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EXISTENCE OF SOLUTIONS OF MULTIPOINT BOUNDARY VALUE PROBLEMS FOR A SECOND ORDER DIFFERENTIAL EQUATION

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ABSTRACT. Assuming the uniqueness of an n-point boundary value problem, for some $n \geq 4$ for

$$y'' = f(x, y, y')$$

existence of unique solutions are proved for all $n \geq 2$.

1. INTODUCTION

We are concerned with the question of the existence of a unique solution of the n-point boundary value problem for the second order differential equation

$$y'' = f(x, y, y') \tag{1}$$

$$y(x_1) = y_1;\ y(x_n) - \sum_{i=2}^{n-1} y(x_i) = y_2. \tag{2}$$

We assume throughout:

(A): $f : (a, b) \times \Re^2 \to \Re$ is continuous.

(B): Solutions of initial value problems for (1) are unique and exist on all of (a, b).

(C): For a given $n \geq 2$ and any points $a < x_1 < x_2 < \cdots < x_n < b$, if y and z are solutions of (1) such that

$$y(x_1) = z(x_1);\ y(x_n) - \sum_{i=2}^{n-1} y(x_i) = z(x_n) - \sum_{i=2}^{n-1} z(x_i) \tag{3}$$

then $y(x) = z(x)$. for all $a < x < b$.

There has been much interest in multipoint boundary value problems for the second order differential equation. See [1] and the references contained therein.

The main theorem of [1] is the following

Theorem 1. *Assume that conditions (A), (B) and (C) hold. Then for each $2 \leq k \leq n$, there is a unique solution of (1) that satisfies (2).*

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2. MAIN RESULT

Our purpose is to complement Theorem 1 with the following

Theorem 2. *Assume conditions (A), (B) and that (C) holds for some $n \geq 4$. Then for each $n \geq 2$, there is a unique solution of (1) that satisfies (2).*

Proof. Suppose (C) holds for $k = n \geq 4$. Assume that it does not hold for $k+1$, i.e. there are two distinct solutions u and v of (1) that satisfy (2) when $n = k+1$.

Define $g := u - v$. Then

$$g(x_1) = 0$$

$$g(x_{k+1}) = \sum_{i=2}^{k} g(x_i).$$

Note that $g(x) \neq 0$ for $x > x_1$, otherwise u and v satisfy (C) for $n = 2$. In that case, it follows from Theorem 1 that $u \equiv v$. Also see [2].

Next note that $g(t) \neq g(s)$ for $t \neq s$ and both t and s greater than x_1, otherwise $g(x_1) = 0$ and $g(s) = g(t)$ so that u and v satisfy (C) for $n = 3$. Again, Theorem 1 implies $u \equiv v$.

Thus, we can assume without loss of generality that $g(x)$ is positive and monotone increasing for $x > x_1$. Since g is also continuous, it follows that there exist points $t_1 < t_2$ in (x_1, x_2) so that

$$g(x_2) = g(t_1) + g(t_2).$$

In that case u and v satisfy (C) for $n = 4$ and consequently applying Theorem 1 again, $u \equiv v$.

The following example shows that for $n = 3$ (C) can hold while it does not hold for $n = 4$.

Example 1. There are examples of (1) with solutions that satisfy (2) for $n = 3$ but do not have solutions that satisfy (2) if $n = 4$.

Consider the linear equation

$$y'' + py' + qy = f. \tag{4}$$

Every solution of (4) is of the form

$$y = c_1 y_1 + c_2 y_2 + z$$

where z is any solution of (4) and y_1 and y_2 are linearly independent solutions of

$$y'' + py' + qy = 0. \tag{5}$$

Suppose $a < x_1 < x_2 < x_3 < b$. We want a solution of (1) that satisfies

$$y(x_1) = \alpha;\ y(x_3) - y(x_2) = \beta \tag{6}$$

Choose y_1 so that $y_1(x_1) = 0$. Since y_1 and y_2 are linearly independent, $y_2(x_1) \neq 0$. It follows that $c_2 = \frac{\alpha - z(x_1)}{y_2(x_1)}$. Now $\beta = y(x_3) - y(x_2) = c_1[y_1(x_3) - y_1(x_2)] + c_2[y_2(x_3) - y_2(x_2)] + [z(x_3 - z(x_2]$. Therefore, (4) has a solution which is unique satisfying (6) provided $y_1(x_3) - y_1(x_2) \neq 0$. This is the case when y_1 is increasing on the interval (a, b).

In the case of the four point problem

$$y(x_1) = \alpha;\ y(x_4) - y(x_3) - y(x_2) = \beta \tag{7}$$

with $a < x_1 < x_2 < x_3 < x_4 < b$, (4) has a solution satisfying (7), provided $y_1(x_4) - y_1(x_3) - y_1(x_2) \neq 0$, which fails to hold for all points in (a, b).

Example 2. There are examples of (1) with solutions that satisfy (2) for $n = 2$ but do not have solutions that satisfy (2) if $n = 3$.

As in Example 1, there is a solution of (4) that satisfies

$$y(x_1) = \alpha;\ y(x_2) = \beta \tag{8}$$

provided $y_1(x_2) \neq 0$, however unless y_1 is monotone on (a, b) there is not always a solution of (4) that satisfies (6).

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