# INVARIANT MANIFOLD REDUCTION FOR STOCHASTIC DYNAMICAL SYSTEMS

AIJUN DU AND JINQIAO DUAN

Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL
60616, USA   (duan@iit.edu)

**ABSTRACT.** Invariant manifolds facilitate the understanding of nonlinear stochastic dynamics. When an invariant manifold is represented approximately by a graph for example, the whole stochastic dynamical system may be reduced or restricted to this manifold. This reduced system may provide valuable dynamical information for the original system. The authors have derived an invariant manifold reduction or restriction principle for systems of Stratonovich or Ito stochastic differential equations.

Two concepts of invariance are considered for invariant manifolds.

The first invariance concept is in the framework of cocycles — an invariant manifold being a random set. The dynamical reduction is achieved by investigating random center manifolds.

The second invariance concept is in the sense of almost sure — an invariant manifold being a deterministic set which is not necessarily attracting. The restriction of the original stochastic system on this deterministic local invariant manifold is still a stochastic system but with reduced dimension.

**Key Words:** Stochastic differential equations; invariant manifolds; random center manifold reduction; almost sure invariance; method of characteristics.

**AMS (MOS Subject Classifications:** 34F05, 34C45, 37H10, 60H10.

## 1. INTRODUCTION

Invariant manifolds provide geometric structures that describe dynamical behavior of nonlinear systems. Dynamical reductions to attracting invariant manifolds or dynamical restrictions to other (not necessarily attracting) invariant manifolds are often sought to gain understanding of nonlinear dynamics.

There have been recent works on invariant manifolds for stochastic or random ordinary differential equations by Carverhill [9], Wanner [33], Arnold [3], Boxler [5, 6], and Mohammed [22], among others. These authors use the (sample-wise) cocycle property for the solution operator of the stochastic differential equations, the Oseledets' multiplicative ergodic theorem [3], and a less-physical but technically convenient random norm, to prove the existence of invariant manifolds. The construction of a random norm needs the knowledge of Oseledets spaces (a kind of eigenspace in

random linear algebra) as well as Lyapunov exponents, both are hardly ever available; see [3], p. 191 and p.379. Random norms are not realistic in this sense, and thus representations of invariant manifolds and the dynamical reductions are difficult to achieve when random norms are used.

Earlier approaches on deriving dynamical reductions on stochastic center-like manifolds by series expansions are considered by Knoblock and Wiesenfeld [18], Schoner and Haken [30], and Xu and Roberts [36].

For stochastic dynamical systems, there are various concepts for invariance in the definition of invariant manifolds [34]. In the framework of cocycles [3], the suitable concept for invariance of a random set is that each orbit starting inside it stays inside it sample-wise, modulo the change of sample due to noise. Another concept is almost sure invariance of a deterministic set under stochastic dynamics, i.e., each orbit starting inside it stays inside it almost surely.

In this paper, we consider invariant manifold reductions or restrictions for Stratonovich and Ito stochastic differential equations in Euclidean spaces.

We first study the system of Stratonovich stochastic differential equations in $\mathbb{R}^n$:

$$(1) \qquad dX = [AX + F^\epsilon(X)]dt + B(X) \circ dW(t), \quad X(0) = x_0,$$

where $X = X(t, \omega)$ is the unknown variable; $A$ is a $n \times n$ matrix with $k$ eigenvalues of zero real parts and $n - k$ eigenvalues of negative real parts; $F^\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ are nonlinear vector and matrix functions (with $\epsilon > 0$ a small parameter), respectively; and $W(t)$ is a standard vector Brownian motion (or Wiener process) taking values in $\mathbb{R}^n$. Moreover, $\circ$ denotes the stochastic differential in the sense of Stratonovich.

Then we consider the following stochastic system defined by Ito stochastic differential equations in $\mathbb{R}^n$:

$$(2) \qquad dX = F(X)dt + B(X)dW(t), \quad X(0) = x_0,$$

where again $F$ and $B$ are vector and matrix functions in $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$, respectively. And $W(t)$ are standard vector Brownian motion in $\mathbb{R}^n$.

Note that the Stratonovich stochastic differential $B(X) \circ dW(t)$ and Ito stochastic differential $B(X)dW(t)$ are interpreted through their corresponding definitions of stochastic integrals [24]:

$$\int_0^T B(X) \circ dW(t) := \text{mean-square} \lim_{\Delta t_j \to 0} \sum_j B(X(\frac{t_{j+1} + t_j}{2}))(W_{t_{j+1}} - W_{t_j}),$$

$$\int_0^T B(X)dW(t) := \text{mean-square} \lim_{\Delta t_j \to 0} \sum_j B(X(t_j))(W_{t_{j+1}} - W_{t_j}).$$

Note the difference in the sums: In Stratonovich integral, the integrand is evaluated at the midpoint $\frac{t_{j+1}+t_j}{2}$ of a subinterval $(t_j, t_{j+1})$, while for Ito integral, the integrand is evaluated at the left end point $t_j$. See [24] for the discussion about the difference in physical modeling by these two kinds of stochastic differential equations. There are also dynamical differences for these two type of stochastic equations, even at linear level [8].

In this paper, we derive an invariant manifold reduction or restriction principle for the above systems of stochastic differential equations.

For the Stratonovich stochastic system (1), we consider random invariant center manifolds. The dynamical reduction is achieved by investigating asymptotic behavior of random center manifolds.

For the Ito stochastic system (2), we study deterministic almost sure invariant manifolds, which are not necessarily attracting. We reformulate the local invariance condition as invariance equations, i.e., first order partial differential equations, and then solve these equations by the method of characteristics. Although the local invariant manifold is deterministic, the restriction of the original stochastic system on this deterministic local invariant manifold is still a stochastic system but with reduced dimension.

This paper is organized as follows: In Section 2, we recall some basic concepts for stochastic dynamical systems. We consider random center manifold reduction in Section 3. Finally, in Section 4, we construct deterministic invariant manifolds by investigating first order partial differential equations via the method of characteristics, and thus obtain dynamical restrictions for stochastic dynamical systems.

## 2. STOCHASTIC DYNAMICAL SYSTEMS

In this section we introduce some definitions in stochastic dynamical systems, as well as recall some usual notations in probability.

We consider stochastic systems in the state space $\mathbb{R}^n$, with the usual metric or distance $d(x, y) = \sqrt{\sum_{j=1}^n (x_j - y_j)^2}$, norm or length $\|x\| = \sqrt{\sum_{j=1}^n x_j^2}$, and the usual scalar product $< x, y > = \sum_{j=1}^n x_j y_j$. All invariant manifolds and their sample versions are in this state space.

Some stochastic processes, such as a Brownian motion, can be described by a canonical (deterministic) dynamical system (see [3], Appendix A). A standard Brownian motion (or Wiener process) $W(t)$ in $\mathbb{R}^n$, with two-sided time $t \in \mathbb{R}$, is a stochastic process with $W(0) = 0$ and stationary independent increments satisfying $W(t) - W(s) \sim \mathcal{N}(0, |t - s|I)$. Here $I$ is the $n \times n$ identity matrix. The Brownian motion can be realized in a canonical sample space of continuous paths passing the

origin at time 0

$$\Omega = C_0(\mathbb{R}, \mathbb{R}^n) := \{\omega \in C(\mathbb{R}, \mathbb{R}^n) : \omega(0) = 0\}.$$

The convergence concept in this sample space is the uniform convergence on bounded and closed time intervals, induced by the following metric

$$\rho(\omega, \omega') := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\omega - \omega'\|_n}{1 + \|\omega - \omega'\|_n}, \text{ where } \|\omega - \omega'\|_n := \sup_{-n \le t \le n} \|\omega(t) - \omega'(t)\|.$$

With this metric, we can define events represented by open balls in $\Omega$. For example, a ball centered at zero with radius 1 is $\{\omega : \rho(\omega, 0) < 1\}$. We define the Borel $\sigma$-algebra $\mathcal{F}$ as the collection of events represented by open balls $A$'s, complements of open balls, $A^c$'s, unions and intersections of $A$'s and/or $A^c$'s, together with the empty event, the whole event (the sample space $\Omega$), and all events formed by doing the complements, unions and intersections forever in this collection.

Taking the (incomplete) Borel $\sigma$-algebra $\mathcal{F}$ on $\Omega$, together with the corresponding Wiener measure $\mathbb{P}$, we obtain the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$, also called the Wiener space. This is similar to the game of gambling with a dice, where the canonical sample space is $\Omega_{dice} = \{1, 2, 3, 4, 5, 6\}$. Moreover, $\mathbb{E}$ denotes the mathematical expectation with respect to probability $\mathbb{P}$.

The canonical *driving* dynamical system describing the Brownian motion is defined as

$$\theta(t) : \Omega \to \Omega, \quad \theta(t)\omega(s) := \omega(t + s) - \omega(t), \ s, t \in \mathbb{R}.$$

Then $\theta(t)$, also denoted as $\theta_t$, is a homeomorphism for each $t$ and $(t, \omega) \rightarrowtail \theta(t)\omega$ is continuous, hence measurable. The Wiener measure $\mathbb{P}$ is invariant and ergodic under this so-called Wiener shift $\theta_t$. In summary, $\theta_t$ satisfies the following properties.

- $\theta_0 = id$,
- $\theta_t \theta_s = \theta_{t+s}$, for all $s, t \in \mathbb{R}$,
- the map $(t, \omega) \mapsto \theta_t \omega$ is measurable and $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

We now introduce an important concept. A filtration is an increasing family of information accumulations, called $\sigma$-algebras, $\mathcal{F}_t$. For each $t$, $\sigma$-algebra $\mathcal{F}_t$ is a collection of events in sample space $\Omega$. One might observe the Wiener process $W_t$ over time $t$ and use $\mathcal{F}_t$ to represent the information accumulated up to and including time $t$. More formally, on $(\Omega, \mathcal{F})$, a filtration is a family of $\sigma$-algebras $\mathcal{F}_s : 0 \le s \le t$ with $\mathcal{F}_s$ contained in $\mathcal{F}$ for each $s$, and $\mathcal{F}_s \subset \mathcal{F}_\tau$ for $s \le \tau$. It is also useful to think $\mathcal{F}_t$ as the $\sigma$-algebra generated by infinite union of $\mathcal{F}_s$'s, which is contained in $\mathcal{F}_t$. So a filtration is often used to represent the change in the set of events that can be measured, through gain or loss of information.

For understanding stochastic differential equations from a dynamical point of view, the natural filtration is defined as a two-parameter family of $\sigma$-algebras generated by increments

$$\mathcal{F}_s^t := \sigma(\omega(\tau_1) - \omega(\tau_2) : s \le \tau_1, \tau_2 \le t), \quad s, t \in \mathbb{R}.$$

This represents the information accumulated from time $s$ up to and including time $t$. This two-parameter filtration allows us to define forward as well as backward stochastic integrals, and thus we can solve a stochastic differential equation from an initial time forward as well as backward in time [3].

The solution operator for the stochastic system (1) or (2) with initial condition $x(0) = x_0$ is denoted as $\varphi(t, \omega, x_0)$.

The dynamics of the system on the state space $\mathbb{R}^n$, over the driving flow $\theta_t$ is described by a cocycle. A cocycle $\varphi$ is a mapping:

$$\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$$

which is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n), \mathcal{F})$-measurable such that

$$\varphi(0, \omega, x) = x \in \mathbb{R}^n,$$
$$\varphi(t_1 + t_2, \omega, x) = \varphi(t_2, \theta_{t_1}\omega, \varphi(t_1, \omega, x)),$$

for $t_1$, $t_2 \in \mathbb{R}$, $\omega \in \Omega$, and $x \in \mathbb{R}^n$. Then $\varphi$, together with the driving dynamical system, is called a *random dynamical system*. Sometimes we also use $\varphi(t, \omega)$ to denote this system.

Under very general conditions, the stochastic differential systems (1) and (2) each generates a random dynamical system in $\mathbb{R}^n$; see [3, 17].

We recall some concepts in dynamical systems. A *manifold M* is a set, which locally looks like an Euclidean space. Namely, a "patch" of the manifold $M$ looks like a "patch" in $\mathbb{R}^n$. For example, curves, torus and spheres in $\mathbb{R}^3$ are one- and two-dimensional differentiable manifolds, respectively. However, a manifold arising from the study of invariant sets for dynamical systems in $\mathbb{R}^n$, can be very complicated. So we give a formal definition of manifolds. For more discussions on differentiable manifolds, see [1, 25].

**Definition 1** (Differentiable manifold and Lipschitz manifold). An $n$-dimensional differentiable manifold $M$, is a connected metric space with an open covering $\{U_\alpha\}$, i.e, $M = \bigcup_\alpha U_\alpha$, such that

(i) for all $\alpha$ , $U_\alpha$ is homeomorphic to the open unit ball in $\mathbb{R}^n$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$, i.e., for all $\alpha$ there exists a homeomorphism of $U_\alpha$ onto B, $h_\alpha : U_\alpha \to B$, and

(ii) if $U_\alpha \cap U_\beta \neq \varnothing$ and $h_\alpha : U_\alpha \to B$, $h_\beta : U_\beta \to B$ are homeomorphisms, then $h_\alpha(U_\alpha \cap U_\beta)$ and $h_\beta(U_\alpha \cap U_\beta)$ are subsets of $\mathbb{R}^n$ and the map

$$\tag{3} h = h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$$

is differentiable, and for all $x \in h_\beta(U_\alpha \cap U_\beta)$, the Jacobian determinant $\det Dh(x) \neq 0$.

If the map (3) is only Lispchitz continuous, then we call $M$ an n-dimensional Lispchitz continuous manifold.

Recall that a homeomorphism of A to B is a continuous one-to-one map of A onto B, $h : A \to B$, such that $h^{-1} : B \to A$ is continuous.

Just as invariant sets are important building blocks for deterministic dynamical systems, invariant sets are basic geometric objects to help understand stochastic dynamics [3]. Here we present two different concepts about invariant sets for stochastic systems: random invariant sets and almost sure invariant sets.

**Definition 2** (Random set). A collection $M = M(\omega)_{\omega \in \Omega}$, of nonempty closed sets $M(\omega)$, $\omega \in \Omega$, contained in $\mathbb{R}^n$, is called a random set if

$$\omega \mapsto \inf_{y \in M(\omega)} d(x, y)$$

is a random variable for any $x \in \mathbb{R}^n$.

**Definition 3** (Tempered absorbing set). A random set $B(\omega)$ is called an tempered absorbing set of $\varphi$ if for any bounded set $K \subset \mathbb{R}^n$ there exists $t_K(\omega)$ such that $\forall t \geq t_K(\omega)$

$$\varphi(t, \theta_{-t}\omega, K) \subset B(\omega).$$

and for all $\varepsilon > 0$

$$\lim_{t \to \infty} e^{-\varepsilon t} d(B(\theta_{-t}\omega)) = 0, \quad a.e. \ \omega \in \Omega,$$

where $d(B) = \sup_{x \in B} d(x, 0)$, with $0 \in \mathbb{R}^n$, is the diameter of $B$.

**Definition 4** (Random invariant set). A random set $M(\omega)$ is called an invariant set for a random dynamical system $\varphi$ if

$$\varphi(t, \omega, M(\omega)) \subset M(\theta_t \omega), \quad t \in \mathbb{R} \ \text{and} \ \omega \in \Omega.$$

**Definition 5** (Random invariant manifold). If a random invariant set $M$ can be represented by a graph of a Lipschitz mapping

$$\gamma^*(\omega, \cdot) : \ H^+ \to H^-, \ \text{with direct sum decomposition} \quad H^+ \oplus H^- = \mathbb{R}^n$$

such that

$$M(\omega) = \{x^+ + \gamma^*(\omega, x^+), x^+ \in H^+\},$$

then $M$ is called a Lipschitz continuous invariant manifold.

We will also consider deterministic invariant sets or manifolds, while the invariance is in the sense of almost-sure (a.s.) [4, 13, 10, 31, 37].

**Definition 6** (Almost sure invariant set)**.** A (deterministic) set $M$ in $\mathbb{R}^n$ is called locally almost surely invariant for (2), if for all $(t_0, x_0) \in \mathbb{R} \times M$, there exists a continuous local weak solution $X^{(t_0, x_0)}$ with lifetime $\tau = \tau(t_0, x_0)$, such that

$$X_{t \wedge \tau}^{(t_0, x_0)} \in M, \ \ \forall t > t_0, \ \ a.s. \ \omega \in \Omega,$$

where $t \wedge \tau = \min(t, \tau)$.

## 3. RANDOM CENTER MANIFOLD REDUCTION

In this section we study an $n-$dimensional system of Stratonovich stochastic differential equations in $\mathbb{R}^n$:

$$(4) \qquad dX = [AX + F^\epsilon(X)]dt + B(X) \circ dW(t), \ \ X(0) = x_0,$$

where $A$ is a $n \times n$ real matrix with $k$ eigenvalues of zero real parts and $n - k$ eigenvalues of negative real parts ($k < n$). Without loss of generality, we assume that the matrix $A$ is in Jordan form (which can be achieved by an invertible linear coordinate transformation in $\mathbb{R}^n$) and that the first $k$ eigenvalues have zero real parts.

Moreover, nonlinear function $F^\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ and nonlinear matrix mapping $B : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ are Lipschitz continuous with Lipschitz constants $L_F^\epsilon$ and $L_B$, respectively. We assume that $F^\epsilon(0) = B(0) = 0$. Here $\epsilon$ is a small parameter so that $F^\epsilon$ can be seen as a small perturbation, that is, we have $L_F^\epsilon \to 0$ as $\epsilon \to 0$. The state space $\mathbb{R}^n$ is the direct sum of the center space $\mathbb{R}_c^n$ (i.e., eigenspace spanned by eigenvectors or generalized eigenvectors corresponding to eigenvalues with zero real parts) and its orthogonal complement $\mathbb{R}_s^n$:

$$\mathbb{R}^n = \mathbb{R}_c^n \oplus \mathbb{R}_s^n,$$

where dimensions dim $\mathbb{R}_c^n = k$ and dim $\mathbb{R}_s^n = n - k$.

Let $X_c$ be the projection of $X \in \mathbb{R}^n$ to the center space $\mathbb{R}_c^n$. Hence every point $X$ in $\mathbb{R}^n$ may be uniquely decomposed as a sum of a vector $X_c$, in the center space $\mathbb{R}_c^n$ and another vector $X_s$, in the orthogonal complement $\mathbb{R}_s^n$. Namely,

$$X = X_c + X_s.$$

Let $F_c^\epsilon(\cdot)$ and $B_c(\cdot)$ be the projections of $F^\epsilon(\cdot)$ and $B(\cdot)$ into the center space $\mathbb{R}_c^n$, respectively.

Note that later on we will truncate the nonlinearity so that it has global Lipschitz constant. For stochastic systems, truncation may not be always appropriate, although sometimes it works fine, such as in considering nonlinear dynamical behavior near fixed points [7]. We have the following result about dynamically reducing an $n$-dimensional stochastic system to a lower $k$-dimensional stochastic system.

**Theorem 1** (Random center manifold reduction). *Given the above assumptions for the Stratonovich stochastic differential equation (4) in $\mathbb{R}^n$. If further assume that the equation (4) generates a dissipative random dynamical system (e.g., having a random absorbing set). Then for sufficiently small $\epsilon$, the long time behavior of (4) can be described by the following $k$-dimensional stochastic system with $k < n$:*

$$(5) \qquad dX_c(t) = AX_c(t) + F_c^\epsilon(X_c(t))dt + B_c(X_c(t)) \circ dW_c(t)$$

*provided (5) is structurally stable. In this reduced lower dimensional stochastic system, $F_c^\epsilon(X_c(t))$ and $B_c(X_c(t))$ denote $F_c^\epsilon(\cdot)$ and $B_c(\cdot)$ evaluated at $X_c$, respectively. Moreover, $W_c(t)$ is the projection of $W(t)$ into the $k$-dimensional center space $\mathbb{R}_c^n$.*

**Remark 1.** We say the long time dynamics of the stochastic equation (4) is described by the stochastic equation (5) if both systems have the same limit sets (and possibly also share some other invariant sets).

**Remark 2.** The random dynamical system $\varphi(t, \omega)$ generated by (5) is called structurally stable, if for any small perturbation (small in the sense of the usual metric in the space of continuous functions) to $F^\epsilon(x)$ and $B(x)$, the perturbed random dynamical system $\Phi(t, \omega)$ is topologically equivalent to $\varphi(t, \omega)$. Namely, there exists a random homeomorphism $h(\omega)$ so that $\Phi(t, \omega) \circ h(\omega) = h(\theta_t \omega) \circ \varphi(t, \omega)$.

**Proof:** The proof for this theorem can be obtained by modifying the proof in [11, 12] or [32] from the infinite dimensional case to the present finite dimensional system (4) in the state space $\mathbb{R}^n$. Here we only highlight some main points.

The first step is to decompose or project the original system (4) into two subsystems, one in the center space $\mathbb{R}_c^n$ and another one in its orthogonal complement $\mathbb{R}_s^n$. The subsystem in the center space $\mathbb{R}_c^n$ is what we would like to keep or retain, while the subsystem in $\mathbb{R}_s^n$ is what we would like to reduce or eliminate.

The second step is to show the existence of a $k$-dimensional (local) random invariant (center) manifold $M(\omega)$. This is achieved by showing that this manifold is represented by a Lipschitz graph, where variable $X_s \in \mathbb{R}_s^n$ can be represented by variable $X_c \in \mathbb{R}_c^n$ (so that we can eliminate $X_s$ inside this random (center) manifold $M(\omega)$). Thus we obtain a reduced lower $k$-dimensional stochastic system on variable $X_c$, which is actually the system (5).

In the third step, we show that the reduced lower $k$-dimensional stochastic system (5) captures the long time dynamics of the original system (4) when $\epsilon$ is sufficiently small. To this end, we use an argument based on cone invariance and asymptotic completeness [26, 27, 19].

This completes the sketch of the proof.

Let us look at an example.

**Example 1.** Consider a system of stochastic differential equations in $\mathbb{R}^2$:

$$dx \quad = \quad -xdt + (xy^2 - x^3 - \frac{1}{2}x)dt + x \circ dW_1(t),$$

$$dy \quad = \quad 0ydt + (-2 + x^2y - y^3 - \frac{1}{2}y)dt + y \circ dW_2(t),$$

where $W_1$ and $W_2$ are independent scalar Brownian motions.

Let $u = (x, y)^T$. Then

$$du = (Au + \tilde{F}(u))dt + Bu \circ dW(t),$$

with

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{F}(u) = \begin{pmatrix} xy^2 - x^3 - \frac{1}{2}x \\ -2 + x^2y - y^3 - \frac{1}{2}y \end{pmatrix}$$

and

$$Bu \circ dW(t) = \begin{pmatrix} x \circ dW_1(t) \\ y \circ dW_2(t) \end{pmatrix}.$$

In order to apply Ito's formula, we rewrite this system in the equivalent Ito's stochastic differential equations (see [24], page 36):

$$dx \quad = \quad -xdt + (xy^2 - x^3)dt + xdW_1(t),$$

$$dy \quad = \quad 0ydt + (-2 + x^2y - y^3)dt + ydW_2(t),$$

where $W_1$ and $W_2$ are independent scalar Brownian motions. Let $u = (x, y)^T$, then

$$du = (Au + F(u))dt + Bu \, dW(t)$$

with

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad F(u) = \begin{pmatrix} xy^2 - x^3 \\ -2 + x^2y - y^3 \end{pmatrix} \quad \text{and} \quad Bu = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Recall the standard scalar product $< u_1, u_2 >= x_1x_2 + y_1y_2$ and norm $\|u\| = \sqrt{x^2 + y^2}$ in $\mathbb{R}^2$. Then, we apply Ito's formula (see [24], page 48) to obtain "energy" estimate

$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\|u\|^2 \quad = \quad \mathbb{E}\langle u, du \rangle + \mathbb{E}\frac{1}{2}\langle du, du \rangle$$

$$= \quad \mathbb{E}\langle u, du \rangle + \mathbb{E}\frac{1}{2}\langle Bu \, dW(t), Bu \, dW(t) \rangle$$

$$= \quad \mathbb{E}\langle u, A(u) + F(u) \rangle + \frac{1}{2}\mathbb{E} \, Trace[Bu \cdot (Bu)^T]$$

$$= \quad -x^2 + x^2y^2 - x^4 - 2y + x^2y^2 - y^4 + \frac{1}{2}(x^2 + y^2)$$

$$= \quad -\frac{1}{2}(x^2 + y^2) + (2x^2y^2 - x^4 - y^4) - y + y^2 - 1 + 1$$

$$= \quad -\frac{1}{2}(x^2 + y^2) - (x^2 - y^2)^2 + (y - 1)^2 - 1$$

$$\leq \quad -\frac{1}{2}E\|u\|^2,$$

if $y$ is near the equilibrium point $(0,0)$ (so that $0 < y < 1$). Note that here $\mathbb{E}$ denotes the expectation with respect to probability $\mathbb{P}$. This estimate will be used to conclude dissipativity for the (truncated) system.

The nonlinear terms $(xy^2 - x^3)$ and $(-2 + x^2y - y^3)$ can be truncated within a disk centered at $(0,0)$ with radius $0 < \epsilon < 1$ (making them zero outside the disk). The truncated nonlinear terms satisfy desired Lipschitz conditions. And, the above "energy" estimate implies the dissipative property for the truncated system.

By Theorem 1, near the equilibrium point $(0,0)$ (i.e., taking $\epsilon$ is small enough), the original two-dimensional system is asymptotically reduced to a one-dimensional stochastic dynamical system

$$dy = (-2 - y^3)dt + y dW_2(t).$$

## 4. INVARIANT MANIFOLD RESTRICTION

Now we consider the stochastic system (2) defined by Ito stochastic differential equations in $\mathbb{R}^n$:

$$(6) \qquad dX = F(X)dt + B(X)dW(t), \quad X(0) = x_0,$$

where $F$ and $B$ are vector and matrix functions in $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$, respectively. We also assume that $F(\cdot) \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ and $B(\cdot) \in C^1(\mathbb{R}^n; \mathbb{R}^{n \times n})$.

We are going to derive representations of invariant finite dimensional manifolds in terms of $A, F$ and $B$, by using the tangency conditions for a deterministic $C^2$ manifold $M$ in $\mathbb{R}^n$:

$$(7) \qquad \mu(\omega, x) := F(\omega, x) \; - \; \frac{1}{2} \sum_j [DB^j(\omega, x)]B^j(\omega, x) \in T_x M,$$

$$(8) \qquad B^j(\omega, x) \; \in \; T_x M, \; j = 1, \cdots, n,$$

where $D$ represents Jacobian operator and $B_j$ is the $j-$th column of the matrix $B$. The above tangency conditions are shown to be equivalent to almost sure local invariance of manifold $M$; see Filipovic ([13]) and related works [4, 21, 37, 10, 2].

The almost sure invariance conditions (7)-(8) for manifold $M$ mean that the $n+1$ vectors, $\mu$ and $B^j$, $j = 1, \cdots, n$, are tangent vectors to $M$. Namely, these $n+1$ vectors are orthogonal to the normal vectors of manifold $M$.

In other words, if the normal vector for $M$ at $x$ is $N(x)$, then the almost sure invariance conditions (7)-(8) become the following *invariance equations* for manifold $M$: For all $x \in M$,

$$(9) \qquad\qquad < \mu(x), N(x) > \; = \; 0,$$

$$(10) \qquad\qquad < B^j(x), N(x) > \; = \; 0, \; j = 1, \cdots, n,$$

where, as before, $< \cdot, \cdot >$ denotes the usual scalar product in $\mathbb{R}^n$.

Invariant manifolds are usually represented as graphs of some functions in $\mathbb{R}^n$. By investigating the above invariance equations (9)-(10), we may be able to find some local invariant manifolds $M$ for the stochastic system (6).

The goal for this section is to present a method to find some of these local invariant manifolds. Although the following result and example are stated for a codimension 1 local invariant manifold, the idea extends to other lower dimensional local invariant manifolds, as long as the normal vectors $N(x)$ (or tangent vectors) may be represented; see tangency conditions (9)-(10) above and (12)-(13) below.

**Theorem 2** (Local invariant manifold restriction). *Let the local invariant manifold $M$ for the stochastic dynamical system (6) be represented as a graph defined by the algebraic equation*

$$(11) \qquad\qquad M: \quad G(x_1, \cdots, x_n) = 0.$$

*Then $G$ satisfies a system of first order (deterministic) partial differential equations and the local invariant manifold $M$ may be found by solving these partial differential equations by the method of characteristics. By restricting the original dynamical system (6) on this local invariant manifold $M$, we obtain a locally valid, reduced lower dimensional system.*

In fact, the normal vector to this graph or surface is, in terms of partial derivatives, $\nabla G(x) = (G_{x_1}, \cdots, G_{x_n})$. Thus the invariance equations (9)-(10) are now

$$(12) \qquad\qquad < \mu(x), \nabla G(x) > \;=\; 0,$$

$$(13) \qquad\qquad < B^j(x), \nabla G(x) > \;=\; 0, \;\; j = 1, \cdots, n,$$

This is a system of first order partial differential equations in $G$. We apply the method of characteristics to solve for $G$, and therefore obtain the invariant manifold $M$, represented by a graph in state space $\mathbb{R}^n$: $G(x_1, \cdots, x_n) = 0$.

In the rest of this section, we first recall the method of characteristics, and then work out an example of finding a local invariant manifold and reduced system.

**Method of Characteristics**: Consider a first order partial differential equation for the unknown scalar function $u$ of n variables $x_1, \cdots, x_n$

$$(14) \qquad\qquad \sum_{j=1}^{n} a_i(x_1, \cdots, x_n) u_{x_i} = c(x_1, \cdots, x_n),$$

with continuous coefficients $a_i$'s and $c$.

Note that the solution surface $u = u(x_1, \ldots, x_n, t)$ in $x_1 \cdots x_n u-$space has normal vectors $N := (u_{x_1}, \cdots, u_{x_n}, -1)$. This partial differential equation implies that the vector $V =: (a_1, \cdots, a_n, c)$ is perpendicular to this normal vector and hence must lie in the tangent plane to the graph of $z = u(x_1, \cdots, x_n)$.

In other words, $(a_1, \cdots, a_n, c)$ defines a vector field in $\mathbb{R}^n$, to which graphs of the solutions must be tangent at each point [23]. Surfaces that are tangent at each point to a vector field in $\mathbb{R}^n$ are called **integral surfaces** of the vector field. Thus to find a solution of equation (14), we should try to find integral surfaces.

How can we construct integral surfaces? We can try using the characteristics curves that are the integral curves of the vector field. That is, $X = (x_1(t), \cdots, x_n(t))$ is a **characteristic** if it satisfies the following system of ordinary differential equations:

$$\frac{dx_1}{dt} = a_1(x_1, \cdots, x_n),$$
$$\cdots$$
$$\frac{dx_n}{dt} = a_n(x_1, \ldots, x_n),$$
$$\frac{du}{dt} = c(x_1, \ldots, x_n).$$

A smooth union of characteristic curves is an integral surface. There may be many integral surfaces. Usually an integral surface is determined by requiring it to contain (or pass through) a given initial curve or an $n-1$ dimensional manifold $\Gamma$:

$$x_i = f_i(s_1, \ldots, s_{n-1}), i = 1, \ldots, n$$
$$u = h(s_1, \ldots, s_{n-1})$$

This generates an $n$-dimensional integral manifold $M$ parameterized by $(s_1, \ldots, s_{n-1}, t)$. The solution $u(x_1, \cdots, x_n)$ is obtained by solving for $(s_1, \ldots, s_{n-1}, t)$ in terms of variables $(x_1, \cdots, x_n)$.

**Remark**: If initial data $\Gamma$ is non-characteristic, i.e., it is nowhere tangent to the vector field $V = (a_1, \cdots, a_n, c)$, and $a_1, \cdots, a_n, c$ are $C^1$ (and thus locally Lipschitz continuous), then there exists a unique integral surface $u = u(x_1, \cdots, x_n)$ containing $\Gamma$, defined at least locally near $\Gamma$.

Now applying the above method of characteristics to (12)-(13), we obtain a solution $G = G(x_1, \cdots, x_n)$. However, the local invariant manifold $M$ that we look for is represented by the equation

$$G(x_1, \cdots, x_n) = 0.$$

Therefore, a skill is needed to make sure that the solution $G = G(x_1, \cdots, x_n)$ actually penetrates the plane $G = 0$ in the $x_1 \cdots x_n G-$space; see Fig. 1. This needs to be achieved by selecting appropriate initial data $\Gamma$. The invariant manifold $M$ we thus obtain is defined at least locally near the initial data $\Gamma$.

We illustrate the method for finding local invariant manifold and the corresponding reduced system by an example.
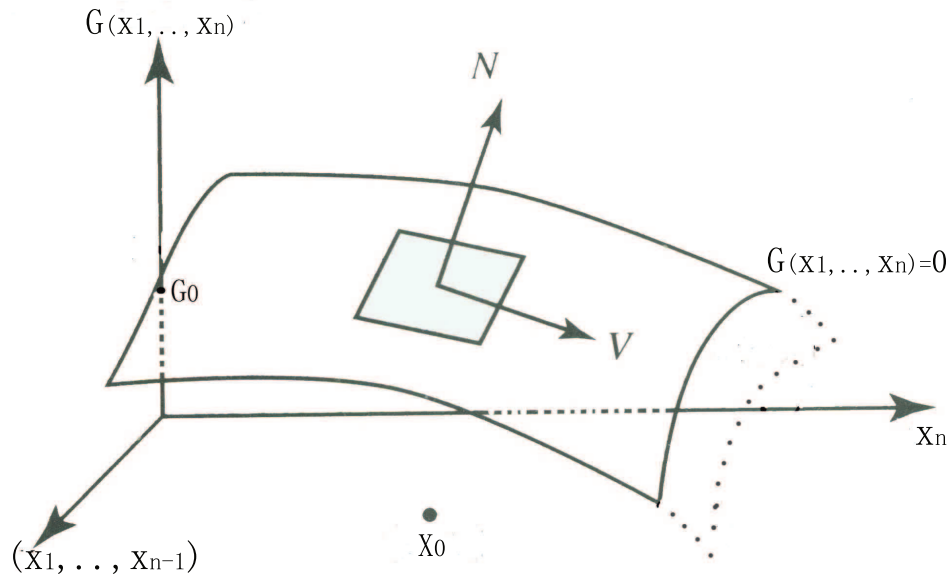
FIGURE 1. Local invariant manifold $M$ is represented by the equation $G(x_1, \cdots, x_n) = 0$ in the $x_1 \cdots x_n-$ space. Namely, $M$ is the intersection of the surface $G = G(x_1, \cdots, x_n)$ with the plane $G = 0$ in $x_1 \cdots x_n G-$space. Here $G(x_1, \cdots, x_n)$ is the solution of (12)-(13) via the method of characteristics. Note that $N = (u_{x_1}, \cdots, u_{x_n}, -1)$ and $V = (a_1, \cdots, a_n, c)$.

**Example 2.**

$$\frac{dx}{dt} = x + x \, \dot{W}_1 + x \, \dot{W}_2,$$

$$\frac{dy}{dt} = 3x + 2y + (x + y) \, \dot{W}_1 + (x + y) \, \dot{W}_2$$

where $W_t^1$ and $W_t^2$ are independent scalar Brownian motions.

We look for a local invariant manifold $M \subset \mathbb{R}^2$. For this illustrative example, the associated tangency conditions (7) and (8) coincide and thus becomes a single *invariance condition*:

(15)                          $(x, x + y)^T \in T_x M$

We represent the invariant manifold $M$ by $G(x, y) = 0$. This surface has normal vector $(G_x, G_y)$. By noticing that normal vector is orthogonal to the tangent surface $T_x M$, we see that the above single invariance condition (15) becomes a single invariance equation:

$$xG_x + (x + y)G_y = 0.$$

We solve this first order partial differential equation with initial curve $\Gamma$ parameterized as $(f(s), g(s), h(s))$. The characteristic equations are

$$\frac{dx}{dt} = x,$$
$$\frac{dy}{dt} = x + y,$$
$$\frac{dG}{dt} = 0.$$

We solve these equations and invoke the initial conditions to find that

$$x = f(s)e^t,$$
$$y = (f(s)t + g(s))e^t,$$
$$G = h(s).$$

This is the general solution with respect to the general initial condition $(x_0(s), y_0(s), G_0(s)) := (f(s), g(s), h(s))$. By solving for $t, s$ in terms of $x, y$, we obtain $G = G(x, y)$.

We illustrate this by a specific choice of initial curve $(f(s), g(s), h(s))$. Note that, in order to obtain a local invariant manifold $G(x, y) = 0$, we also need to pick initial curve so that $G$ actually takes both positive and negative values, and thus the invariant manifold $G(x, y) = 0$ is defined on some set in the $xy-$plane. In other words, the continuous function $G(x, y)$ satisfies $\max\{G\} * \min\{G\} \leq 0$ locally.

For example, taking $\Gamma : (x_0(s), y_0(s), G_0(s)) = (1, s, s)$, we then have

$$x = e^t,$$
$$y = (t + s)e^t,$$
$$G = s.$$

Thus $s = \frac{y}{x} - \ln(x)$ and $G(x, y) = \frac{y}{x} - \ln(x)$. Thus an invariant manifold $M$ is $G(x, y) = 0$, i.e.,

$$\frac{y}{x} - \ln(x) = 0.$$

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