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RECIPROCITY PRINCIPLE FOR EVEN-ORDER DYNAMIC EQUATIONS WITH MIXED DERIVATIVES

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ABSTRACT. We establish the equivalence of oscillatory behavior of a certain pair of even-order time scale dynamic equations with mixed derivatives. For the special time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$, this statement is usually referred to as the *reciprocity principle.*

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1. INTRODUCTION

To motivate our research, consider the even-order two-term differential equation

$$(-1)^n \left(r(t)y^{(n)}\right)^{(n)} = q(t)y \tag{1}$$

with positive continuous functions r, q. This equation frequently appears in the spectral theory of one-term differential operator $Ly := \frac{(-1)^n}{q(t)} \left(r(t)y^{(n)}\right)^{(n)}$, see e.g. [15]. If y is a solution of (1), then $z := ry^{(n)}$ is a solution of the so-called *reciprocal equation*

$$(-1)^n \left(\frac{1}{q(t)} z^{(n)}\right)^{(n)} = \frac{1}{r(t)} z. \tag{2}$$

If $n = 1$, i.e., we consider the second order Sturm-Liouville equations, it follows immediately from the Rolle mean value theorem that a solution y of (1) oscillates if and only if its quasiderivative $z = ry'$ (a solution of (2)) oscillates. Hence, if $n = 1$, the original equation (1) is oscillatory if and only if the reciprocal equation (2) is oscillatory.

A natural question is whether the same statement holds also for higher order equations. This problem was solved in the affirmative way in [1] using the relationship of even order self-adjoint differential equations to linear Hamiltonian systems. More precisely, it was proved in [1] that the linear Hamiltonian system

$$x' = A(t)x + B(t)u, \quad u' = -C(t)x - A^T(t)u \tag{3}$$

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with nonnegative definite symmetric matrices B, C, is nonoscillatory if and only if the *reciprocal system*

$$y' = -A^T(t)y + C(t)z, \quad z' = -B(t)y + A(t)z, \tag{4}$$

which results from (3) upon the transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix},$$

is also nonoscillatory. By an easy computation one can find that equation (1) can be written in the form (3) with the diagonal matrices (see [19])

$$B(t) = \operatorname{diag}\left\{0, \dots, 0, \frac{1}{r(t)}\right\}, \quad C(t) = \operatorname{diag}\{q(t), 0, \dots, 0\}.$$

and that the last entry z_n of the vector z in (4) is a solution of the reciprocal equation (2). The above mentioned equivalence of oscillatory behavior of (3) and (4) (the so-called *reciprocity principle* for Hamiltonian systems) then implies the same statement about oscillation of (1) and (2).

The discrete version of the reciprocity principle (see [6, 11]) concerns the linear Hamiltonian difference system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = -C_k x_{k+1} - A_k^T u_k \tag{5}$$

and when applied to the even-order two-term equations, it says that the equation

$$(-1)^n \Delta^n \left(r_k \Delta^n y_k\right) = q_k y_{k+n}$$

with positive sequences r, q, is nonoscillatory if and only if the equation

$$(-1)^n \Delta^n \left(\frac{1}{q_k} \Delta^n y_k\right) = \frac{1}{r_{k+n}} y_{k+n}$$

is nonoscillatory.

The theory of dynamic equations on time scales is the unifying theory for differential, difference, quantum equations, and all other differential equations and systems defined over nonempty closed subsets of the real line. Recently, several papers appeared where higher order equations containing both delta and nabla derivatives are investigated (see, e.g., [4, 14] and the references given therein). The idea to consider equations with mixed derivatives enabled to deal successfully with problems like self-adjointness, symmetry, etc., which seemed to be a hard problem for equations and operators containing delta derivatives only.

In this paper we deal with the higher order dynamic equation

$$(-1)^n \widetilde{D}_n^{\nabla} \left(r(t) D_n^{\Delta} y\right) = q(t) y, \tag{6}$$

where

$$D_k^{\Delta} y := \begin{cases} y^{\Delta\nabla\ldots\Delta\nabla} & k \text{ even}, \\ y^{\Delta\nabla\ldots\nabla\Delta} & k \text{ odd}, \end{cases} \qquad \widetilde{D}_k^{\nabla} y := \begin{cases} y^{\Delta\nabla\ldots\Delta\nabla} & k \text{ even}, \\ y^{\nabla\Delta\ldots\Delta\nabla} & k \text{ odd}, \end{cases}$$

where the subscript k indicates the total number of delta and nabla derivatives, and its reciprocal equation

$$(-1)^n D_n^{\Delta}\left(\frac{1}{q(t)}\widetilde{D}_n^{\nabla} z\right) = \frac{1}{r(t)} z, \tag{7}$$

where r, q are positive rd-continuous functions. We use the recently established fact that these equations can be written in the form of the so-called symplectic dynamic system whose oscillation theory is relatively well developed, see [12], to prove that (6) is nonoscillatory if and only if (7) is nonoscillatory.

The paper is organized as follows. In the next section we recall some concepts of the time scale theory and essentials of the oscillation theory of symplectic dynamic systems. Section 3 contains the main result of the paper, the reciprocity principle for (6) and (7), and the last section is devoted to remarks and comments concerning the results of the paper.

2. **AUXILIARY RESULTS**

We suppose that the reader is familiar with the elements of the time scale calculus. Nevertheless, we recall some basic concepts of this theory. The operators σ, ρ, μ, ν are the right and left jump operators and forward and backward graininess, respectively. A point t is called an rd-point (ld-point, rs-point, ls-point) if $\mu(t) = t$ ($\rho(t) = t$, $\mu(t) > t$, $\rho(t) < t$) and this point is said to be a *dense* point if it is both rd and ld. We will also use the usual notation $f^{\sigma} := f \circ \sigma$, $f^{\rho} := f \circ \rho$. Delta and nabla derivatives are defined by the formulas

$$\begin{aligned} f^{\Delta}(t) &= \begin{cases} \lim_{s\to t} \frac{f(s)-f(t)}{s-t}, & \text{if } \mu(t) = 0, \\ \frac{f^{\sigma}(t)-f(t)}{\mu(t)} & \text{if } \mu(t) > 0, \end{cases} \\ f^{\nabla}(t) &= \begin{cases} \lim_{s\to t} \frac{f(s)-f(t)}{s-t} & \text{if } \nu(t) = 0, \\ \frac{f(t)-f^{\rho}(t)}{\rho(t)} & \text{if } \nu(t) > 0. \end{cases} \end{aligned}$$

If $f : \mathbb{T} \to \mathbb{R}$ is a Δ-differentiable function for which $\lim_{s\to t-} f^{\Delta}(s)$ exists finite at ld-points, then we have

$$f^{\nabla}(t) = \begin{cases} \lim_{s\to t-} f^{\Delta}(s) & t \text{ is ld- and rs-point}, \\ f^{\Delta}(\rho(t)) & \text{otherwise}. \end{cases} \tag{8}$$

Similarly, if $\lim_{s\to t+} f^\nabla(s)$ exists finite at rd-points, we have

$$f^\Delta(t) = \begin{cases} \lim_{s\to t+} f^\nabla(s) & t \text{ is rd- and ls-point,} \\ f^\nabla(\sigma(t)) & \text{otherwise.} \end{cases} \tag{9}$$

In particular, if f^Δ and f^∇ are continuous, we have $f^\Delta(t) = f^\nabla(\sigma(t))$, $f^\nabla(t) = f^\Delta(\rho(t))$.

A symplectic dynamic system is the first order dynamic system

$$z^\Delta = \mathcal{S}(t)z \tag{10}$$

with $z \in \mathbb{R}^{2n}$, $\mathcal{S} : \mathbb{T} \to \mathbb{R}^{2n\times 2n}$, and this matrix satisfies the identity

$$\mathcal{S}^T(t)\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\mathcal{S}^T(t)\mathcal{J}\mathcal{S}(t) = 0, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{11}$$

I being the $n \times n$ identity matrix. The characteristic property of this system is that its fundamental matrix $Z(t) \in \mathbb{R}^{2n\times 2n}$ is symplectic (i.e., $Z^T(t)\mathcal{J}Z(t) = \mathcal{J}$) whenever it has this property at one point of $\mathbb{T}$. Basic qualitative properties of symplectic systems are summarized in [12], see also [10, Chap. X]. If we write the matrix $\mathcal{S}$ in the form $\mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$ with $n \times n$ matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, then (11) translates as

$$\begin{aligned} &\mathcal{C} - \mathcal{C}^T + \mu(\mathcal{A}^T\mathcal{C} - \mathcal{C}^T\mathcal{A}) = 0, \\ &\mathcal{B}^T - \mathcal{B} + \mu(\mathcal{B}^T\mathcal{D} - \mathcal{D}^T\mathcal{B}) = 0, \\ &\mathcal{A}^T + \mathcal{D} + \mu(\mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B}) = 0. \end{aligned} \tag{12}$$

Identity (11) is equivalent to the identity

$$\mathcal{S}(t)\mathcal{J} + \mathcal{J}\mathcal{S}^T(t) + \mu(t)\mathcal{S}(t)\mathcal{J}\mathcal{S}^T(t) = 0, \tag{13}$$

and this identity in terms of the matrices $\mathcal{A}, \dots, \mathcal{D}$ reads

$$\begin{aligned} &\mathcal{C} - \mathcal{C}^T + \mu(\mathcal{C}\mathcal{D}^T - \mathcal{D}\mathcal{C}^T) = 0, \\ &\mathcal{B}^T - \mathcal{B} + \mu(\mathcal{A}\mathcal{B}^T - \mathcal{B}\mathcal{A}^T) = 0, \\ &\mathcal{A}^T + \mathcal{D} + \mu(\mathcal{D}\mathcal{A}^T - \mathcal{C}\mathcal{B}^T) = 0. \end{aligned} \tag{14}$$

Now we recall some definitions of the oscillation theory of symplectic dynamic systems. A $2n \times n$ matrix solution $\binom{X}{U}$ of (10) is said to be a *conjoined basis* if $X^T(t)U(t) = U^T(t)X(t)$ and $\operatorname{rank} \binom{X(t)}{U(t)} = n$. A conjoined basis $\binom{X}{U}$ has no *focal points* in a time scale interval $(a, b]$ if $X(t)$ is invertible at all dense points $t \in (a, b]$, and

$$\operatorname{Ker} X^\sigma(t) \subseteq \operatorname{Ker} X(t) \quad \text{and} \quad X(t)[X^\sigma(t)]^\dagger\mathcal{B}(t) \geq 0 \tag{15}$$

hold for $t \in [a, \rho(b)]$. Here Ker, †, and $\geq$ denote the kernel, Moore-Penrose generalized inverse and nonnegative definiteness of the matrix indicated. System (10) is called *disconjugate* on $[a,b]$ if the conjoined basis given by the initial condition $X(a)=0$, $U(a)=I$ has no focal point in $(a,b]$. Equivalently, disconjugacy of (10) can be defined using the concept of the generalized zero of a vector solution as follows. A vector solution $z=\binom{x}{u}$ of (10) has a *generalized zero* in the interval $(t,\sigma(t)]$ if one of the following conditions is satisfied: (a) t is dense and $x(t)=0$, (b) t is right-scattered, $x(t)\neq 0$, $x^\sigma(t)\in \operatorname{Im}\mathcal{B}(t)$, and $x^T(t)\mathcal{B}^\dagger(t)x^\sigma(t)\leq 0$, or (c) t is right-dense, $x(t)\neq 0$, and $\mathcal{B}(t)\not\geq 0$. System (10) is disconjugate on $[a,b]$ if and only if $\mathcal{B}(t)\geq 0$ at all right-dense points $t\in(a,b]$, and any solution $z=\binom{x}{u}$ of with $x(a)=0$ satisfies $x(t)\neq 0$ for all dense points $t\in(a,b]$, and

$$x(t)\neq 0 \text{ and } x^\sigma(t)\in \operatorname{Im}\mathcal{B}(t) \text{ imply } x^T(t)\mathcal{B}^\dagger(t)x^\sigma(t)>0 \tag{16}$$

at all right-scattered $t\in[a,\rho(b)]$. In other words, (10) is disconjugate on $[a,b]$ if and only if no solution $z=\binom{x}{u}$ of (10) with $x(a)=0$ has any generalized zero in $(a,b]$. If the time scale under consideration is unbounded from above, system (10) is said to be *nonoscillatory* if there exists $T\in\mathbb{T}$ such that (10) is disconjugate on $[T,\infty)$, in the opposite case system is said to be *oscillatory*. System (10) is said to be *dense-normal* on $[a,b]$ if for any dense point $s\in[a,b]$ the trivial solution $z=\binom{x}{u}\equiv\binom{0}{0}$ is the only solution for which $x(t)\equiv 0$ on $[a,s]$. We say that symplectic dynamic system (10) has the *Hamiltonian structure*, if the matrix $I+\mu\mathcal{A}$ is invertible. In that case, (10) can be written in the form of the Hamiltonian time scale system

$$x^\Delta = A(t)x^\sigma + B(t)u,\quad u^\Delta = -C(t)x^\sigma - A^T(t)u, \tag{17}$$

where

$$A=(I+\mu\mathcal{A})^{-1}\mathcal{A},\quad B=(I+\mu\mathcal{A})^{-1}\mathcal{B},\quad C=-\mathcal{C}(I+\mu\mathcal{A})^{-1}. \tag{18}$$

Now we define oscillatory properties of equations (6) and (7). We say that the interval $(t,\sigma(t)]$ contains the *generalized zero point of multiplicity* n of a solution y of (6) if $D_k^\Delta y(t)=0$, $k=0,\dots,n-1$ if t is right dense and, if $\sigma(t)>t$, $D_k^\Delta y(\sigma(t))=0$, $k=0,\dots,n-2$, and $D_{n-1}^\Delta y(t)D_{n-1}^\Delta y(\sigma(t))\leq 0$. This definition is in agreement with the definition given in [8], only higher order delta derivatives appearing in that paper are replaced by mixed derivatives contained in the operators D_k^Δ and the position of actual zeros is to be shifted if $\sigma^2(t):=\sigma\circ\sigma(t)>\sigma(t)$. Concerning the definition of the generalized zero of multiplicity n of a solution of (7), we have to distinguish between even and odd n, since for n odd, the first derivative applied to z is the nabla derivative, while for n even it is the delta derivative. Therefore, the definition is the same for n even and for n odd the operators D_k^Δ are to be replaced by the operators D_k^∇, where

$$D_k^\nabla y := \begin{cases} y^{\nabla\Delta\dots\nabla\Delta} & k \text{ even},\\ y^{\nabla\Delta\dots\Delta\nabla} & k \text{ odd},\end{cases}$$

the lower index k again indicated the number of derivatives.

We finish this preparatory section with the reciprocity principle for linear Hamiltonian time scale systems (17) as it is proved in [16]. In this statement, system (17) and its below given reciprocal system (19) (both these systems are special cases of symplectic dynamic system (10)) are said to be *eventually identically normal* if there exists $T \in \mathbb{T}$ such that this system is strongly normal on $[T, s]$ for every dense $s > T$, and, when there is no dense point in (T, ∞), the following condition holds: there exists $l \in \mathbb{N}$ such that for any $t_1 > T$, if $x^{\sigma^k}(t_1) = 0$ for all $k = 0, \dots, l$, then $(x(t), u(t)) \equiv 0$ on (t_1, ∞). Here $\sigma^k := \sigma \circ \cdots \circ \sigma$ is the k-times iterated σ operator and $\sigma^0(t) = t$. Eventual identical normality of the below given reciprocal system (19) is defined analogously.

Proposition 1. *Suppose that* (17) *and its reciprocal system*

$$y^\Delta = -A^T(t)y + C(t)z^\sigma, \quad z^\Delta = -B(t)y + A(t)z^\sigma \tag{19}$$

are eventually identically normal and the matrices $B(t), C(t)$ are nonnegative definite for large t. Then system (17) *is nonoscillatory if and only if the reciprocal system* (19) *is nonoscillatory.*

3. RECIPROCITY PRINCIPLE FOR DYNAMIC EQUATIONS

Now we are ready to present the main result of the paper which reads as follows. Throughout the rest of the paper we suppose that a time scale under consideration is unbounded from above.

Theorem 1. *Suppose that r, q are positive rd-continuous functions. Then equation* (6) *is nonoscillatory if and only if reciprocal equation* (7) *is nonoscillatory.*

Proof: The proof is based on Proposition 1. We will show details of the proof for n even, while for n odd we will only mention technical differences in computations. Therefore, suppose that n is even. The substitution

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} y \\ y^\Delta \\ \vdots \\ D_{n-2}^\Delta y \\ D_{n-1}^\Delta y \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} -D_{n-1}^\Delta(r(t)D_n^\Delta y) \\ D_{n-2}^\Delta(r(t)D_n^\Delta y) \\ \vdots \\ -(r(t)D_n^\Delta y)^\Delta \\ r(t)D_n^\Delta y \end{bmatrix},$$

converts (6) into the symplectic dynamic system (10) with the matrices (see [14])

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & & & 0 & 0 \\ 0 & 0 & 1 & \mu & \cdots & & & 0 & 0 \\ 0 & 0 & 0 & 1 & \ddots & & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 1 & \mu \\ & & & & & & 0 & 0 & 1 \\ 0 & \cdots & & & & & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{B} = \frac{1}{r^\sigma} \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & -\mu & 1 \end{bmatrix}, \quad \mathcal{C} = -q^\sigma \begin{bmatrix} 1 & \mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & & 0 \end{bmatrix},$$

$$\mathcal{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & & 0 & 0 \\ -1 & 0 & 0 & 0 & & & 0 & 0 \\ \mu & -1 & 0 & 0 & & & 0 & 0 \\ 0 & 0 & -1 & 0 & & & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \ddots & \vdots \\ & & & & -1 & 0 & 0 & 0 \\ & & & & \mu & -1 & 0 & 0 \\ 0 & \cdots & & & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Next we show that the definition of the generalized zero of multiplicity n of a solution y of (6) complies with the definition of the generalized zero of the associated vector solution of (10) with the above given matrices $\mathcal{A}, \ldots, \mathcal{D}$. If t is a right dense point then this fact is trivial according to the relationship between the function y (and its derivatives $D_\nu^\Delta y$) and the vector x. If $\sigma(t) > t$, the condition $x^\sigma(t) \in \operatorname{Im} \mathcal{B}(t)$ means that $x_1^\sigma = 0, \ldots, x_{n-1}^\sigma = 0$, i.e., $D_\nu^\Delta y(\sigma(t)) = 0$, $\nu = 0, \ldots, n-2$. By a direct computation one can verify that the Moore-Penrose generalized inverse $\mathcal{B}^\dagger$ is

$$\mathcal{B}^\dagger = \frac{r^\sigma}{1+\mu^2} \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & & 0 \\ 0 & \cdots & 0 & -\mu \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

and hence

$$x^T(t)\mathcal{B}^\dagger(t)x^\sigma(t) = \frac{r^\sigma(t)}{1+\mu^2(t)} D_{n-1}^\Delta y(\sigma(t)) \left[-\mu(t) D_{n-2}^\Delta y(t) + D_{n-1}^\Delta y(t)\right]. \tag{20}$$

Since n is even, we have $D^{\Delta}_{n-1}y(t) = (D^{\Delta}_{n-2}y(t))^{\Delta}$ and from the definition of delta derivative (taking into account that $D^{\Delta}_{n-2}y(\sigma(t)) = 0$)

$$D^{\Delta}_{n-1}y(t) = \frac{1}{\mu(t)}\left[D^{\Delta}_{n-2}y(\sigma(t)) - D^{\Delta}_{n-2}y(t)\right] = -\frac{D^{\Delta}_{n-2}y(t)}{\mu(t)}.$$

Hence, the expression in brackets in (20) equals $(1+\mu^2(t))D^{\Delta}_{n-1}y(t)$, which means that also in this case definitions comply. Note also that the possibility (c) in the definition of the generalized zero of a vector solution cannot happen since $r(t) > 0$. Concerning the situation for reciprocal equation (7) and the system reciprocal to (10) (which results from (10) upon the transformation $y = u$, $z = -x$)

$$y^{\Delta} = \mathcal{D}(t)y - \mathcal{C}(t)z, \quad z^{\Delta} = -\mathcal{B}(t)y + \mathcal{A}(t)z, \tag{21}$$

the relationship is again trivial if $\sigma(t) = t$. In case $t < \sigma(t)$, since $y = (y_1, \dots, y_n)^T = (u_1, \dots, u_n)^T = (-D^{\Delta}_{n-1}u_n, D^{\Delta}_{n-1}u_n, \dots, -u_n^{\Delta}, u_n)^T$, the condition $y^{\sigma}(t) \in \operatorname{Im}\mathcal{C}(t)$ reads

$$u_n(\sigma(t)) = u_n^{\Delta}(\sigma(t)) = \cdots = D^{\Delta}_{n-2}u_n(\sigma(t)).$$

Since

$$\mathcal{C}^{\dagger} = -\frac{1}{q^{\sigma}(1+\mu^2)}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & & 0 \end{bmatrix},$$

using similar computations as above

$$-y^T(t)\mathcal{C}^{\dagger}(t)y^{\sigma}(t) = q^{\sigma}(t)D^{\Delta}_{n-1}u_n(t)D^{\Delta}_{n-1}u_n(\sigma(t)).$$

Since $u_n = rD^{\Delta}_n y$ is the solution of the reciprocal equation (7), also generalized zeros of multiplicity n of solutions of (7) comply with generalized zeros of the vector solutions of reciprocal symplectic system (21) (in this definition, the matrix $-\mathcal{C}$ plays the role of the matrix $\mathcal{B}$).

To finish the proof, it is sufficient now to convert systems (10) and (21) to Hamiltonian systems (3) and (4) and to verify that these systems satisfy assumptions of Proposition 1. To this end, we first compute the matrix $(I+\mu\mathcal{A})^{-1}$ which appears in formulas (18). We have

$$(I+\mu\mathcal{A})^{-1} = \sum_{i=0}^{\infty}\mu^i\mathcal{A}^i = \sum_{i=0}^{n-1}\mu^i\mathcal{A}^i \tag{22}$$

since $A^i = 0$ for $i \geq n$. Substituting into (22) we find that

$$(I + \mu\mathcal{A})^{-1} = \begin{bmatrix} 1 & -\mu & \mu^2 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & -\mu & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & -\mu & \mu^2 & 0 & \dots & 0 \\ \vdots & & & \ddots & \ddots & & & \\ & & & & & & & \\ 0 & \dots & & & 0 & 1 & -\mu & \mu^2 \\ 0 & \dots & & & & 0 & 1 & -\mu \\ 0 & \dots & & & & & 0 & 1 \end{bmatrix}.$$

Consequently,

$$B = (I + \mu\mathcal{A})^{-1}\mathcal{B} = \frac{1}{r}\begin{bmatrix} 0 & \dots & & & 0 \\ \vdots & & & \vdots & \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & \mu^2 & -\mu \\ 0 & \dots & 0 & -\mu & 1 \end{bmatrix}$$

and

$$C = -\mathcal{C}(I + \mu\mathcal{A})^{-1} = q^\sigma \operatorname{diag}\{1, 0, \dots, 0\}.$$

which shows that both these matrices are nonnegative definite. Finally, the assumption of eventual identical normality is trivially satisfied since $x(t) \equiv 0$ on some interval means, in particular, that the first entry of this vector $x_1(t) = y(t) \equiv 0$ in some interval and hence all derivatives of y (which are entries of the vectors x, u in (10)) are identically zero.

Now, we can summarize our proof for n even. If equation (6) is nonoscillatory, the associated symplectic dynamic system is nonoscillatory as well since oscillation is defined via generalized zeros of multiplicity n and generalized zeros of vector solutions, respectively, and these two concepts comply. The symplectic system associated with (6) has the Hamiltonian structure and, when converting to linear Hamiltonian system (17), assumptions of Proposition 1 are satisfied. Hence, the reciprocal system (19), and, in turn, the reciprocal system (21) with the matrices $\mathcal{A}, \dots, \mathcal{D}$ given at the beginning of the proof, is also nonoscillatory. Finally, since the definitions of oscillation of this system and of (7) are the same (definitions of generalized zero of multiplicity n and of generalized "vector zero" comply), reciprocal equation (7) is also nonoscillatory. Since all arguments can be reversed, we have equivalence of oscillatory nature of (6) and (7) for n even.

Concerning the differences in the proof when n is odd, the matrix $\mathcal{B}$ is slightly different

$$\mathcal{B} = \frac{1}{r} \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \mu \\ 0 & \dots & 0 & 1 \end{bmatrix},$$

but also in this case the definition of generalized zero of multiplicity n for (6) and (7) complies with the definition of the generalized zero of vector-valued solutions of associated symplectic dynamic systems as can be verified by a direct computation which is very similar to the computation for n even. Consequently, the statement of the theorem holds also for n odd and the proof is complete. □

4. REMARKS AND COMMENTS

(i) As we have already mentioned in the first section, the reciprocity principle has applications in the spectral theory of differential operators. To be more specific, consider the differential operator

$$\ell(y) := (-1)^n (r(t) y^{(n)})^{(n)}. \tag{23}$$

It is known that this operator has the spectrum discrete and bounded below in $\mathcal{L}^2[T, \infty)$, $T \in \mathbb{R}$ (the so-called property $\mathbb{B}D$), if and only if the equation

$$\ell(y) = \lambda y \tag{24}$$

is nonoscillatory for every $\lambda > 0$. Applying the nonoscillation criterion of Glazman [15] to *reciprocal equation* to (24), Lewis [18] resolved Tkachenko's conjecture (see [15]) in the affirmative way that the operator ℓ has the property $\mathbb{B}D$ if and only if

$$\lim_{t \to \infty} t^{2n-1} \int_t^\infty r^{-1}(s)\, ds = 0. \tag{25}$$

The discrete version of this statement can be found in [11]. The spectral theory of differential operators on time scales is not developed yet (as far as we know), but we believe that the reciprocity principle established in this paper could be of the same importance in this spectral theory as in the continuous and discrete cases.

(ii) There are other possibilities how to "distribute" the delta and nabla derivatives to get a (formally) self-adjoint differential operator. A comprehensive treatment of this problem can be found in the paper [4]. A very special case of equations considered there is the pair of $2n$-order equations

$$(-1)^n \left(r(t) y^{\Delta\Delta \dots \Delta\nabla} \right)^{\nabla\nabla \dots \nabla\Delta} = q(t) y \tag{26}$$

and

$$(-1)^n \left(\frac{1}{q(t)} z^{\nabla\nabla\ldots\nabla\Delta} \right)^{\Delta\Delta\ldots\Delta\nabla} = \frac{1}{r(t)} z. \tag{27}$$

One can find immediately that (27) is related to (26) by the reciprocity transformation $z = ry^{\Delta\Delta\ldots\Delta\nabla}$. We believe that oscillatory nature of equations (26) and (27) is also the same (under the assumption $r(t) > 0$, $q(t) > 0$). However, the technical computations are different from those performed in our paper, so we formulated this reciprocity principle for (26) and (27) only as a conjecture.

(iii) A "frequently asked question" is what is the relationship between the reciprocal equation and the adjoint equation. The answer can be formulated as follows. Consider the adjoint system to (10)

$$z^\Delta = -S^T(t) z^\sigma. \tag{28}$$

This definition of adjointness is the same as in the continuous and discrete case, we ask the matrix $Z = (X^T)^{-1}$ to be the fundamental matrix of the adjoint system if X is a fundamental matrix of the original system. By a direct computation we find that adjoint system is really (28) and expanding the sigma operator as $z^\sigma = z + \mu z^\Delta$, we get the system

$$z^\Delta = -(I + \mu S(t))^{-1} S^T(t) z. \tag{29}$$

Now, if we again consider the matrix S consisting of matrices $\mathcal{A}, \ldots, \mathcal{D}$, the symplecticity of the matrix $(I + \mu S^T)$ implies that

$$\begin{aligned} -(I+\mu S^T)^{-1} S^T &= \begin{bmatrix} I + \mu\mathcal{D} & -\mu C \\ -\mu\mathcal{B} & I + \mu\mathcal{A} \end{bmatrix} \begin{bmatrix} \mathcal{A}^T & \mathcal{C}^T \\ \mathcal{B}^T & \mathcal{D}^T \end{bmatrix} \\ &= - \begin{bmatrix} \mathcal{A}^T + \mu\mathcal{D}\mathcal{A}^T - \mu\mathcal{C}\mathcal{B}^T & \mathcal{C}^T + \mu\mathcal{D}\mathcal{C}^T - \mu\mathcal{C}\mathcal{D}^T \\ -\mu\mathcal{C}\mathcal{A}^T + \mathcal{B}^T + \mu\mathcal{A}\mathcal{B}^T & -\mu\mathcal{B}\mathcal{C}^T + \mathcal{D}^T + \mu\mathcal{A}\mathcal{D}^T \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{D} & -\mathcal{C} \\ -\mathcal{B} & \mathcal{A} \end{bmatrix} \end{aligned}$$

where the last equality follows from (14). Consequently, reciprocal and adjoint systems are *the same* systems. Of course, the reciprocal and adjoint equations to (6) *are not the same*. The adjoint equation is the equation which is satisfied by the *first* entry z_1 of the component z of a solution $\binom{y}{z} \in \mathbb{R}^{2n}$ of (21), while the reciprocal equation is the equation for the *last* entry y_n of y.

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