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CONTROLLABILITY OF NONLINEAR INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACE WITH NONLOCAL CONDITIONS

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ABSTRACT. Sufficient conditions for controllability of nonlinear integrodifferential systems in a separable Banach space with nonlocal conditions are established. The results are obtained using a compactness type hypothesis involving the Hausdorff-measure of noncompactness, Kakutani's fixed-point theorem and Schauder's fixed-point theorem.

1. INTRODUCTION

The problem of controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional space has been extensively studied. The first works on the controllability of infinite-dimensional control systems are due to Fattorini (Refs. 1-2). Fattorini considered systems defined in a Hilbert space, with the linear operator being self-adjoint, densely defined and generator of a C_0-semigroup. His analysis was based the so-called ordered representation theory of a Hilbert space for self-adjoint operators. Soon thereafter appeared several papers on the controllability of linear systems with bounded operators. The most detailed study was conducted by Triggiani (Refs. 3-5). Naito published a series of papers dealing with the controllability of semilinear and Volterra equations (see Refs. 6-9). Here we focus on some works on the controllability of infinite-dimensional control systems proposed Quinn-Carmichael (Ref. 10), Benchohra-Ntouyas (Ref. 11-12), Han-Park (Ref. 13), their approaches were mainly based on variety of fixed-point theory. Our study is also in this direction. On the other hand the nonlocal condition, as generalization of the classical initial condition, was motivated by physical problems. The pioneering work on nonlocal conditions was due to Byszewski (Ref. 14). In the few past years, several papers have been devoted to study the existence of solutions for differential equations with nonlocal conditions. Among others, we refer to the papers by Balachandran-Chandrasekaran (Ref. 15-16), Balachandran-Ilamaran (Ref. 17), Ntouyas-Tsamatos (Refs. 18-19) and Ntouyas (Ref. 20). The purpose of this paper is to examine the controllability of nonlinear integrodifferential inclusion

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systems in Banach Space with nonlocal conditions. Using a compactness type hypothesis involving the Hausdorff-measure of noncompactness , the Kakutani's fixed-point theorem and Schauder's fixed-point theorem, we establish two controllability results which involve convex-valued fields and the other nonconvex valued ones. As a especial case of this paper, Li and Xue [21] studied the controllability of evolution inclusions with nonlocal conditions.

2. PRELIMINARIES AND BASIC HYPOTHESES

Let X be a Banach space and $T = [0, b]$ $(0 < b < \infty)$. we shall use the notations

$$\begin{aligned} P_f(X) &= \{B \subseteq X : \text{nonempty, closed}\} \\ P_k(X) &= \{B \subseteq X : \text{nonempty, compact}\} \\ P_b(X) &= \{B \subseteq X : \text{nonempty, bounded}\} \\ P_{fc}(X) &= \{B \subseteq X : \text{nonempty, closed, convex }\} \\ P_{kc}(X) &= \{B \subseteq X : \text{nonempty, compact,convex }\} \\ P_{wkc}(X) &= \{B \subseteq X : \text{nonempty, w-compact, convex}\} \end{aligned}$$

A multifunction $F : T \to P_f(X)$ is said to be measurable, if for every $z \in X$ the function $t \to d(z, F(t)) = \inf\{||z - x|| : x \in F(t)\}$ is measurable. This is equivalent to saying that $GrF = \{(t, x) \in T \times X : x \in F(t)\} \in \Sigma \times \mathcal{B}(\mathrm{X})$ with Σ being the Lebesgue σ-field of T and being $\mathcal{B}(\mathrm{X})$ the Borel σ-field of X (graph measurability), or that there exists a sequence $f_n : T \to X,\ n \geq 1$ of measurable functions such that $F(t) = \overline{\{f_n(t) : n \geq 1\}}$ for all $t \in T$. A graph measurable multifunction $F : T \times X \times X \to P_k(X)$ has the property that if $x : T \to X$ and $y : T \to X$ are measurable, then $t \to F(t, x(t), y(t))$ is graph measurable, i.e. $GrF(\cdot, x(\cdot), y(\cdot)) \in \Sigma \times \mathcal{B}(\mathrm{X}) \times \mathcal{B}(\mathrm{X}) \times \mathcal{B}(\mathrm{X})$. So By Aumann's selection theorem we can find a measurable function $g : T \to X$ such that $g(t) \in F(t, x(t), y(t))$ a.e. on T. We denote by S^1_F the set of all selectors of $F(\cdot)$ that belong to the Lebesgue-Bochner space $L^1(T, X)$, i.e., $S^1_F = \{f(\cdot) \in L^1(T, X) : f(t) \in F(t)\ \text{ a.e. on } T\}$. It is easy to see that this is closed and it is nonempty if and only if $\inf[||x|| : x \in F(t)] \in L^1(T, R)$. Using this set we can define an integral for multifunctions, i.e.,

$$\int_T F(t)dt = \left\{ \int_T f(t)dt : f(\cdot) \in S^1_F \right\},$$

where the vector valued integral is in the sense of Bochner. We say that $F(\cdot)$ is integrably bounded if and only if $F(\cdot)$ is measurable and $|F(\cdot)| \in L^1(T, R)$, where $|F(t)| = \sup[||x|| : x \in F(t)]$. Finally the set S^1_F is decomposable in the sense that if $(f_1, f_2, A) \in S^1_F \times S^1_F \times \Sigma$, then $f_1\chi_A + f_2\chi_{A^c} \in S^1_F$.

Let Y, Z be Hausdorff topological spaces and let $G : Y \to 2^Z \setminus \{\emptyset\}$. We say that $G(\cdot)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)), if for all $C \subseteq Z$ nonempty closed, $G^-(C) = \{y \in Y : G(y) \cap C \neq \emptyset\}$ (resp. $G^+(C) = \{y \in Y : G(y) \subseteq C\}$) is closed in Y . If Y, Z are both metric spaces, then the above definition of lower semicontinuity is equivalent to saying that for all $z \in Z$, $y \to d_Z(z, G(y)) = \inf[d_Z(z,v) : v \in G(y)]$ is upper semicontinuity as R_+-valued function. Also if $G(\cdot)$ is closed valued and $\overline{G(Y)}$ is compact, then the above definition of upper semicontinuity is equivalent to saying that $G(\cdot)$ has a closed graph in $Y \times Z$.

In the sequel, X denotes a separable Banach space. The Hausdorff-measure of noncompactness $\beta : P_b(X) \to R_+$ is defined by

$$\beta(A) = \inf\{r > 0 : A \subset \bigcup_{i=1}^{n(r)} B(x_i^r, r) \,\text{for}\, x_i^r \in X\}.$$

For convenience we recall some properties of β (see Ref. 22). $\mathcal{L}$(X) denotes the Banach space of bounded linear operators from X into itself with the usual operator norm. $C(T, X)$ denotes the Banach space of continuous functions from T into X with the usual supremum norm.

Proposition 2.1 Let A, $B \in P_b(X)$, $\lambda \in R$, then

(i) $\beta(A) = 0 \Leftrightarrow A$ is relatively compact;

(ii) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$;

(iii) $\beta(\text{con}(A)) = \beta(A)$;

(iv) $\beta(\overline{A}) = \beta(A)$;

(v) $\beta(A + B) \leq \beta(A) + \beta(B)$;

(vi) $\beta(\lambda A) = |\lambda|\beta(A)$.

(vii) if $U \in \mathcal{L}$(X), then $\beta(UA) \leq ||U||_{\mathcal{L}}\beta(A)$.

(viii) if $A \in P_b(C(T, X))$ is equicontinuous, then

$$\beta_C(A) = \sup_{t\in T}\beta(A(t)),$$

where $\beta_C(\cdot)$ is the Hausdorff measure of noncompactness in $C(T, X)$.

We will also need the following property and lemma.

Proposition 2.2 [Ref. 23] If multifunction $F : T \to P_b(X)$ is measurable and integrably bounded, then $\beta(F(t))$ is integrable and for every measurable set $I \subseteq T$

$$\beta(\int_I F(s)ds) \leq \int_I \beta(F(s))ds.$$

Lemma 2.1 [Ref. 24] Let$\{A_n : n \geq 1\} \subset P_{fb}(X)$ and $A_{n+1} \subset A_n$ for $n \geq 1$. If $\beta(A_n) \to 0 (n \to \infty)$,the $A = \bigcap_{n=1}^{\infty} A_n$ is a nonempty, compact subset of X.

Consider the nonlinear integrodifferential system

$$\begin{aligned} x'(t) &\; -A(t)x(t) \in F(t, x(t), V(x)(t)) + (Bu)(t) \text{ a.e. on } T \\ x(0) &\; +M(x) = x_0, \end{aligned} \tag{1}$$

where $\{A(t)\}_{t\in T}$ is a family of linear operators that generate an evolution operator $U : \Delta = \{(t, s) \in T \times T : 0 \leq s \leq t \leq b\} \to \mathcal{L}$(X). $F : T \times X \times X \to 2^X \setminus \emptyset$ is a multifunction,$V : C(T, X) \to C(T, X)$ is the Volterra integral operator corresponding to the kernel $K(t, s)$, i.e. $V(x)(t) = \int_0^t K(t, s)x(s)ds$,where $K : \Delta = \{(t, s) \in T \times T : 0 \leq s \leq t \leq b\} \to \mathcal{L}(\mathrm{X})$ is a strongly continuous kernel; i.e. it is continuous from Δ into $\mathcal{L}(\mathrm{X})$. Also, the control function $u(\cdot)$ is given in $L^2(T, U)$ and B is a linear operator from U to X, where U is a separable Banach space and $L^2(T, U)$ is a Banach space of admissible control function with norm $||u||_{L^2} = \left(\int_0^b ||u(t)||^2 dt\right)^{\frac{1}{2}}$.

Definition 2.1 A function $x(\cdot) \in C(T, X)$ such that

$$x(t) = U(t, 0)x(0) + \int_0^t U(t, s)(f + Bu)(s)ds, \ t \in T$$

with $f \in S^1_{F(\cdot,x(\cdot),V(x)(\cdot))}$ and $x(0) + M(x) = x_0$ is called a mild solution of (1).

Definition 2.2 The system (1) is said to be nonlocally controllable on T if, for every $x_0, x_1 \in X$, there exists a control $u \in L^2(T, U)$ such that the mild solution $x(\cdot)$ of (1) satisfies $x(b) + M(x) = x_1$.

For the proof of the main results in Section 3, we shall need the following hypotheses:

H(1): $\{A(t)\}$ is a family of linear, densely defined operators that generate a strongly continuous evolution operator $U : \Delta = \{(t, s) \in T \times T : 0 \leq s \leq t \leq b\} \to \mathcal{L}(\mathrm{X})$.

H(2): $F : T \times X \times X \to P_{kc}(X)$ is a multifunction such that: (i) $(t, x, y) \to F(t, x, y)$ is graph measurable; (ii) for every $t \in T$, $(x, y) \to F(t, x, y)$ is u.s.c. from $X \times X$ into X_w, where X_w denotes the Banach space X with the weak topology ; (iii) $|F(t, x)| = \sup\{||v|| : v \in F(t, x, y), ||x|| \leq n, \ ||y|| \leq m\} \leq \varphi_n(t) + \psi_m(t)$ a.e, with $\varphi_n(\cdot)$, $\psi_m(\cdot) \in L^1(T, R)$ and

$$\underline{\lim} \frac{1}{n} \int_0^b \varphi_n(s)ds = \underline{\lim} \frac{1}{m} \int_0^b \psi_m(s)ds = 0.$$

H(3): for $A_1, A_2 \in P_b(X)$, $\beta(F(t, A_1, A_2)) \leq k(t)(\beta(A_1) + \beta(A_2))$ with $k(\cdot) \in L^1(T, R)$.

H(4): bounded linear operator $B : U \to X$ is compact and $||B||_{\mathcal{L}} \leq M_1$. $M : C(T, X) \to X$ is a compact operator such that

$$\lim_{||y|| \to \infty} \frac{||M(y)||}{||y||} = 0.$$

H(5): the linear operator $W : L^2(T, U) \to X$, defined by

$$Wu = \int_0^b U(b, s)Bu(s)ds,$$

has an invertible operator W^{-1} which takes values in $L^2(T, U) \setminus \ker W$ and there exist positive constant M_2 such that $||W^{-1}|| \leq M_2$. For $A \in P_{fb}(X)$, $|W^{-1}A|(t) = \sup\{||f(t)|| : f \in W^{-1}(A)\} \leq \phi_n(t)$ a.e. on T with $\phi_n(\cdot) \in L^1(T, R)$ and $|A| = \sup\{||x|| : x \in A\} \leq n$.

3. MAIN RESULTS

Theorem 3.1 If hypotheses H(1)-H(5) hold, the problem (1) is nonlocally controllable on T.

Proof Using hypothesis H(5) for an arbitrary function $y(\cdot) \in C(T, X)$, define the control

$$u_f(t) = W^{-1}\left[x_1 - M(y) - U(b,0)(x_0 - M(y)) - \int_0^b U(b,s)f(s)ds\right](t),$$

where $f(\cdot) \in S^1_{F(\cdot,y(\cdot),V(y)(\cdot))}$. We shall show that, when using this control, the multifunction $R : C(T,X) \to 2^{C(T,X)}$, defined by

$$R(y) = \left\{x \in C(T,X) : x(t) = U(t,0)(x_0 - M(y)) + \gamma(f + Bu_f)(t), f \in S^1_{F(\cdot,y(\cdot),V(y(\cdot))}\right\},$$

where $\gamma(f + Bu_f) \in C(T,X)$ is defined by

$$\gamma(f + Bu_f)(t) = \int_0^t U(t,s)(f + Bu_f)(s)ds,$$

has a fixed point. This fixed point is then a solution of the system (1). Clearly, $x_1 - M(y) \in (R(y))(b)$.

Step 1 We claim that $R(\cdot)$ has nonempty, closed, convex valued.

Let $y(\cdot) \in C(T,X)$. From H(2)(i), $(t,x,y) \to F(t,x,y)$ is graph measurable, then $t \to F(t,y(t),V(y)(t))$ is graph measurable. So By Aumann's selection theorem we can find a measurable function $f : T \to X$ such that $f(t) \in F(t,y(t),V(y)(t))$ a.e. on T; i.e. $f \in S^1_{F(\cdot,y(\cdot),V(y)(\cdot))}$. Therefore for $y \in C(T,X)$, $R(y) \neq \emptyset$. Clearly $R(\cdot)$ is convex valued and because of Proposition 3.1 of Papageorgiou (Ref. 25) $S^1_{F(\cdot,y(\cdot))} \in P_{wkc}(L^1(T,X))$ for every $y \in C(T,X)$, we also deduce that $R(\cdot)$ is closed valued, then $R : C(T,X) \to P_{fc}(C(T,X))$.

Step 2 There exists a positive integer $n_0 \geq 1$ such that $R(B_{n_0}) \subseteq B_{n_0}$, where $B_{n_0} = \{y \in C(T,X) : ||y||_C \leq n_0\}$.

Suppose not. Then we can find $y_n \in C(T,X)$, $x_n \in R(y_n)$ such that $||y_n||_C \leq n$ and $||x_n||_C > n$. Then we have for every $n \geq 1$,

$$x_n(t) = U(t,0)(x_0 - M(y_n)) + \gamma(f_n + Bu_{f_n})(t)$$

for some $f_n \in S^1_{F(\cdot,y_n(\cdot),V(y_n)(\cdot))}$. So we get

$$n < ||x_n||_C \leq M_3(||x_0|| + ||M(y_n)||) + ||\gamma(f_n)||_C + ||\gamma(Bu_{f_n}||_C \tag{2}$$

where $M_3 > 0$ is such that $||U(t,s)||_{\mathcal{L}} \leq M_3$. Note that

$$||V(y_n)||_C = \max_{t\in[0,b]} ||\int_0^t K(t,s)y_n(s)ds|| \leq Lbn,$$

where $||K(s,t)|| \leq L$, for all $(S,t) \in \Delta$. Let $m = [Lbn] + 1$, then

$$||\gamma(f_n)||_C = \sup_{t\in[0,b]} ||\gamma(f_n)(t)|| \quad \leq \quad \sup_{t\in[0,b]} \int_0^t ||U(t,s)||_{\mathcal{L}} \cdot ||f_n(s)||ds$$

$$\leq M_3 \int_0^b [\varphi_n(s) + \psi_m(s)]ds, \tag{3}$$

$$||\gamma(Bu_{f_n})||_C \leq \sup_{t\in[0,b]} \int_0^t ||U(t,s)||_{\mathcal{L}} \cdot ||B||_{\mathcal{L}} \cdot ||u_{f_n}(s)||ds \leq M_2 M_3 b^{\frac{1}{2}} ||u_{f_n}||_{L^2} \tag{4}$$

and

$$\begin{aligned} ||u_{f_n}||_{L^2} &= ||W^{-1}\left[x_1 - M(y_n) - U(b,0)(x_0 - M(y_n)) - \gamma(f_n)(b)\right]|| \\ &\leq M_1 \left[||x_1|| + M_3||x_0|| + (1+M_3)||M(y_n)|| + M_3 \int_0^b [\varphi_n(s) + \psi_m(s)]ds \right] \quad (5) \end{aligned}$$

Hence by (2)-(5) we have

$$\begin{aligned} n &< (M_3 + M_1M_2M_3^2 b^{\frac{1}{2}})||x_0|| + M_1M_2M_3b^{\frac{1}{2}}||x_1|| \\ &+ (1 + M_1M_2b^{\frac{1}{2}} + M_1M_2M_3b^{\frac{1}{2}})M_3||M(y_n)|| \\ &+ (M_3 + M_1M_2M_3^2 b^{\frac{1}{2}}) \int_0^b [\varphi_n(s) + \psi_m(s)]ds \\ \Rightarrow 1 &< \frac{1}{n}\left[C_1 + C_2||M(y_n)|| + C_3 \int_0^b [\varphi_n(s) + \psi_m(s)]ds\right] \quad (6) \end{aligned}$$

where $C_1 = (M_3 + M_1M_2M_3^2b^{\frac{1}{2}})||x_0|| + M_1M_2M_3b^{\frac{1}{2}}||x_1||$, $C_2 = (1 + M_1M_2b^{\frac{1}{2}} + M_1M_2M_3b^{\frac{1}{2}})M_3$ and $C_3 = M_3 + M_1M_2M_3^2b^{\frac{1}{2}}$. Observe that H(2)(iii)and H(4). So by passing to the limit as $n \to \infty$ in inequality (6), we get $1 \leq 0$, a contradiction. Thus we conclude that there exists $n_0 \geq 1$ such that $R(B_{n_0}) \subseteq B_{n_0}$.

Step 3 $R(B_{n_0})$ is equicontinuous.

To this end, let $x \in R(B_{n_0})$ and $t_2, t_1 \in T$, $t_2 > t_1 > 0$, $m_0 = [Lbn_0] + 1$. We have for some $y \in B_{n_0}$, $f \in S^1_{F(\cdot,y(\cdot),V(y)(\cdot))}$ and any $\varepsilon > 0$ such that $t_1 - \varepsilon > 0$,

$$\begin{aligned} ||x(t_2) - x(t_1)|| &= ||\gamma(f + Bu_f)(t_2) - \gamma(f + Bu_f)(t_1)|| \\ &= ||\int_0^{t_2} U(t_2,s)[f(s) + (Bu_f)(s)]ds - \int_0^{t_1} U(t_1,s)[f(s) + (Bu_f)(s)]ds|| \\ &\leq \int_{t_1}^{t_2} ||U(t_2,s)||_{\mathcal{L}} \cdot ||f(s) + (Bu_f)(s)||ds \\ &+ \int_0^{t_1} ||U(t_2,s) - U(t_1,s)||_{\mathcal{L}} \cdot ||f(s) + (Bu_f)(s)||ds \\ &\leq M_3 \int_{t_1}^{t_2} [\varphi_{n_0}(s) + \psi_{m_0}(s) + M_2||u_f(s)||]ds \\ &+ \int_{t_1-\varepsilon}^{t_1} ||U(t_2,s) - U(t_1,s)||_{\mathcal{L}} \cdot ||f(s) + (Bu_f)(s)||ds \\ &+ \int_0^{t_1-\varepsilon} ||U(t_2,s) - U(t_1,s)||_{\mathcal{L}} \cdot ||f(s) + (Bu_f)(s)||ds \\ &\leq M_3 \int_{t_1}^{t_2} [\varphi_{n_0}(s) + \psi_{m_0}(s) + M_2||u_f(s)||]ds \end{aligned}$$

$$
\begin{aligned}
&+ \quad 2M_3 \int_{t_1-\varepsilon}^{t_1} [\varphi_{n_0}(s) + \psi_{m_0}(s) + M_2||u_f(s)||]ds \\
&+ \quad \int_0^{t_1-\varepsilon} ||U(t_2,s) - U(t_1,s)||_{\mathcal{L}}[\varphi_{n_0}(s) + \psi_{m_0}(s) + M_2||u_f(s)||]ds \\
&\leq \quad M_3 \int_{t_1}^{t_2} [\varphi_{n_0}(s) + \psi_{m_0}(s)]ds + 2M_3 \int_{t_1-\varepsilon}^{t_1} [\varphi_{n_0}(s) + \psi_{m_0}(s)]ds \\
&+ \quad \int_0^{t_1-\varepsilon} ||U(t_2,s) - U(t_1,s)||_{\mathcal{L}}[\varphi_{n_0}(s) + \psi_{m_0}(s)]ds \\
&+ \quad \left(\int_0^{t_1-\varepsilon} ||U(t_2,s) - U(t_1,s)||^2_{\mathcal{L}} ds\right)^{\frac{1}{2}} M_2||u_f||_{L^2} + M_2M_3(t_2-t_1)^{\frac{1}{2}}||u_f||_{L^2} \\
&+ \quad 2M_2M_3\varepsilon^{\frac{1}{2}}||u_f||_{L^2}.
\end{aligned}
$$

and

$$
||u_f||_{L^2} \quad \leq \quad M_1\left[M_3||x_0|| + ||x_1|| + (M_3+1)|M(B_{n_0})| + M_3\int_0^b [\varphi_{n_0}(s) + \psi_{m_0}(s)]ds\right] = M_4,
$$

where $|M(B_{n_0})| = \sup[||M(y)|| : y \in B_{n_0}]$ is bounded (since M is a compact operator). Also, we know that $t \to U(t,s)$ is continuous in the operator norm topology, uniformly $s \in T$ such that $t - s$ is bounded away from zero.

Now, given $\varepsilon' > 0$, by absolute continuity of the Lebesgue integral we can choose $\varepsilon > 0$, such that

$$
2M_3 \int_{t_1-\varepsilon}^{t_1} [\varphi_{n_0}(s) + \psi_{m_0}(s)]ds + 2M_2M_3M_4\varepsilon^{\frac{1}{2}} < \frac{\varepsilon'}{2}.
$$

By the continuity property of $U(\cdot, s)$ mentioned above and absolute continuity of the Lebesgue integral, we can find $\delta > 0$ such that if $t_2 - t_1 < \delta$, we have

$$
\begin{aligned}
M_3 \quad & \int_{t_1}^{t_2} [\varphi_{n_0}(s) + \psi_{m_0}(s)]ds + \int_0^{t_1-\epsilon} ||U(t_2,s) - U(t_1,s)||_{\mathcal{L}}[\varphi_{n_0}(s) + \psi_{m_0}(s)]ds \\
+ \quad & \left(\int_0^{t_1-\epsilon} ||U(t_2,s) - U(t_1,s)||^2_{\mathcal{L}} ds\right)^{\frac{1}{2}} M_2M_4 + M_2M_3M_4(t_2-t_1)^{\frac{1}{2}} < \frac{\varepsilon'}{2}.
\end{aligned}
$$

So $R(B_{n_0})$ is equicontinuous.

Let$A_1 = \overline{\text{conv}} R(B_{n_0})$, then A_1 is also equicontinuous. We define sequence $A_{n+1} = \overline{\text{conv}} R(A_n)$ $(n \geq 1)$, then $A_{n+1} \subset A_n$ and A_n is equicontinuous, closed and convex subset of $C(T, X)$ when $n \geq 1$.

Step 4 $\beta_C(A_n) \to 0$, as $n \to \infty$.

Let $A \in P_{fb}(B_{n_0})$ be equicontinuous. In what follows, we set $A(t) = \{x(t) : x(\cdot) \in A\}$ and $\overline{V(A)(s)} = \overline{\{\int_0^s K(s,\tau)x(\tau)d\tau : x \in A\}}$. Observe that $s \to A(s)$ is measurable, since if $\{x_n\}_{n\geq 1} \subseteq A$ is dense in A, then from the continuity of the evaluation map, we have that $A(s) = \overline{\{x_n(s) : n \geq 1\}}$, establishing the measurability of $A(\cdot)$. Similarly, using Theorem 3.1 of Kandilakis-Papageorgiou (Ref. 26), we have that $\overline{V(A)(s)} = \overline{\{\int_0^s K(s,\tau)x_n(\tau)d\tau : n \geq 1\}} \Rightarrow s \to \overline{V(A)(s)}$ is measurable. As in proof Theorem 3.2

of Papageorgiou (Ref. 27), we have that $s \to H(s) = \overline{\text{conv}}F(s, A(s), \overline{V(A)(s)})$ is measurable for the Lebesgue $\sigma-$field on T. By H(2)(iii), $|H(s)| = \sup\{||h|| : h \in H(s)\} \leq \varphi_{n_0}(s) + \psi_{m_0}(s)$ a.e. on T, then $H(\cdot)$ is integrably bounded. Let

$$\hat{B} = \{x_1 - M(x) - U(b,0)(x_0 - M(x)) - \int_0^b U(b,s)f(s)ds : f \in S^1_{F(\cdot,x(\cdot),V(x)(\cdot))}, x \in A\},$$

we can easily check that $\hat{B}$ is bounded subset in X. Let $|\hat{B}| \leq n_1$ and $\hat{W} = \{f \in L^2(T,U) : |f(t)| \leq \phi_{n_1}(t) \text{ a.e}\}$. Also, $L^2(T,U)$ is separable, there exists $\{g_n : n \geq 1\} \subseteq \hat{W}$ such that it is dense in $\hat{W}$. Let $G(t) = \overline{\{g_n(t) : n \geq 1\}}$, then $G(\cdot)$ is integrable bonded. By H(5)(ii), $\{u_f \in L^2(T,U) : f \in S^1_{F(\cdot,x(\cdot),V(x(\cdot))}, x \in A\} \subseteq S^1_{G(\cdot)}$. We have

$$R(A)(t) \subseteq U(t,0)(x_0 - M(A)) + \int_0^t U(t,s)[(H(s) + BG(s)]ds.$$

By Proposition 2.1-2.2, H(3) and note that M, B are compact operators, we have

$$\begin{aligned}
\beta(R(A)(t)) &\leq \beta(U(t,0)(x_0 - M(A))) + \beta(\int_0^t U(t,s)[H(s) + BG(s)]ds) \\
&\leq \int_0^t \beta(U(t,s)H(s))ds + \int_0^t \beta(U(t,s)BG(s))ds \\
&\leq M_3 \int_0^t \beta(H(s))ds = M_3 \int_0^t \beta(F(s, A(s), \overline{V(A)(s)}))ds \\
&\leq M_3 \int_0^t k(s)[\beta(A(s)) + \beta(\overline{V(A)(s)})]ds \qquad (7)\\
&= M_3 \int_0^t k(s)[\beta(A(s)) + \beta(V(A)(s))]ds.
\end{aligned}$$

From the definition of the Volterra integral operator $V(\cdot)$, we have

$$\begin{aligned}
\beta(V(A)(s)) &= \beta[\int_0^s K(s,\tau)A(\tau)d\tau] \\
&\leq \int_0^s \beta(K(s,\tau)A(\tau))d\tau \\
&\leq \int_0^s L\beta(A(\tau))d\tau
\end{aligned}$$

$$\Rightarrow \int_0^t \beta(V(A)(s))ds \leq \int_0^t \int_0^s L\beta(A(\tau))d\tau ds \leq Lb \int_0^t \beta(A(\tau))d\tau.$$

$$\beta(R(A)(t)) \leq \int_0^t k(s)M_3(1 + Lb)\beta(A(s))ds.$$

We choose $\lambda > M_3(1 + Lb)$ and let $\psi(A) = \sup_{t\in T}[e^{-\lambda \int_0^t k(s)ds}\beta(A(t)]$. Using the properties of $\beta(\cdot)$ and the fact that $A \subset B_{n_0}$ is equicontinuous, we can easily check that $\psi(\cdot)$is a sublinear measure of noncompactness, in the sense of Banas-Goebel(Ref. 22). We have

$$\beta(R(A)(t)) \leq \int_0^t k(s)M_3(1 + Lb)e^{-\lambda \int_0^s k(\tau)d\tau} e^{\lambda \int_0^s k(\tau)d\tau}\beta(A(s))ds$$

$$
\begin{aligned}
&\le \int_0^t k(s)M_3(1+Lb)\psi(A)e^{\lambda\int_0^s k(\tau)d\tau}ds \\
&= \frac{M_3(1+Lb)\psi(A)}{\lambda}\int_0^t d(e^{\lambda\int_0^s k(\tau)d\tau}) \\
&= \frac{M_3(1+Lb)}{\lambda}e^{\lambda\int_0^t k(s)ds}\psi(A).
\end{aligned}
$$

Then

$$
\begin{aligned}
\beta(R(A)(t))e^{-\lambda\int_0^t k(s)ds} &\le \frac{M_3(1+Lb)}{\lambda}\psi(A),\ t\in T \\
\Rightarrow \psi(R(A)) &\le \frac{M_3(1+Lb)}{\lambda}\psi(A).
\end{aligned}
$$

Set $\rho = \frac{M_3(1+Lb)}{\lambda} < 1$ and note that

$$
\psi(A) = \sup_{t\in T}[e^{-\lambda\int_0^t k(s)ds}\beta(A(t))] \ge m\sup_{t\in T}\beta(A(t)) = m\beta_C(A),
$$

where $m = e^{-\lambda\int_0^b k(s)ds} > 0$, then

$$
\begin{aligned}
\beta_C(A_{n+1}) &\le \frac{1}{m}\psi(A_{n+1}) = \frac{1}{m}\psi(\overline{\text{conv}}R(A_n)) \le \frac{\rho}{m}\psi(A_n) \\
\Rightarrow \beta_C(A_{n+1}) &\le \frac{\rho^n}{m}\psi(A_1),\ \text{for}\, n\ge 1 \Rightarrow \beta_C(A_n)\to 0,\ \text{as}\, n\to\infty.
\end{aligned}
$$

Invoking Lemma 2.1, we get $A_0 = \bigcap_{n=1}^{\infty} A_n$ is a nonempty, compact convex subset of $C(T,X)$ and $R(A_0)\subset A_0$.

Step 5 $R: A_0 \to P_{kc}(A_0)$ is u.s.c.

To this end, we only need to show that $R(\cdot)$ has a closed graph. Let $\{y_m : m\ge 1\}\subseteq A_0$, $y_m\to y$ in $C(T,X)$ and $x_m\in R(y_m)$, $x_m\to x$ in $C(T,X)$. Then by definition we have

$$
x_m(t) = U(t,0)(x_0 - M(y_m)) + \gamma(f_m + Bu_{f_m})(t)
$$

with $f_m\in S^1_{F(\cdot,y_m(\cdot),V(y_m)(\cdot))}$. Let $G(t) = \overline{\text{conv}}\bigcup_{m\ge 1}F(t,y_m(t),V(y_m)(t))$. Because of hypothesis H(2)(ii) and Theorem 7.4.2 of Klein-Thompson (Ref. 28), we have that $G(t)\in P_{wkc}(X)$, $G(\cdot)$ is clearly measurable (since for each $m\ge 1$ $t\to F(t,y_m(t),$ $V(y_m(t))$ is measurable) and $|G(t)|\le \varphi_{n_0}(t)+\psi_{m_0}(t)$ a.e on T. So from Proposition 3.1 of Papageorgiou (Ref. 25), we have that $S^1_G\in P_{wkc}(L^1(T,X))$ and since $\{f_m\}_{m\ge1}\subseteq S^1_G$, we may assume by to a subsequence if necessary, that $f_m\to f$ in $L^1(T,X)_w$. The invoking Theorem 3.1 of Papageorgiou (Ref. 29), we get that

$$
\begin{aligned}
f(t)\in \overline{\text{conv}}\, w-\overline{\lim}\{f_m(t)\}_{m\ge1} &\subseteq \overline{\text{conv}}\, w-\overline{\lim}F(t,y_m(t),V(y_m)(t)) \\
&\subseteq F(t,y(t),V(y)(t))\,\text{a.e. on}\, T,
\end{aligned}
$$

the last inclusion being a consequence of hypothesis H(2)(ii). Therefore $f\in S^1_{F(\cdot,y(\cdot),V(y)(\cdot))}$. Also we can easily verify that for every $t\in T$, $\gamma(f_m+Bu_{f_m})(t)\to\gamma(f+Bu_f)(t)$ in X_w, as $m\to\infty$. Hence

$$
U(t,0)(x_0-M(y_m))+\gamma(f_m+Bu_{f_m})(t)\to U(t,0)(x_0-M(y))+\gamma(f+Bu_f)(t)
$$

in X_w as $m \to \infty$, $t \in T$. Therefore we get

$$x(t) = U(t,0)(x_0 - M(y)) + \gamma(f + Bu_f)(t), t \in T$$

with $f \in S^1_{F(\cdot,y(\cdot),V(y)(\cdot))}$, i.e., $R(\cdot)$ has a closed graph and so is u.s.c. By the Kakutani' fixed point theorem, we deduce that $R(\cdot)$ has a fixed point and thus the system (1) is nonlocally controllable on T.

Now we consider the nonconvex version of the above result. Our hypothesis on the orientor field is now the following:

H(2′): $F : T\times X\times X \to P_f(X)$ is a multifunction such that: (i) $(t,x,y) \to F(t,x,y)$ is graph measurable; (ii) for every $t \in T$, $(x,y) \to F(t,x,y)$ is l.s.c. from $X \times X$ into X; (iii) $|F(t,x)| = \sup\{||v|| : v \in F(t,x,y), ||x|| \le n,\ ||y|| \le m\} \le \varphi_n(t) + \psi_m(t)$ a.e, with $\varphi_n(\cdot)$, $\psi_m(\cdot) \in L^1(T,R)$ and

$$\underline{\lim}\frac{1}{n}\int_0^b \varphi_n(s)ds = \underline{\lim}\frac{1}{m}\int_0^b \psi_m(s)ds = 0.$$

Theorem 3.2 If hypotheses H(1), H(2′),H(3)-H(5)hold, then problem(1) is nonlocally controllable on T.

Proof Consider the multivalued Nemitsky operator $N : C(T,X) \to 2^{L^1(T,X)}$ defined by $N(x) = S^1_{F(\cdot,x(\cdot),V(x)(\cdot))}$. we shall show that $N(\cdot)$ has nonempty, closed, decomposable values and is l.s.c from $C(T,X)$ to $L^1(T,X)$.

The nonemptiness, closedness and decomposability of the values of $N(\cdot)$ are easy to check. To check the lower semicontinuity of $N(\cdot)$, we need to show that for every $u \in L^1(T,X)$, $x \to d(u,N(x))$ is an upper semicontinuous R_+-valued function. To this end, we have

$$\begin{aligned} d(u,N(x)) &= \inf\left[||u-v||_1 : v \in N(x)\right] \\ &= \inf\left[\int_0^b ||u(t)-v(t)||\, dt : v \in N(x)\right] \\ &= \int_0^b \inf\left[||u(t)-v|| : v \in F(t,x(t),V(x)(t))\right] dt \\ &= \int_0^b d(u(t),F(t,x(t),V(x)(t)))\, dt \end{aligned}$$

(see Ref. 30)). We shall show that every $\lambda \ge 0$, the superlevel set $U_\lambda = \{x \in C(T,X) : d(u,N(x)) \ge \lambda\}$ is closed in $C(T,X)$. For this purpose let $\{x_n\}_{n\ge1} \subseteq U_\lambda$ and assume that $x_n \to x$ in $C(T,X)$, then for all $t \in T$, $x_n(t) \to x(t)$ and $V(x_n)(t) \to V(x)(t)$ in X. By virtue of hypothesis H(2′) (ii), $(x,y) \to d(u(t),F(t,x,y))$ is an upper semicontinuous R_+-valued function. So via Fatou's Lemma, we have

$$\begin{aligned} \lambda &\le \overline{\lim} d\left(u,N(x_n)\right) = \overline{\lim}\int_0^b d\left(u(t),F(t,x_n(t),V(x_n)(t))\right)\, dt \\ &\le \int_0^b \overline{\lim} d\left(u(t),F(t,x_n(t),V(x_n)(t))\right)\, dt \end{aligned}$$

$$\leq \int_0^b d\left(u(t), F(t, x(t), V(x)(t))\right) dt = d\left(u, N(x)\right).$$

Therefore $x \in U_\lambda$ and this proves the lower semicontinuity of $N(\cdot)$. This allows us to apply Theorem 3 of Bressan-Colombo (Ref. 31) and obtain a continuous map $r : C(T,X) \to L^1(T,X)$ such that $r(x) \in N(x)$ for every $x \in C(T,X)$. Consider the map $\pi : C(T,X) \to C(T,X)$ defined by

$$\pi(x)(t) = U(t,0)(x_0 - M(x)) + \gamma(r(x) + Bu_{r(x)}).$$

As in the the proof of Step 2-3 of Theorem 3.1, we can show that there exist n_0 such that $\pi(B_{n_0}) \subseteq B_{n_0}$ and $\pi(B_{n_0})$ is equicontinuous.

Let$A_1 = \overline{\text{conv}}\pi(B_{n_0})$, then A_1 is also equicontinuous. Consider the sequence defined by $A_{n+1} = \overline{\text{conv}}\pi(A_n)(n \geq 1)$, then $A_{n+1} \subset A_n$ and A_n is equicontinuous, closed and convex subset of $C(T,X)$ when $n \geq 1$. As in the proof of Step 4 of Theorem 3.1, $A_0 = \bigcap_{n=1}^{\infty} A_n$ is a nonempty, compact convex subset of $C(T,X)$ and $\pi(A_0) \subset A_0$.

Finally, we show that $\pi : A_0 \to A_0$ is continuous. Let $\{x_n : n \geq 1\} \subset A_0$ and $x_n \to x$ in $C(T,X)$. Then by definition

$$\pi(x_n)(t) = U(t,0)(x_0 - M(x)) + \gamma(r(x_n) + Bu_{r(x_n)}).$$

Since $r(\cdot)$ and B are continuous map, we only need to show that $u_{r(x_n)} \to u_{r(x)}$ in $L^2(T,U)$, as $n \to \infty$. Note that

$$u_{r(x_n)} = W^{-1}\left[x_1 - M(x_n) - U(b,0)(x_0 - M(x_n)) - \int_0^b U(b,s)r(x_n)ds\right],$$

by the continuity property of operators W^{-1}, M, $U(b,\cdot)$ and map $r(\cdot)$, then $u_{r(x_n)} \to u_{r(x)}$ in $L^2(T,U)$, as $n \to \infty$. So $\pi : A_0 \to A_0$ is continuous. Thus applying Schauder' fixed point theorem, there exist $x \in A_0$ such that $x = \pi(x)$. Then system (1) is nonlocally controllable on T.

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