3.4. *The conclusion of Theorem* 3.3 *is valid if* (H_1) *is replaced by*

(H_1^*) $g_0 = \infty$ or $\sum\limits_{k=1}^{m} I_0(k)\phi_1(x_k) = \infty;$ $\qquad g_\infty = \infty$ or $\sum\limits_{k=1}^{m} I_\infty(k)\phi_1(x_k) = \infty.$

Theorem 3.5. *Assume that* (H_2) *and* (H_4) *are satisfied, then problem* (1.1) *has at least two positive solutions* u_1 *and* u_2 *with*

$$0 < ||u_1|| < p < ||u_2||.$$

Proof According to Lemma 3.2, we have that

$$(3.9) \qquad i(\Phi, K_p, K) = 0.$$

Since (H_2) holds, there exists $0 < \varepsilon < \min\{\lambda_1 - g^0, \lambda_1 - g^\infty\}$ such that

$$(3.10) \qquad (\lambda_1 - \varepsilon - g^0)\int_0^1 \left(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)}\right)\phi_1(x)dx > \sum_{k=1}^{m}(I^0(k)+\varepsilon)\phi_1(x_k),$$

and

$$(3.11) \qquad (\lambda_1 - \varepsilon - g^\infty)\int_0^1 \left(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)}\right)\phi_1(x))dx > \sum_{k=1}^{m}(I^\infty(k)+\varepsilon)\phi_1(x_k).$$

One can find $0 < r_0 < p$ such that

$$(3.12) \qquad g(x,u) \leq (g^0+\varepsilon)u, \quad I_k(u) \leq (I^0(k)+\varepsilon)u,\ \forall\ x \in [0,1],\ 0 \leq u \leq r_0.$$

Let $r \in (0, r_0)$. Now we prove that $\mu\Phi u \neq u$ for any $x \in \partial K_r$ and $0 < \mu \leq 1$. If this is not true, then there exist $u_0 \in \partial K_r$ and $0 < \mu_0 \leq 1$ such that $\mu_0 \Phi u_0 = u_0$. Then $u_0(x)$ satisfies equation (3.3). Multiply equation (3.3) by $\phi_1(x)$ and integrate from 0 to 1, using (3.12), to obtain

$$\begin{aligned}\lambda_1 \int_0^1 u_0(x)\phi_1(x)dx &= \mu_0 \sum_{k=1}^m I_k(u_0(x_k))\phi_1(x_k) + \mu_0 \int_0^1 \phi_1(x) g(x, u_0(x))dx \\ &\leq \sum_{k=1}^m (I^0(k)+\varepsilon) u_0(x_k)\phi_1(x_k) + \int_0^1 \phi_1(x) u_0(x) dx (g^0+\varepsilon),\end{aligned}$$

i.e.

$$(\lambda_1 - g^0 - \varepsilon)\int_0^1 u_0(x)\phi_1(x)dx \leq \sum_{k=1}^m (I^0(k)+\varepsilon)u_0(x_k)\phi_1(x_k) \leq r\sum_{k=1}^m (I^0(k)+\varepsilon)\phi_1(x_k).$$

Since $u_0(x) \geq \min\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\}||u_0|| \geq \left(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)}\right) r$, and so from the above inequality we see that

$$(\lambda_1 - g^0 - \varepsilon)\int_0^1 \left(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)}\right)\phi_1(x)dx \leq \sum_{k=1}^m (I^0(k)+\varepsilon)\phi_1(x_k),$$

which is a contradiction. By Lemma 2.4, we have

$$i(\Phi, K_r, K) = 1. \tag{3.13}$$

On the other hand, from (H_2), there exist $H > p$ such that

$$g(x,u) \leq (g^\infty + \varepsilon)u,\ I_k(u) \leq (I^\infty(k)+\varepsilon)u\ \forall\ x \in [0,1],\ u \geq H.$$

Let $C = \max\limits_{0\leq u\leq H}\max\limits_{0\leq x\leq 1}|g(x,u) - (g^\infty+\varepsilon)u| + \sum\limits_{k=1}^m \max\limits_{0\leq u\leq H}|I_k(u) - (I^\infty(k)+\varepsilon)u|$. It is clear that

$$g(x,u) \leq (g^\infty+\varepsilon)u + C, \quad I_k(u) \leq (I^\infty(k)+\varepsilon)u + C,\ \forall\ x\in[0,1],\ u \geq 0. \tag{3.14}$$

Next we show that if R is large enough, then $\mu\Phi u \neq u$ for any $u \in \partial K_R$ and $0 < \mu \leq 1$. In fact, if there exist $u_0 \in \partial K_R$ and $0 < \mu_0 \leq 1$ such that $\mu_0\Phi u_0 = u_0$, then $u_0(x)$ satisfies equation (3.3). Multiply equation (3.3) by $\phi_1(x)$ and integrate from 0 to 1, using (3.14), to obtain

$$\begin{aligned}\lambda_1 \int_0^1 u_0(x)\phi_1(x)dx &= \mu_0 \sum_{k=1}^m I_k(u_0(x_k))\phi_1(x_k) + \mu_0\int_0^1 g(x,u_0(x))\phi_1(x)dx \\ &\leq \sum_{k=1}^m (I^\infty(k)+\varepsilon)\phi_1(x_k)u_0(x_k) + \int_0^1 \phi_1(x)u_0(x)dx(g^\infty+\varepsilon) \\ &+ \quad C(\sum_{k=1}^m \phi_1(x_k) + \int_0^1 \phi_1(x)dx),\end{aligned}$$

i.e.,

$$(3.15)\quad \begin{aligned}(\lambda_1 - g^\infty - \varepsilon)\textstyle\int_0^1 u_0(x)\phi_1(x)dx &\le \sum_{k=1}^m (I^\infty(k)+\varepsilon)\phi_1(x_k)u_0(x_k)\\ &\quad + C(\sum_{k=1}^m \phi_1(x_k) + \textstyle\int_0^1 \phi_1(x)dx)\\ &\le \|u_0\| \sum_{k=1}^m (I^\infty(k)+\varepsilon)\phi_1(x_k)\\ &\quad + C(\sum_{k=1}^m \phi_1(x_k) + \textstyle\int_0^1 \phi_1(x)dx).\end{aligned}$$

Also we have $\int_0^1 u_0(x)\phi_1(x)dx \ge \|u_0\| \int_0^1 (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\phi_1(x)dx$, and this together with (3.15) yields

$$\|u_0\| \le \frac{C(\sum_{k=1}^m \phi_1(x_k) + \int_0^1 \phi_1(x)dx)}{(\lambda_1 - g^\infty - \varepsilon)\int_0^1 (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\phi_1(x)dx - \sum_{k=1}^m (I^\infty(k)+\varepsilon)\phi_1(x_k)} =: \bar{R}.$$

Let $R = \max\{p, \bar{R}\}$, then for any $x \in \partial K_R$ and $0 < \mu \le 1$, we have $\mu\Phi u \ne u$. Thus

$$(3.16)\quad i(\Phi, K_R\ , K) = 1.$$

In view of (3.9), (3.13) and (3.16), we obtain

$$i(\Phi, K_R \setminus \bar{K}_p, K) = 1,\ i(\Phi, K_p \setminus \bar{K}_r, K) = -1.$$

Then Φ has fixed points u_1 and u_2 in $K_p \setminus \bar{K}_r$ and $K_R \setminus \bar{K}_p$, respectively, which means $u_1(x)$ and $u_2(x)$ are positive solution of problem (1.1) and $0 < \|u_1\| < p < \|u_2\|$. □

Corollary 3.6. *The conclusion of Theorem* 3.5 *is valid if* (H_2) *is replaced by*

(H_2^*) $g^0 = 0$ *and* $I^0(k) = 0$, $k = 1, 2, \ldots, m$; $g^\infty = 0$ *and* $I^\infty(k) = 0$, $k = 1, 2, \ldots, m$.

The proof of the following two Theorems follows the ideas in the proof of Theorems 3.3 and 3.5. Here we omit it here.

Theorem 3.7. *Assume the following condition is satisfied:*

$$g_0 + \frac{\sigma \sum_{k=1}^m I_0(k)\phi_1(x_k)}{\int_0^1 \phi_1(x)dx} > \lambda_1, \quad g^\infty + \frac{\sum_{k=1}^m I^\infty(k)\phi_1(x_k)}{\int_0^1 (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\phi_1(x)dx} < \lambda_1.$$

Then (1.1) *has at least one positive solution.*

Corollary 3.8. *Assume the following condition is satisfied:*

$$g_0 = \infty \ \textit{or} \ \sum_{k=1}^m I_0(k)\phi_1(x_k) = \infty, \quad g^\infty = 0 \ \textit{and} \ I^\infty(k) = 0,\ k = 1, \ldots, m$$

Then (1.1) *has at least one positive solution.*

Theorem 3.9. *Assume the following condition is satisfied:*

$$g^0 + \frac{\sum\limits_{k=1}^{m} I^0(k)\phi_1(x_k)}{\int_0^1 (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\phi_1(x)dx} < \lambda_1, \quad g_\infty + \frac{\sigma \sum\limits_{k=1}^{m} I_\infty(k)\phi_1(x_k)}{\int_0^1 \phi_1(x)dx} > \lambda_1.$$

Then (1.1) *has at least one positive solution.*

Corollary 3.10. *Assume that*

$$g^0 = 0 \textit{ and } I^0(k) = 0,\ k = 1, \ldots, m; \quad g_\infty = \infty \textit{ or } \sum_{k=1}^{m} I^\infty(k)\phi_1(x_k) = \infty.$$

Then (1.1) *has at least one positive solution.*

Example 3.11. Consider the following impulsive boundary value problem

$$\begin{cases} Lu + Au^\alpha + Bu^\beta = 0,\ x \in I',\ 0 < \alpha < 1 < \beta,\ A > 0,\ B > 0, \\ -\Delta(pu')|_{x=x_k} = c_k u(x_k),\ c_k \geq 0, \\ R_1(u) = \alpha_1 u(0) + \beta_1 u'(0) = 0, \\ R_2(u) = \alpha_2 u(1) + \beta_2 u'(1) = 0, \end{cases} \tag{3.17}$$

here $Lu = (p(x)u')' + q(x)u$. Assume that (S_1) is satisfied. Then problem (3.17) has at least two positive solutions u_1 and u_2 with

$$0 < ||u_1|| < 1 < ||u_2||$$

provided

$$A + B < \frac{1}{d}(1 - \sum_{k=1}^{m} G(x_k, x_k)c_k),\ d = \int_0^1 G(y, y)dy. \tag{3.18}$$

Proof To see this we will apply Theorem 3.3 (or Corollary 3.4).

By (3.18), $\eta > 0$ is chosen such that

$$A + B < \eta < \frac{1}{d}(1 - \sum_{k=1}^{m} G(x_k, x_k)c_k).$$

Set

$$g(x, u) = Au^\alpha + Bu^\beta.$$

Note

$$g_0 = \infty,\ g_\infty = \infty,$$

so (H_1) (or (H_1^*)) holds.

Let $\eta_k = c_k$, then $\eta,\ \eta_k$ satisfy

$$\eta \int_0^1 G(y, y)dy + \sum_{k=1}^{m} G(x_k, x_k)\eta_k < 1.$$

Let $p = 1$, then for $0 \leq u \leq p$, we have

$$g(x, u) = Au^\alpha + Bu^\beta \leq A + B < \eta p = \eta,$$

and

$$I_k(u) = c_k u = \eta_k u \leq \eta_k p,$$

thus (H_3) holds. The result now follows from Theorem 3.3 (or Corollary 3.4). □

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