Dynamic Systems and Applications 21 (2012) 251-268

FRACTIONAL FUNDAMENTAL LEMMA OF ORDER $\alpha \in (n-\frac{1}{2}, n)$ WITH $n \in \mathbb{N}$, $n \geq 2$

DARIUSZ IDCZAK AND MAREK MAJEWSKI

Faculty of Mathematics and Computer Science University of Lodz Banacha 22,
90-238 Lodz, Poland

ABSTRACT. In the paper, we derive a fractional fundamental lemma for functions of one variable with Riemann-Liouville derivatives of order $\alpha \in (n-\frac{1}{2}, n)$ where $n \in \mathbb{N}$, $n \geq 2$. To prove this lemma we derive a theorem on the integral representation of a function possessing the fractional derivative of order $\alpha > 0$ and a theorem on the fractional integration by parts of high order. Case of $n = 1$ is studied in [D. Idczak, *Fractional fundamental lemma of order* $\alpha \in (\frac{1}{2}, 1)$, to appear].

AMS (MOS) Subject Classification. 34A08, 49K15

1. INTRODUCTION

In the classical calculus of variations and in the variational theory of ordinary differential equations of high order the following result is known as the fundamental lemma or du Bois-Reymond lemma of order n (cf. [10]).

Lemma 1.1 (fundamental lemma of order n). *If* $a_n \in L^2([a,b], \mathbb{R}^m)$, $a_{n-1}, \ldots, a_0 \in L^1([a,b], \mathbb{R}^m)$ *and*

$$\int_a^b (a_n(t)D^n h(t) + \cdots + a_1(t)D^1 h(t) + a_0(t)h(t))dt = 0$$

for any function $h : [a,b] \to \mathbb{R}^m$ *which is absolutely continuous together with the classical derivatives* $D^1 h, \ldots, D^{(n-1)} h$*, such that* $D^{(n)} h \in L^2([a,b], \mathbb{R}^m)$ *and*

$$h(a) = D^1 h(a) = \cdots = D^{n-1} h(a) = 0,$$
$$h(b) = D^1 h(b) = \cdots = D^{n-1} h(b) = 0,$$

then the functions appearing below in the brackets are absolutely continuous and

$$D^1(\ldots(D^1(D^1((D^1(a_n) - a_{n-1}) + a_{n-2}) + \cdots + (-1)^{n-1} a_1) + (-1)^n a_0 = 0$$

a.e. on $[a,b]$.

The project was financed with funds of National Science Centre, granted on the basis of decision DEC-2011/01/B/ST7/03426.

Received November 15, 2011 1056-2176 $15.00 ©Dynamic Publishers, Inc.

Remark 1.2. More precisely, each of the above functions appearing in brackets is equal a.e. on $[a,b]$ to some absolutely continuous function. In the next (as in the lemma), we shall identify functions that are equal a.e. on $[a,b]$.

The aim of the paper is to derive a counterpart of Lemma 1.1 for the Riemann-Liouville fractional derivatives of order $\alpha \in (n-\frac{1}{2}, n)$ where $n \in \mathbb{N}$, $n \geq 2$. Case of $n = 1$ is studied in [6]. The presented paper contains an extension of the result obtained in [6] to the case of any $n \in \mathbb{N}$. Our investigations are based on a theorem on the integral representation of a function possessing fractional derivative of order $\alpha > 0$ and on a theorem on the fractional integration by parts for the Riemann-Liouville derivatives of high order, obtained in the paper. To the best knowledge of the authors the fractional fundamental lemma of high order nor theorem on the fractional integration by parts of high order nor theorem on the integral representation of a function possessing fractional derivatives of order $\alpha > 0$ have not been proved up to now. The obtained fundamental lemma can be used to study the Euler-Lagrange equations for functionals depending on fractional derivatives of high order. Some formulations of the Euler-Lagrange equations for such functionals have been given by other Authors (cf. [1], [2], [5], [8]) but as we read in [3]: "*For a given Lagrangian, there are several proposed methods to obtain the fractional Euler-Lagrange equations and the corresponding Hamiltonians. However, this issue is not yet completely clarified and it requires more further detailed analysis.*"

The paper is organized as follows. In section 2 we recall some definitions and facts from the fractional differential calculus. In section 3 we derive a theorem on the integral representation of a function possessing the fractional derivative of order $\alpha > 0$ and in section 4 - a theorem on the integration by parts for the Riemann-Liouville derivatives of high order. Section 5 is devoted to some technical result which is used in the proof of the main result. In section 6 we give the proof of a fractional fundamental lemma and some its generalizations.

2. PRELIMINARIES

Let $n \in \mathbb{N}$, $n \geq 2$. By $AC^n([a,b],\mathbb{R}^m)$ we denote the space of all functions $f : [a,b] \to \mathbb{R}^m$ that are absolutely continuous together with the classical derivatives $D^1 f, \ldots, D^{n-1} f$. It is known that $f \in AC^n([a,b],\mathbb{R}^m)$ if and only if there exist constants c_0, $c_1, \ldots, c_{n-1} \in \mathbb{R}^m$ and a function $\varphi \in L^1([a,b],\mathbb{R}^m)$ such that

$$f(t) = c_0 + \frac{c_1}{1!}(t-a) + \cdots + \frac{c_{n-1}}{(n-1)!}(t-a)^{n-1} + \int_a^t \int_a^{t_1} \ldots \int_a^{t_{n-1}} \varphi(\tau) d\tau dt_{n-1} \ldots dt_1, \ t \in [a,b].$$

Consequently, if $f \in AC^n([a,b],\mathbb{R}^m)$, then

$$f(a) = c_0,\ D^1 f(a) = c_1, \dots,\ D^{n-1} f(a) = c_{n-1},$$
$$D^n f(t) = \varphi(t),\ t \in [a,b] \text{ a.e.}$$

By $AC_0^n([a,b],\mathbb{R}^m)$ we denote the space of all functions $f \in AC^n([a,b],\mathbb{R}^m)$ that satisfy the conditions

$$f(a) = 0,\ D^1 f(a) = 0, \dots,\ D^{n-1} f(a) = 0.$$

Of course, $f \in AC_0^n([a,b],\mathbb{R}^m)$ if and only if there exists a function $\varphi \in L^1([a,b],\mathbb{R}^m)$ such that

$$f(t) = \int_a^t \int_a^{t_1} \dots \int_a^{t_{n-1}} \varphi(\tau) d\tau dt_{n-1} \dots dt_1,\ t \in [a,b]. \tag{2.1}$$

By $AC^{n,p}([a,b],\mathbb{R}^m)$, $AC_0^{n,p}([a,b],\mathbb{R}^m)$ $(1 \le p < \infty)$ we denote the spaces of functions possessing the appropriate representations with $\varphi \in L^p([a,b],\mathbb{R}^m)$; of course, $AC^{n,1}([a,b],\mathbb{R}^m) = AC^n([a,b],\mathbb{R}^m)$, $AC_0^{n,1}([a,b],\mathbb{R}^m) = AC_0^n([a,b],\mathbb{R}^m)$.

Let $\alpha > 0$, $\varphi \in L^1([a,b],\mathbb{R}^m)$. By a left-sided Riemann-Liouville fractional integral of φ on the interval $[a,b]$ we mean a function $I_{a+}^\alpha \varphi$ given by

$$(I_{a+}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}} d\tau,\ t \in [a,b] \text{ a.e.}$$

Analogously, by a right-sided Riemann-Liouville fractional integral of φ on the interval $[a,b]$ we mean a function $I_{b-}^\alpha f$ given by

$$(I_{b-}^\alpha \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{\varphi(\tau)}{(\tau-t)^{1-\alpha}} d\tau,\ t \in [a,b] \text{ a.e.}$$

We shall use

Theorem 2.1. *(a) If $\alpha > 0$ and $p \ge 1$, then $I_{a+}^\alpha \varphi \in L^p([a,b],\mathbb{R}^m)$ for any $\varphi \in L^p([a,b],\mathbb{R}^m)$*

(b) If $\alpha > 0$, $p \ge 1$ and $p > \frac{1}{\alpha}$, then the function $I_{a+}^\alpha \varphi$ is continuous on $[a,b]$ for any $\varphi \in L^p([a,b],\mathbb{R}^m)$.

Remark 2.2. Theorem 2.1(a) follows from [9, Th. 2.6], Theorem 2.1(b) can be deduced from the results obtained in [4] (cf. also [9, Th. 3.6]). Analogous theorem holds true for the right-sided integral.

In [9, Corollary of Th. 3.5] (cf. also [7, Lemma 2.7 (a)]) the following theorem on the integration by parts is given.

Theorem 2.3. *If $\alpha > 0$ and $p \ge 1$, $q \ge 1$ and $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ (additionally, we assume that $p > 1$ and $q > 1$ when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$), then*

$$\int_a^b \varphi(t)(I_{a+}^\alpha \psi)(t) dt = \int_a^b (I_{b-}^\alpha \varphi)(t) \psi(t) dt.$$

for $\varphi \in L^p([a,b],\mathbb{R}^m)$, $\psi \in L^q([a,b],\mathbb{R}^m)$.

Now, let $n-1<\alpha<n$ for some $n\in\mathbb{N}$, $f\in L^1([a,b],\mathbb{R}^m)$. We say that f possesses the left-sided Riemann-Liouville derivative $D^\alpha_{a+}f$ of order α on the interval $[a,b]$ if $I^{n-\alpha}_{a+}f\in AC^n([a,b],\mathbb{R}^m)$. By this derivative we mean the derivative $D^n(I^{n-\alpha}_{a+}f)$, i.e.

$$(D^\alpha_{a+}f)(t)=\frac{1}{\Gamma(n-\alpha)}D^n(\int_a^t\frac{f(\tau)}{(t-\tau)^{1-n+\alpha}}d\tau)(t),\ t\in[a,b] \text{ a.e.}$$

Similarly, we say that f possesses the right-sided Riemann-Liouville derivative $D^\alpha_{b-}f$ of order α on the interval $[a,b]$ if $I^{n-\alpha}_{b-}f\in AC^n([a,b],\mathbb{R}^m)$. By this derivative we mean the function $(-1)^nD^{(n)}(I^{n-\alpha}_{b-}f)$, i.e.

$$(D^\alpha_{b-}f)(t)=\frac{1}{\Gamma(n-\alpha)}(-1)^nD^{(n)}(\int_t^b\frac{f(\tau)}{(\tau-t)^{1-n+\alpha}}d\tau)(t),\ t\in[a,b] \text{ a.e.}$$

Additionally, we put $D^0_{a+}f=f$, $D^0_{b-}f=f$.

Theorem 2.3 implies the next theorem (cf. [9, Corollary 2 of Th. 2.4] and also [7, Lemma 2.7 (b)]).

Theorem 2.4. *If* $\alpha>0$, $p\ge1$, $q\ge1$ *and* $\frac{1}{p}+\frac{1}{q}\le 1+\alpha$ *(additionally, we assume that* $p>1$ *and* $q>1$ *when* $\frac{1}{p}+\frac{1}{q}=1+\alpha$*), then*

$$\int_a^b f(t)(D^\alpha_{b-}g)(t)dt=\int_a^b(D^\alpha_{a+}f)(t)g(t)dt.$$

for $f\in I^\alpha_{a+}(L^p([a,b],\mathbb{R}^m))$, $g\in I^\alpha_{b-}(L^q([a,b],\mathbb{R}^m))$.

Remark 2.5. The previous theorem is an analogue of the following classical one: if the functions $g,\ h:[a,b]\to\mathbb{R}$ belongs to $AC^n([a,b],\mathbb{R}^m)$ and

$$g(b)D^{n-1}h(b)=D^1g(b)D^{n-2}h(b)=\cdots=D^{n-1}g(b)h(b)=0,$$

$$g(a)D^{n-1}h(a)=D^1g(a)D^{n-2}h(a)=\cdots=D^{n-1}g(a)h(a)=0,$$

then

$$\int_a^b g(t)D^nh(t)dt=(-1)^n\int_a^b D^ng(t)h(t)dt.$$

This classical result follows from the following more general theorem: if the functions $g,\ h:[a,b]\to\mathbb{R}$ belongs to $AC^n([a,b],\mathbb{R}^m)$, then

$$\int_a^b g(t)D^nh(t)dt=\sum_{k=1}^n(-1)^{k-1}D^{k-1}g(b)D^{n-k}h(b)$$
$$-\sum_{k=1}^n(-1)^{k-1}D^{k-1}g(a)D^{n-k}h(a)+(-1)^n\int_a^b D^ng(t)h(t)dt.$$

3. INTEGRAL REPRESENTATION

We shall prove

Theorem 3.1. *If $n-1<\alpha<n$, $n\in\mathbb{N}$, $n\geq 2$, and $f\in L^1([a,b],\mathbb{R}^m)$, then f has the left-sided Riemann-Liouville derivative $D^\alpha_{a+}f$ of order α on the interval $[a,b]$ if and only if there exist constants c_0, c_1,...,$c_{n-1}\in\mathbb{R}^m$ and a function $\varphi\in L^1([a,b],\mathbb{R}^m)$ such that*

$$f(t)=\frac{c_0}{\Gamma(\alpha-n+1)}(t-a)^{\alpha-n}+\frac{c_1}{\Gamma(\alpha-n+2)}(t-a)^{\alpha-n+1}+\dots+\frac{c_{n-1}}{\Gamma(\alpha)}(t-a)^{\alpha-1}+I^\alpha_{a+}\varphi(t),\ t\in[a,b]\ a.e. \tag{3.1}$$

In such a case

$$D^0(I^{n-\alpha}_{a+}f)(a)=c_0,\ D^1(I^{n-\alpha}_{a+}f)(a)=c_1,\dots,D^{n-1}(I^{n-\alpha}_{a+}f)(a)=c_{n-1}$$

and

$$(D^\alpha_{a+}f)(t)=\varphi(t),\ t\in[a,b]\ a.e.$$

Proof. Let us assume that f has the left-sided Riemann-Liouville derivative $D^\alpha_{a+}f$ of order α on the interval $[a,b]$, i.e. $I^{n-\alpha}_{a+}f\in AC^n([a,b],\mathbb{R}^m)$. This means that there exist constants c_0, c_1,...,$c_{n-1}\in\mathbb{R}^m$ and a function $\varphi\in L^1([a,b],\mathbb{R}^m)$ such that

$$I^{n-\alpha}_{a+}f(t)=c_0+\frac{c_1}{1!}(t-a)+\dots+\frac{c_{n-1}}{(n-1)!}(t-a)^{n-1}+\int_a^t\int_a^{t_1}\dots\int_a^{t_{n-1}}\varphi(\tau)d\tau dt_{n-1}\dots dt_1,\ t\in[a,b].$$

Of course, in such a case

$$D^0(I^{n-\alpha}_{a+}f)(a)=c_0,\ D^1(I^{n-\alpha}_{a+}f)(a)=c_1,\dots,\ D^{n-1}(I^{n-\alpha}_{a+}f)(a)=c_{n-1} \tag{3.2}$$

and

$$(D^\alpha_{a+}f)(t)=D^n(I^{n-\alpha}_{a+}f)(t)=\varphi(t),\ t\in[a,b]\ \text{a.e.}$$

Let us define on (a,b) a function

$$g(t)=\frac{c_0}{\Gamma(\alpha-n+1)}(t-a)^{\alpha-n}+\frac{c_1}{\Gamma(\alpha-n+2)}(t-a)^{\alpha-n+1}+\dots+\frac{c_{n-1}}{\Gamma(\alpha)}(t-a)^{\alpha-1}.$$

From [7, formula (2.1.16)] it follows that

$$\begin{aligned}I^{n-\alpha}_{a+}((\cdot-a)^{\alpha-n})(t)&=I^{n-\alpha}_{a+}((\cdot-a)^{\alpha-n+1-1})(t)=\tfrac{\Gamma(\alpha-n+1)}{\Gamma(1)}(t-a)^0,\\ I^{n-\alpha}_{a+}((\cdot-a)^{\alpha-n+1})(t)&=I^{n-\alpha}_{a+}((\cdot-a)^{\alpha-n+2-1})(t)=\tfrac{\Gamma(\alpha-n+2)}{\Gamma(2)}(t-a)^1,\\ &\vdots\\ I^{n-\alpha}_{a+}((\cdot-a)^{\alpha-1})(t)&=\tfrac{\Gamma(\alpha)}{\Gamma(n)}(t-a)^{n-1}.\end{aligned} \tag{3.3}$$

So,

$$I_{a+}^{n-\alpha}g(t) = \frac{c_0}{\Gamma(1)}(t-a)^0 + \frac{c_1}{\Gamma(2)}(t-a)^1 + \cdots + \frac{c_{n-1}}{\Gamma(n)}(t-a)^{n-1}. \tag{3.4}$$

Consequently,

$$I_{a+}^{n-\alpha}f(t) = I_{a+}^{n-\alpha}(f-g)(t) + \frac{c_0}{\Gamma(1)}(t-a)^0 + \frac{c_1}{\Gamma(2)}(t-a)^1 + \cdots + \frac{c_{n-1}}{\Gamma(n)}(t-a)^{n-1}.$$

Since $I_{a+}^{n-\alpha}f,\ I_{a+}^{n-\alpha}g \in AC^n([a,b],\mathbb{R}^m)$, therefore $I_{a+}^{n-\alpha}(f-g) \in AC^n([a,b],\mathbb{R}^m)$. Of course (cf. (3.2), (3.4)),

$$D^0(I_{a+}^{n-\alpha}(f-g))(a) = 0,\ D^1(I_{a+}^{n-\alpha}(f-g))(a) = 0, \ldots,$$
$$D^{n-1}(I_{a+}^{n-\alpha}(f-g))(a) = 0.$$

This means that $I_{a+}^{n-\alpha}(f-g) \in AC_0^n([a,b],\mathbb{R}^m)$, i.e. (cf. (2.1)) there exists a function $\varphi \in L^1([a,b],\mathbb{R}^m)$ such that

$$(I_{a+}^{n-\alpha}(f-g))(t) = \int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} \varphi(\tau)d\tau dt_{n-1}\ldots dt_1 = (I_{a+}^n\varphi)(t),$$
$$t \in [a,b].$$

Thus,

$$(I_{a+}^{n-\alpha}f)(t) = (I_{a+}^{n-\alpha}g)(t) + (I_{a+}^n\varphi)(t) = (I_{a+}^{n-\alpha}g)(t) + (I_{a+}^{n-\alpha}I_{a+}^{\alpha}\varphi)(t),$$
$$t \in [a,b] \text{ a.e.}$$

So,

$$\begin{aligned} f(t) &= D_{a+}^{n-\alpha}(I_{a+}^{n-\alpha}f)(t) = D_{a+}^{n-\alpha}(I_{a+}^{n-\alpha}g)(t) + D_{a+}^{n-\alpha}(I_{a+}^{n-\alpha}I_{a+}^{\alpha}\varphi)(t) \\ &= g(t) + (I_{a+}^{\alpha}\varphi)(t) = \frac{c_0}{\Gamma(\alpha-n+1)}(t-a)^{\alpha-n} + \frac{c_1}{\Gamma(\alpha-n+2)}(t-a)^{\alpha-n+1} \\ &\quad + \cdots + \frac{c_{n-1}}{\Gamma(\alpha)}(t-a)^{\alpha-1} + (I_{a+}^{\alpha}\varphi)(t),\ t \in [a,b] \text{ a.e.} \end{aligned}$$

Now, let us assume that f has the representation (3.1). From (3.3) and from the fact that

$$(I_{a+}^{n-\alpha}I_{a+}^{\alpha}\varphi)(t) = (I_{a+}^n\varphi)(t) = \int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} \varphi(\tau)d\tau dt_{n-1}\ldots dt_1,$$
$$t \in [a,b] \text{ a.e.}$$

it follows that $I_{a+}^{n-\alpha}f$ belongs to $AC^n([a,b],\mathbb{R}^m)$. □

Remark 3.2. The above theorem can be also deduced from [7, Corollary 2.1, Lemma 2.5 (b), Lemma 2.6 (b)] but, to our best knowledge, it has not been formulated up to now.

By $AC^{\alpha,p}_{a+}([a,b],\mathbb{R}^m)$ with $n-1<\alpha<n$, $n\in\mathbb{N}$, $n\geq 2$, we denote the set of all functions $f:[a,b]\to\mathbb{R}^m$ that have representation (3.1) with $\varphi\in L^p([a,b],\mathbb{R}^m)$.

Remark 3.3. The space $AC^{\alpha,1}_{a+}([a,b],\mathbb{R}^m)$ can be treated as a fractional counterpart of the space $AC^n([a,b],\mathbb{R}^m)$.

In an analogous way one can prove

Theorem 3.4. *If $n-1<\alpha<n$, $n\in\mathbb{N}$, $n\geq 2$, and $g\in L^1([a,b],\mathbb{R}^m)$, then g has the left-sided Riemann-Liouville derivative $D^{\alpha}_{a+}g$ of order α on the interval $[a,b]$ if and only if there exist constants d_0, d_1,…,$d_{n-1}\in\mathbb{R}^m$ and a function $\psi\in L^1([a,b],\mathbb{R}^m)$ such that*

$$
g(t)=\frac{d_0}{\Gamma(\alpha-n+1)}(b-t)^{\alpha-n}+\frac{d_1}{\Gamma(\alpha-n+2)}(b-t)^{\alpha-n+1}+\cdots+\frac{d_{n-1}}{\Gamma(\alpha)}(b-t)^{\alpha-1}+I^{\alpha}_{b-}\psi(\tau),\ \ t\in[a,b]\ \text{a.e.} \tag{3.5}
$$

In such a case

$$
D^0(I^{n-\alpha}_{b-}g)(b)=d_0,\ \ D^1(I^{n-\alpha}_{b-}g)(b)=-d_1,\dots,D^{n-1}(I^{n-\alpha}_{b-}g)(b)=(-1)^{n-1}d_{n-1}
$$

and

$$
(D^{\alpha}_{b-}g)(t)=\psi(t),\ \ t\in[a,b]\ \text{a.e.}
$$

By $AC^{\alpha,q}_{b-}([a,b],\mathbb{R}^m)$ with $n-1<\alpha<n$, $n\in\mathbb{N}$, $n\geq 2$, we denote the set of all functions $g:[a,b]\to\mathbb{R}^m$ that have representation (3.5) with $\psi\in L^q([a,b],\mathbb{R}^m)$.

4. **INTEGRATION BY PARTS**

The following fractional counterpart of the second theorem formulated in Remark 2.5 holds true.

Theorem 4.1. *If $n-1<\alpha<n$, $n\in\mathbb{N}$, $n\geq 2$, $n-\alpha<1-\frac{1}{p}$, $n-\alpha<1-\frac{1}{q}$, then*

$$
\begin{aligned}
\int_a^b f(t)(D^{\alpha}_{b-}g)(t)dt&=\sum_{i=0}^{n-1}(D^iI^{n-\alpha}_{a+}f)(a)(I^{\alpha-n+1+i}_{b-}D^{\alpha}_{b-}g)(a)\\
&\quad-\sum_{i=0}^{n-1}(-1)^i(D^iI^{n-\alpha}_{b-}g)(b)(I^{\alpha-n+1+i}_{a+}D^{\alpha}_{a+}f)(b)+\int_a^b(D^{\alpha}_{a+}f)(t)g(t)dt
\end{aligned}
$$

for $f\in AC^{\alpha,p}_{a+}([a,b],\mathbb{R}^m)$, $g\in AC^{\alpha,q}_{b-}([a,b],\mathbb{R}^m)$.

Proof. Let

$$
f(t)=\frac{c_0}{\Gamma(\alpha-n+1)(t-a)^{n-\alpha}}+\frac{c_1}{\Gamma(\alpha-n+2)}(t-a)^{\alpha-n+1}+\cdots+\frac{c_{n-1}}{\Gamma(\alpha)}(t-a)^{\alpha-1}+I^{\alpha}_{a+}\varphi(\tau)d\tau,
$$

$$g(t) = \frac{d_0}{\Gamma(\alpha-n+1)(b-t)^{n-\alpha}} + \frac{d_1}{\Gamma(\alpha-n+2)}(b-t)^{\alpha-n+1} + \cdots + \frac{d_{n-1}}{\Gamma(\alpha)}(b-t)^{\alpha-1} + I^{\alpha}_{b-}\psi(\tau)d\tau,$$

where $c_0,\dots,c_{n-1},d_0,\dots,d_{n-1} \in \mathbb{R}^m$, $\varphi \in L^p([a,b],\mathbb{R}^m)$, $\psi \in L^q([a,b],\mathbb{R}^m)$. We have (the third integral given below exists because $n-\alpha < 1-\frac{1}{p}$; integral $\int_a^b (I^{\alpha}_{a+}\varphi)(t)\psi(t)dt$ exists by Theorem 2.1 (b))

$$\begin{aligned}\int_a^b f(t)(D^{\alpha}_{b-}g)(t)dt &= \int_a^b f(t)\psi(t)dt \\ &= \frac{c_0}{\Gamma(\alpha-n+1)}\int_a^b \frac{\psi(t)}{(t-a)^{n-\alpha}}dt + \frac{c_1}{\Gamma(\alpha-n+2)}\int_a^b (t-a)^{\alpha-n+1}\psi(t)dt \\ &\quad + \cdots + \frac{c_{n-1}}{\Gamma(\alpha)}\int_a^b (t-a)^{\alpha-1}\psi(t)dt + \int_a^b (I^{\alpha}_{a+}\varphi)(t)\psi(t)dt \\ &= c_0(I^{\alpha-n+1}_{b-}\psi)(a) + c_1(I^{\alpha-n+2}_{b-}\psi)(a) + \cdots + c_{n-1}(I^{\alpha}_{b-}\psi)(a) \\ &\quad + \int_a^b (I^{\alpha}_{a+}\varphi)(t)\psi(t)dt.\end{aligned}$$

Similarly,

$$\int_a^b (D^{\alpha}_{a+}f)(t)g(t)dt = d_0(I^{\alpha-n+1}_{a+}\varphi)(b) + d_1(I^{\alpha-n+2}_{a+}\varphi)(b) + \dots + d_{n-1}(I^{\alpha}_{a+}\varphi)(b) + \int_a^b \varphi(t)(I^{\alpha}_{b-}\psi)(t)dt.$$

Since (cf. Theorem 2.3)

$$\int_a^b (I^{\alpha}_{a+}\varphi)(t)\psi(t)dt = \int_a^b \varphi(t)(I^{\alpha}_{b-}\psi)(t)dt,$$

therefore

$$\begin{aligned}\int_a^b f(t)(D^{\alpha}_{b-}g)(t)dt &= \sum_{i=0}^{n-1} c_i(I^{\alpha-n+1+i}_{b-}\psi)(a) - \sum_{i=0}^{n-1} d_i(I^{\alpha-n+1+i}_{a+}\varphi)(b) \\ &\quad + \int_a^b (D^{\alpha}_{a+}f)(t)g(t)dt = \sum_{i=0}^{n-1}(D^i(I^{n-\alpha}_{a+}f)(a))(I^{\alpha-n+1+i}_{b-}D^{\alpha}_{b-}g)(a) \\ &\quad - \sum_{i=0}^{n-1}(-1)^i(D^i(I^{n-\alpha}_{b-}g)(b))(I^{\alpha-n+1+i}_{a+}D^{\alpha}_{a+}f)(b) + \int_a^b (D^{\alpha}_{a+}f)(t)g(t)dt\end{aligned}$$

and the proof is completed. □

5. ON SOME DETERMINANT

Let us denote

$$W_{\alpha,n}=\begin{bmatrix} \frac{1}{\Gamma(\alpha-n+1)}I_{a+}^{\alpha-n+1}((b-\cdot)^{\alpha-n})(b) & \dots & \frac{1}{\Gamma(\alpha)}I_{a+}^{\alpha-n+1}((b-\cdot)^{\alpha-1})(b)\\ \frac{1}{\Gamma(\alpha-n+1)}I_{a+}^{\alpha-n+2}((b-\cdot)^{\alpha-n})(b) & \dots & \frac{1}{\Gamma(\alpha)}I_{a+}^{\alpha-n+2}((b-\cdot)^{\alpha-1})(b)\\ \vdots & & \vdots\\ \frac{1}{\Gamma(\alpha-n+1)}I_{a+}^{\alpha-1}((b-\cdot)^{\alpha-n})(b) & \dots & \frac{1}{\Gamma(\alpha)}I_{a+}^{\alpha-1}((b-\cdot)^{\alpha-1})(b)\\ \frac{1}{\Gamma(\alpha-n+1)}I_{a+}^{\alpha}((b-\cdot)^{\alpha-n})(b) & \dots & \frac{1}{\Gamma(\alpha)}I_{a+}^{\alpha}((b-\cdot)^{\alpha-1})(b) \end{bmatrix}$$

where $\alpha\in(n-\frac{1}{2},n)$, $n\in\mathbb{N}$, $n\geq 2$ (assumptions on α guarantee that the entries of the above $n\times n$ dimensional matrix are well defined).

In the proof of the main result of the paper the following proposition plays the fundamental role.

Proposition 5.1. *The matrix $W_{\alpha,n}$ is nonsingular.*

Proof. Clearly, our theorem holds true for $n=2$ because

$$\det W_{\alpha,2}=\frac{1}{(\Gamma(\alpha-1)\Gamma(\alpha))^2}\det\begin{bmatrix} \frac{(b-a)^{2\alpha-3}}{2\alpha-3} & \frac{(b-a)^{2\alpha-2}}{2\alpha-2}\\ \frac{(b-a)^{2\alpha-2}}{2\alpha-2} & \frac{(b-a)^{2\alpha-1}}{2\alpha-1} \end{bmatrix}.$$

So, let us assume that $n\geq 3$. It is easy to see that determinant of $W_{\alpha,n}$ is different from zero if and only if determinant of the matrix

$$\begin{bmatrix} \int_a^b(b-\tau)^{2\alpha-2n}d\tau & \int_a^b(b-\tau)^{2\alpha-(2n-1)}d\tau & \dots & \int_a^b(b-\tau)^{2\alpha-(n+1)}d\tau\\ \int_a^b(b-\tau)^{2\alpha-(2n-1)}d\tau & \int_a^b(b-\tau)^{2\alpha-(2n-2)}d\tau & \dots & \int_a^b(b-\tau)^{2\alpha-n}d\tau\\ \vdots & \vdots & & \vdots\\ \int_a^b(b-\tau)^{2\alpha-(n+2)}d\tau & \int_a^b(b-\tau)^{2\alpha-(n+1)}d\tau & \dots & \int_a^b(b-\tau)^{2\alpha-3}d\tau\\ \int_a^b(b-\tau)^{2\alpha-(n+1)}d\tau & \int_a^b(b-\tau)^{2\alpha-n}d\tau & \dots & \int_a^b(b-\tau)^{2\alpha-2}d\tau \end{bmatrix}$$

$$=\begin{bmatrix} \frac{(b-a)^{2\alpha-2n+1}}{2\alpha-2n+1} & \frac{(b-a)^{2\alpha-(2n-1)+1}}{2\alpha-(2n-1)+1} & \dots & \frac{(b-a)^{2\alpha-(n+1)+1}}{2\alpha-(n+1)+1}\\ \frac{(b-a)^{2\alpha-(2n-1)+1}}{2\alpha-(2n-1)+1} & \frac{(b-a)^{2\alpha-(2n-2)+1}}{2\alpha-(2n-2)+1} & \dots & \frac{(b-a)^{2\alpha-n+1}}{2\alpha-n+1}\\ \vdots & \vdots & & \vdots\\ \frac{(b-a)^{2\alpha-(n+2)+1}}{2\alpha-(n+2)+1} & \frac{(b-a)^{2\alpha-(n+1)+1}}{2\alpha-(n+1)+1} & \dots & \frac{(b-a)^{2\alpha-3+1}}{2\alpha-3+1}\\ \frac{(b-a)^{2\alpha-(n+1)+1}}{2\alpha-(n+1)+1} & \frac{(b-a)^{2\alpha-n+1}}{2\alpha-n+1} & \dots & \frac{(b-a)^{2\alpha-2+1}}{2\alpha-2+1} \end{bmatrix}.$$

is nonzero. Now, we multiply $(n-1)$-th row by $(b-a)^1$, $(n-2)$-th row by $(b-a)^2$, $\dots$, first row by $(b-a)^{n-1}$ and next we divide the n-th column by $(b-a)^{2\alpha-2+1}$, $(n-1)$-th column by $(b-a)^{2\alpha-3+1}$, $\dots$,first column $(b-a)^{2\alpha-(n+1)+1}$. In such a way we obtain the following matrix

$$\begin{bmatrix} \frac{1}{\gamma} & \frac{1}{\gamma+1} & \cdots & \frac{1}{\gamma+(n-1)} \\ \frac{1}{\gamma+1} & \frac{1}{\gamma+2} & \cdots & \frac{1}{\gamma+n} \\ \vdots & \vdots & & \vdots \\ \frac{1}{\gamma+(n-2)} & \frac{1}{\gamma+(n-1)} & \cdots & \frac{1}{\gamma+(2n-3)} \\ \frac{1}{\gamma+(n-1)} & \frac{1}{\gamma+n} & \cdots & \frac{1}{\gamma+(2n-2)} \end{bmatrix}$$

where $\gamma = 2\alpha - 2n + 1$. Of course, matrix $W_{\alpha,n}$ is nonsingular if and only if the last one is.

Let us multiply the first row by γ, the second row by $\gamma + 1$, $\ldots$, n-th row by $\gamma + (n-1)$. We obtain

$$\begin{bmatrix} 1 & \frac{\gamma}{\gamma+1} & \cdots & \frac{\gamma}{\gamma+(n-1)} \\ 1 & \frac{\gamma+1}{\gamma+2} & \cdots & \frac{\gamma+1}{\gamma+n} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{\gamma+(n-2)}{\gamma+(n-1)} & \cdots & \frac{\gamma+(n-2)}{\gamma+(2n-3)} \\ 1 & \frac{\gamma+(n-1)}{\gamma+n} & \cdots & \frac{\gamma+(n-1)}{\gamma+(2n-2)} \end{bmatrix}.$$

Since the second column can be written down as

$$\begin{bmatrix} \frac{\gamma}{\gamma+1} \\ \frac{\gamma+1}{\gamma+2} \\ \vdots \\ \frac{\gamma+(n-2)}{\gamma+(n-1)} \\ \frac{\gamma+(n-1)}{\gamma+n} \end{bmatrix} = \begin{bmatrix} \frac{\gamma+1-1}{\gamma+1} \\ \frac{\gamma+1+1-1}{\gamma+2} \\ \vdots \\ \frac{\gamma+(n-2)+1-1}{\gamma+(n-1)} \\ \frac{\gamma+(n-1)+1-1}{\gamma+n} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{-1}{\gamma+1} \\ \frac{-1}{\gamma+2} \\ \vdots \\ \frac{-1}{\gamma+(n-1)} \\ \frac{-1}{\gamma+n} \end{bmatrix},$$

therefore $\det W_{\alpha,n} \neq 0$ if and only if

$$\det \begin{bmatrix} 1 & \frac{1}{\gamma+1} & \cdots & \frac{\gamma}{\gamma+(n-1)} \\ 1 & \frac{1}{\gamma+2} & \cdots & \frac{\gamma+1}{\gamma+n} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{1}{\gamma+(n-1)} & \cdots & \frac{\gamma+(n-2)}{\gamma+(2n-3)} \\ 1 & \frac{1}{\gamma+n} & \cdots & \frac{\gamma+(n-1)}{\gamma+(2n-2)} \end{bmatrix} \neq 0.$$

Working in the similar way with the remaining columns we assert that $\det W_{\alpha,n} \neq 0$ if and only if

$$\det \begin{bmatrix} 1 & \frac{1}{\gamma+1} & \cdots & \frac{1}{\gamma+(n-1)} \\ 1 & \frac{1}{\gamma+2} & \cdots & \frac{1}{\gamma+n} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{1}{\gamma+(n-1)} & \cdots & \frac{1}{\gamma+(2n-3)} \\ 1 & \frac{1}{\gamma+n} & \cdots & \frac{1}{\gamma+(2n-2)} \end{bmatrix} \neq 0.$$

Now, we multiply the first row by $\gamma+1$, the second row - by $\gamma+2$, ..., the last row - by $\gamma+n$ and repeat the previous procedure to obtain the matrix

$$\begin{bmatrix} \gamma+1 & 1 & \frac{1}{\gamma+2} & \cdots & \frac{1}{\gamma+(n-1)} \\ \gamma+2 & 1 & \frac{1}{\gamma+3} & \cdots & \frac{1}{\gamma+n} \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma+(n-1) & 1 & \frac{1}{\gamma+n} & \cdots & \frac{1}{\gamma+(2n-3)} \\ \gamma+n & 1 & \frac{1}{\gamma+(n+1)} & \cdots & \frac{1}{\gamma+(2n-2)} \end{bmatrix},$$

etc. In effect, the matrix $W_{\alpha,n}$ is nonsingular if and only if the matrix

$$\begin{bmatrix} (\gamma+1)\ldots(\gamma+n) & \ldots & (\gamma+(n-1))(\gamma+n) & (\gamma+n) \\ (\gamma+2)\ldots(\gamma+(n+1)) & \ldots & (\gamma+n)(\gamma+(n+1)) & (\gamma+(n+1)) \\ \vdots & & \vdots & \vdots \\ (\gamma+(n-1))\ldots(\gamma+(2n-2)) & \ldots & (\gamma+(2n-3))(\gamma+(2n-2)) & (\gamma+(2n-2)) \\ (\gamma+n)\ldots(\gamma+(2n-1)) & \ldots & (\gamma+(2n-2)(\gamma+(2n-1)) & (\gamma+(2n-1)) \end{bmatrix}$$

is nonsingular. If we divide the first row by $(\gamma+1)\ldots(\gamma+n)$, the second row - by $(\gamma+2)\ldots(\gamma+(n+1))$, ..., the last row - by $(\gamma+n)\ldots(\gamma+(2n-1))$, we obtain the matrix

$$\begin{bmatrix} 1 & \frac{1}{\delta-2n+1} & \cdots & \frac{1}{(\delta-2n+1)\ldots(\delta-2n+(n-2))} & \frac{1}{(\delta-2n+1)\ldots(\delta-2n+(n-1))} \\ 1 & \frac{1}{\delta-2n+2} & \cdots & \frac{1}{(\delta-2n+2)\ldots(\delta-2n+(n-1))} & \frac{1}{(\delta-2n+2)\ldots(\delta-2n+n)} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \frac{1}{\delta-2n+(n-1)} & \cdots & \frac{1}{(\delta-2n+(n-1))\ldots(\delta-2n+(2n-4))} & \frac{1}{(\delta-2n+(n-1))\ldots(\delta-2n+(2n-3))} \\ 1 & \frac{1}{\delta-2n+n} & \cdots & \frac{1}{(\delta-2n+n)\ldots(\delta-2n+(2n-3))} & \frac{1}{(\delta-2n+n)\ldots(\delta-2n+(2n-2))} \end{bmatrix}$$

where $\delta=\gamma+2n=2\alpha-2n+1+2n=2\alpha+1$.

Let us denote

$$Z_k = \begin{bmatrix} 1 & \frac{1}{\delta-2k+1} & \cdots & \frac{1}{(\delta-2k+1)\ldots(\delta-2k+(k-2))} & \frac{1}{(\delta-2k+1)\ldots(\delta-2k+(k-1))} \\ 1 & \frac{1}{\delta-2k+2} & \cdots & \frac{1}{(\delta-2k+2)\ldots(\delta-2k+(k-1))} & \frac{1}{(\delta-2k+2)\ldots(\delta-2k+k)} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \frac{1}{\delta-2k+(k-1)} & \cdots & \frac{1}{(\delta-2k+(k-1))\ldots(\delta-2k+(2k-4))} & \frac{1}{(\delta-2k+(k-1))\ldots(\delta-2k+(2k-3))} \\ 1 & \frac{1}{\delta-2k+k} & \cdots & \frac{1}{(\delta-2k+k)\ldots(\delta-2k+(2k-3))} & \frac{1}{(\delta-2k+k)\ldots(\delta-2k+(2k-2))} \end{bmatrix}$$

for $k=2,\ldots,n$. Of course, to show that the matrix $W_{\alpha,n}$ has determinant different from zero it is sufficient to show that $\det Z_n \neq 0$. To this end, first we shall show that $\det Z_2 \neq 0$ and next we shall show that the following implication holds true

$$\det Z_{k-1} \neq 0 \Rightarrow \det Z_k \neq 0$$

for any $k=3,\ldots,n$.

Indeed,

$$Z_2 = \begin{bmatrix} 1 & \frac{1}{\delta-4+1} \\ 1 & \frac{1}{\delta-4+2} \end{bmatrix},$$

so $\det Z_2 \neq 0$. Moreover, we have (below, we write $a \approx b$ for $a, b \in \mathbb{R}$ when $a = b = 0$ or $ab \neq 0$)

$$\det Z_k \approx \det Z_{k-1}$$

for any $k = 3, \ldots, n$. Really, let us fix $k = 3,\ldots,n$ and replace in the matrix Z_k the k-th row by the difference of k-th row and $(k-1)$-th row, the $(k-1)$-th row - by the difference of $(k-1)$-th row and $(k-2)$-th row, ..., the second row - by the difference of the second row and the first one. Then

$$\det Z_k = \det \begin{bmatrix} 1 & \frac{1}{\delta-2k+1} & \cdots & \frac{1}{(\delta-2k+1)\ldots(\delta-2k+(k-1))} \\ 0 & \frac{-1}{(\delta-2k+1)(\delta-2k+2)} & \cdots & \frac{-k+1}{(\delta-2k+1)\ldots(\delta-2k+k)} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{-1}{(\delta-2k+(k-1))(\delta-2k+k)} & \cdots & \frac{-k+1}{(\delta-2k+(k-1))\ldots(\delta-2k+(2k-2))} \end{bmatrix}$$

$$\approx \det \begin{bmatrix} 1 & \frac{1}{(\delta-2k+3)} & \cdots & \frac{1}{(\delta-2k+3)\ldots(\delta-2k+k)} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{1}{(\delta-2k+(k+1))} & \cdots & \frac{1}{(\delta-2k+(k+1))\ldots(\delta-2k+(2k-2))} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & \frac{1}{(\delta-2(k-1)+1)} & \cdots & \frac{1}{(\delta-2(k-1)+1)\ldots(\delta-2(k-1)+((k-1)-1)} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{1}{(\delta-2(k-1)+(k-1))} & \cdots & \frac{1}{(\delta-2(k-1)+(k-1))\ldots(\delta-2(k-1)+(2(k-1)-2)} \end{bmatrix}$$

$$= \det Z_{k-1}.$$

So, if $\det Z_{k-1} \neq 0$, then $\det Z_k \neq 0$ and the proof is completed. □

6. FRACTIONAL FUNDAMENTAL LEMMA

The main result of the paper is the following

Lemma 6.1 (fundamental lemma of order $\alpha \in (n - \frac{1}{2}, n)$). *If $n - \frac{1}{2} < \alpha < n$, $n \in \mathbb{N}$, $n \geq 2$, $c \in L^2([a,b], \mathbb{R}^m)$ and*

$$\int_a^b c(t)(D^\alpha_{a+}h)(t)dt = 0$$

for any $h \in AC^{\alpha,2}_{a+}([a,b], \mathbb{R}^m)$ such that

(6.1) $$D^0(I^{n-\alpha}_{a+}h)(a) = 0,\ D^1(I^{n-\alpha}_{a+}h)(a) = 0, \ldots, D^{n-1}(I^{n-\alpha}_{a+}h)(a) = 0,$$

(6.2) $$(I^{\alpha-n+1}_{a+}D^\alpha_{a+}h)(b) = 0, \ldots, (I^{\alpha-1}_{a+}D^\alpha_{a+}h)(b) = 0, (I^{\alpha}_{a+}D^\alpha_{a+}h)(b) = 0,$$

then there exist constants $d_0, d_1,\dots,d_{n-1} \in \mathbb{R}^m$ *such that*

$$c(t) = \frac{d_0}{\Gamma(\alpha-n+1)}(b-t)^{\alpha-n} + \frac{d_1}{\Gamma(\alpha-n+2)}(b-t)^{\alpha-n+1} + \dots + \frac{d_{n-1}}{\Gamma(\alpha)}(b-t)^{\alpha-1},\ t\in[a,b] \text{ a.e.} \tag{6.3}$$

and, consequently,

$$(D^\alpha_{b-}c)(t) = 0,\ t\in[a,b].$$

Proof. Let us consider a function

$$p_{\alpha-1}(t) = \frac{d_0}{\Gamma(\alpha-n+1)}(b-t)^{\alpha-n} + \frac{d_1}{\Gamma(\alpha-n+2)}(b-t)^{\alpha-n+1} + \frac{d_{n-1}}{\Gamma(\alpha)}(b-t)^{\alpha-1},\ t\in(a,b).$$

Clearly,

$$(D^\alpha_{b-}p_{\alpha-1})(t) = 0,\ t\in(a,b).$$

Moreover, for any $h \in AC^{\alpha,2}_{a+}([a,b],\mathbb{R}^m)$ satisfying (6.1), (6.2), we have

$$\begin{aligned} 0 &= \int_a^b c(t)(D^\alpha_{a+}h)(t)dt - 0 = \int_a^b c(t)(D^\alpha_{a+}h)(t)dt - \int_a^b (D^\alpha_{b-}p_{\alpha-1})(t)h(t)dt \\ &= \int_a^b c(t)(D^\alpha_{a+}h)(t)dt - \int_a^b p_{\alpha-1}(t)(D^\alpha_{a+}h)(t)dt \\ &\quad + \sum_{i=0}^{n-1}(-1)^i(D^i(I^{n-\alpha}_{b-}p_{\alpha-1})(b))(I^{\alpha-n+1+i}_{a+}D^\alpha_{a+}h)(b) \\ &\quad - \sum_{i=0}^{n-1}(D^i(I^{n-\alpha}_{a+}h)(a))(I^{\alpha-n+1+i}_{b-}0)(a) \\ &= \int_a^b (c(t)-p_{\alpha-1}(t))(D^\alpha_{a+}h)(t)dt. \end{aligned} \tag{6.4}$$

Now, we shall show that there exist constants $d_0, d_1,\dots,d_{n-1} \in \mathbb{R}^m$ such that the function

$$\overline{h}(t) = I^\alpha_{a+}(c-p_{\alpha-1})(t),\ t\in[a,b], \tag{6.5}$$

belongs to $AC^{\alpha,2}_{a+}([a,b],\mathbb{R}^m)$ and satisfies (6.1), (6.2). The facts that each function $\overline{h}$ given by (6.5) belongs to $AC^{\alpha,2}_{a+}([a,b],\mathbb{R}^m)$ and satisfies (6.1) are obvious. So, it is sufficient to show that there exist constants $d_0, d_1,\dots,d_{n-1} \in \mathbb{R}^m$ such that

$$\begin{gathered} (I^{\alpha-n+1}_{a+}p_{\alpha-1})(b) = (I^{\alpha-n+1}_{a+}c)(b) \\ (I^{\alpha-n+2}_{a+}p_{\alpha-1})(b) = (I^{\alpha-n+2}_{a+}c)(b) \\ \vdots \\ (I^{\alpha-1}_{a+}p_{\alpha-1})(b) = (I^{\alpha-1}_{a+}c)(b) \\ (I^{\alpha}_{a+}p_{\alpha-1})(b) = (I^{\alpha}_{a+}c)(b) \end{gathered}$$

i.e. such that

$$W_{\alpha,n}\begin{bmatrix} d_0 \\ \vdots \\ d_{n-1} \end{bmatrix} = \begin{bmatrix} (I_{a+}^{\alpha-n+1}c)(b) \\ (I_{a+}^{\alpha-n+2}c)(b) \\ \vdots \\ (I_{a+}^{\alpha-1}c)(b) \\ (I_{a+}^{\alpha}c)(b) \end{bmatrix}$$

Existence of such constants follows directly from the Proposition 5.1. Thus, from (6.4)

$$0 = \int_a^b (c(t) - p_{\alpha-1}(t))(D_{a+}^{\alpha}\overline{h})(t)dt = \int_a^b (c(t) - p_{\alpha-1}(t))^2 dt$$

and the proof is completed. □

Remark 6.2. It is easy to see that if $n - \frac{1}{2} < \alpha < n$, $n \in \mathbb{N}$, $n \geq 2$, a function $h \in AC_{a+}^{\alpha,2}([a,b],\mathbb{R}^m)$ satisfies (6.1) (i.e. $h \in I_{a+}^{\alpha}(L^2([a,b],\mathbb{R}^m))$), then

$$(I_{a+}^{\alpha}D_{a+}^{\alpha}h)(b) = h(b), (I_{a+}^{\alpha-1}D_{a+}^{\alpha}h)(b) = D^1h(b), \ldots, \\ (I_{a+}^{\alpha-n+1}D_{a+}^{\alpha}h)(b) = D^{n-1}h(b)$$

and the conditions (6.2) can be replaced by the following ones

$$h(b) = 0,\ D^1h(b) = 0, \ldots, D^{n-1}h(b) = 0$$

Indeed, let $h = I_{a+}^{\alpha}\varphi$ with some $\varphi \in L^2([a,b],\mathbb{R}^m)$ and $i \in \{0,\ldots,n-1\}$. From [7, Property 2.2] it follows that

$$I_{a+}^{\alpha-n+1+i}D_{a+}^{\alpha}h = I_{a+}^{\alpha-(n-1-i)}D_{a+}^{\alpha}h = D^{n-1-i}I_{a+}^{\alpha}D_{a+}^{\alpha}h = D^{n-1-i}h$$

a.e. on $[a,b]$. Since $\alpha - n + 1 + i > \frac{1}{2}$, the function $I_{a+}^{\alpha-n+1+i}D_{a+}^{\alpha}h = I_{a+}^{\alpha-n+1+i}\varphi$ is continuous on $[a,b]$. Moreover, since $h = I_{a+}^{\alpha}\varphi = I_{a+}^{n-1-i}I_{a+}^{\alpha-n+1+i}\varphi$ on $[a,b]$ (h and $I_{a+}^{n-1-i}I_{a+}^{\alpha-n+1+i}\varphi$ are continuous on $[a,b]$), therefore $D^{n-1-i}h = I_{a+}^{\alpha-n+1+i}\varphi = I_{a+}^{\alpha-n+1+i}D_{a+}^{\alpha}h$ on $[a,b]$.

Using Lemma 6.1 we obtain

Lemma 6.3. *If $n - \frac{1}{2} < \alpha < n$, $n \in \mathbb{N}$, $n \geq 2$, a_0, $a_1 \in L^2([a,b],\mathbb{R}^m)$ and*

$$\int_a^b (a_1(t)(D_{a+}^{\alpha}h)(t) - a_0(t)h(t))dt = 0$$

for any $h \in AC_{a+}^{\alpha,2}([a,b],\mathbb{R}^m)$ satisfying (6.1) and (6.2), i.e.

$$D^0(I_{a+}^{n-\alpha}h)(a) = 0, \ldots, D^{n-1}(I_{a+}^{n-\alpha}h)(a) = 0,$$

$$(I_{a+}^{\alpha-n+1}D_{a+}^{\alpha}h)(b) = 0, \ldots, (I_{a+}^{\alpha-1}D_{a+}^{\alpha}h)(b) = 0, (I_{a+}^{\alpha}D_{a+}^{\alpha}h)(b) = 0$$

then there exist constants d_0, d_1,...,$d_{n-1} \in \mathbb{R}^m$ *such that*

$$a_1(t) = \frac{d_0}{\Gamma(\alpha-n+1)}(b-t)^{\alpha-n} + \frac{d_1}{\Gamma(\alpha-n+2)}(b-t)^{\alpha-n+1} + \cdots + \frac{d_{n-1}}{\Gamma(\alpha)}(b-t)^{\alpha-1} + (I^{\alpha}_{b-}a_0)(t), \ t \in [a,b] \ a.e.$$

and, consequently,

$$(D^{\alpha}_{b-}a_1)(t) = a_0(t), \ t \in [a,b] \ a.e..$$

Proof. Let us consider a function

$$p(t) = I^{\alpha}_{b-}a_0(t), \ t \in [a,b] \text{ a.e.}$$

For any $h \in AC^{\alpha,2}_{a+}([a,b],\mathbb{R}^m)$ satisfying (6.1) and (6.2) we have

$$\begin{aligned}
&\int_a^b (a_1(t)(D^{\alpha}_{a+}h)(t)dt - \int_a^b a_0(t)h(t)dt \\
&= \int_a^b a_1(t)(D^{\alpha}_{a+}h)(t)dt - \int_a^b (D^{\alpha}_{b-}p)(t)h(t)dt \\
&= \int_a^b a_1(t)(D^{\alpha}_{a+}h)(t)dt - \sum\nolimits_{i=0}^{n-1}(D^i(I^{n-\alpha}_{a+}h)(a))(I^{\alpha-n+1+i}_{b-}D^{\alpha}_{b-}p)(a) \\
&+ \sum_{i=0}^{n-1}(-1)^i(D^iI^{n-\alpha}_{b-}p)(b)(I^{\alpha-n+1+i}_{a+}D^{\alpha}_{a+}h)(b) - \int_a^b p(t)(D^{\alpha}_{a+}h)(t)dt \\
&= \int_a^b (a_1(t)-p(t))(D^{\alpha}_{a+}h)(t)dt.
\end{aligned}$$

Lemma 6.1 implies the existence of constants d_0, d_1,...,$d_{n-1} \in \mathbb{R}^m$ such that (6.3) holds true. □

Before we prove a general version of the fractional fundamental lemma we shall give some useful proposition.

Proposition 6.4. *Let* $n-\frac{1}{2} < \alpha < n$, $n \in \mathbb{N}$, $n \geq 2$. *If* $h \in AC^{\alpha,2}_{a+}([a,b],\mathbb{R}^m)$ *satisfies (6.1) and (6.2), then* $h \in AC^{\alpha-i,2}_{a+}([a,b],\mathbb{R}^m)$ *and*

$$D^0(I^{(n-i)-(\alpha-i)}_{a+}h)(a) = 0, \ldots, D^{(n-i)-1}(I^{(n-i)-(\alpha-i)}_{a+}h)(a) = 0,$$

$$(I^{(\alpha-i)-(n-i)+1}_{a+}D^{\alpha-i}_{a+}h)(b) = 0, \ldots, (I^{\alpha-i}_{a+}D^{\alpha-i}_{a+}h)(b) = 0$$

for any $i = 1,\ldots,n-1$. *Moreover,* $D^{\alpha-i}_{a+}h \in I^i_{a+}(L^2([a,b],\mathbb{R}^m))$.

Proof. Let $h \in AC^{\alpha,2}_{a+}([a,b],\mathbb{R}^m)$ satisfies (6.1), (6.2) and i be a positive integer belonging to the set $\{1,\ldots,n-1\}$. Since $n-\frac{1}{2} < \alpha < n$, therefore

$$n-i-\frac{1}{2} < \alpha-i < n-i \text{ for } i = 1,\ldots,n-1$$

and

$$I^{(n-i)-(\alpha-i)}_{a+}h = I^{n-\alpha}_{a+}h \in AC^{n,2} \subset AC^{n-i,2}.$$

This means that h possesses a (continuous) derivative $D^{\alpha-i}_{a+}h$ and, consequently, belongs to $AC^{\alpha-i,2}_{a+}([a,b],\mathbb{R}^m)$. Moreover,

$$(6.6)\qquad \begin{cases} D^0(I^{(n-i)-(\alpha-i)}_{a+}h)(a) = (I^{n-\alpha}_{a+}h)(a) = 0, \\ \vdots \\ D^{(n-i)-1}(I^{(n-i)-(\alpha-i)}_{a+}h)(a) = D^{(n-i)-1}(I^{n-\alpha}_{a+}h)(a) = 0 \end{cases}$$

Of course, function $h \in AC^{\alpha,2}_{a+}([a,b],\mathbb{R}^m)$ satisfying (6.1) is of the form

$$h = I^{\alpha}_{a+}\varphi$$

where $\varphi \in L^2([a,b],\mathbb{R}^m)$. Since

$$I^{\alpha}_{a+}\varphi = I^{\alpha-i}_{a+}(I^{i}_{a+}\varphi),$$

therefore

$$h = I^{\alpha-i}_{a+}\psi$$

where $\psi = I^{i}_{a+}\varphi$. So, from (6.2)

$$(6.7)\qquad \begin{cases} (I^{(\alpha-i)-(n-i)+1}_{a+}D^{\alpha-i}_{a+}h)(b) = (I^{\alpha-n+1}_{a+}I^{i}_{a+}\varphi)(b) = (I^{\alpha-(n-i-1)}_{a+}D^{\alpha}_{a+}h)(b) = 0 \\ \vdots \\ (I^{\alpha-i}_{a+}D^{\alpha-i}_{a+}h)(b) = (I^{\alpha-i}_{a+}I^{i}_{a+}\varphi)(b) = (I^{\alpha}_{a+}D^{\alpha}_{a+}h)(b) = 0 \end{cases}$$

Thus, the function h belongs to $AC^{\alpha-i,2}_{a+}([a,b],\mathbb{R}^m)$ with $D^{\alpha-i}_{a+}h = I^{i}_{a+}\varphi \in I^{i}_{a+}(L^2([a,b],\mathbb{R}^m))$ and satisfies (6.6) as well as (6.7). □

Now, we are in the position to prove

Lemma 6.5 (general fundamental lemma of order $\alpha \in (n-\frac{1}{2}, n)$)**.** *If $n-\frac{1}{2} < \alpha < n$, $n \in \mathbb{N}$, $n \geq 2$, a_0, a_1, $b_1,\dots,b_{n-1} \in L^2([a,b],\mathbb{R}^m)$ and*

$$(6.8)\qquad \int_a^b (a_1(t)(D^{\alpha}_{a+}h)(t) - b_1(t)(D^{\alpha-1}_{a+}h)(t) - \dots \\ - b_{n-1}(t)(D^{\alpha-(n-1)}_{a+}h)(t) - a_0(t)h(t))dt = 0$$

for any $h \in AC^{\alpha,2}_{a+}([a,b],\mathbb{R}^m)$ satisfying (6.1) and (6.2), then there exists the derivative $D^{\alpha-(n-1)}_{b-}a_1$, the functions appearing below in brackets are absolutely continuous and

$$D^1(\dots(D^1(D^1(D^{\alpha-(n-1)}_{b-}a_1 - I^{n-\alpha}_{b-}b_1) + I^{n-\alpha}_{b-}b_2) + \dots \\ + (-1)^{n-1}I^{n-\alpha}_{b-}b_{n-1}) = (-1)^{n-1}a_0$$

a.e. on $[a,b]$ (the operator D^1 acts $(n-1)$ times).

Proof. Let us fix $i = 1, \ldots, n-1$ and denote $g_i = I^i_{b-} b_i$, $f_i = D^{\alpha-i}_{a+} h$. Then $D^i_{b-} g_i = b_i$, $f_i \in I^i_{a+}(L^2([a,b], \mathbb{R}^m))$ (cf. Proposition 6.4) and for any $h \in AC^{\alpha,2}_{a+}([a,b], \mathbb{R}^m)$ satisfying (6.1) and (6.2) we have

$$\int_a^b b_i(t)(D^{\alpha-i}_{a+} h)(t)dt = \int_a^b (D^i_{b-} g_i)(t) f_i(t)dt = \int_a^b g_i(t)(D^i_{a+} f_i)(t)dt$$
$$= \int_a^b (I^i_{b-} b_i)(t)(D^i_{a+} D^{\alpha-i}_{a+} h)(t)dt = \int_a^b (I^i_{b-} b_i)(t)(D^\alpha_{a+} h)(t)dt$$

(in the last equality we used [7, Property 2.3 (a)]). So, from (6.8) we obtain

$$\int_a^b ((a_1(t) - (I^1_{b-} b_1)(t) - \cdots - (I^{n-1}_{b-} b_{n-1})(t))(D^\alpha_{a+} h)(t) - a_0(t)h(t))dt = 0.$$

Now, Lemma 6.3 implies the existence of constants d_0, d_1,…,$d_{n-1} \in \mathbb{R}^m$ such that

$$a_1(t) - (I^1_{b-} b_1)(t) - \cdots - (I^{n-1}_{b-} b_{n-1})(t)$$
$$= \frac{d_0}{\Gamma(\alpha - n + 1)}(b-t)^{\alpha-n} + \frac{d_1}{\Gamma(\alpha - n + 2)}(b-t)^{\alpha-n+1}$$
$$+ \cdots + \frac{d_{n-1}}{\Gamma(\alpha)}(b-t)^{\alpha-1} + (I^\alpha_{b-} a_0)(t), \ t \in [a,b] \text{ a.e.}$$

Let us denote $\beta = \alpha - (n-1)$. The above equality can be written down as

$$a_1(t) - (I^1_{b-} b_1)(t) - \cdots - (I^{n-1}_{b-} b_{n-1})(t)$$
$$= \frac{d_0}{\Gamma(\alpha - n + 1)}(b-t)^{\beta-1} + \frac{d_1}{\Gamma(\alpha - n + 2)}(b-t)^{\beta}$$
$$+ \cdots + \frac{d_{n-1}}{\Gamma(\alpha)}(b-t)^{\beta+(n-2)} + (I^{\beta+(n-1)}_{b-} a_0)(t), \ t \in [a,b] \text{ a.e.}$$

All terms appearing on the left side of the above equality, excluding a_1, and all terms appearing on the right side of the above equality, excluding the first one, have derivative D^β_{b-} as absolutely continuous functions. The term $\frac{d_0}{\Gamma(\alpha-n+1)}(b-t)^{\beta-1}$ has zero derivative D^β_{b-} (cf. [7, Corollary 2.1 (b)]). So there exists derivative $D^\beta_{b-} a_1$ and

$$(D^\beta_{b-} a_1)(t) = (I^{1-\beta}_{b-} b_1)(t) + \cdots + (I^{(n-1)-\beta}_{b-} b_{n-1})(t)$$
$$+ \frac{d_1}{\Gamma(\alpha - n + 2)} \frac{\Gamma(\beta+1)}{\Gamma(1)}(b-t)^0 + \frac{d_2}{\Gamma(\alpha - n + 3)} \frac{\Gamma(\beta+2)}{\Gamma(2)}(b-t)^1$$
$$+ \cdots + \frac{d_{n-1}}{\Gamma(\alpha)} \frac{\Gamma(\beta+(n-1))}{\Gamma(n-1)}(b-t)^{(n-2)} + (I^{n-1}_{b-} a_0)(t), \ t \in [a,b] \text{ a.e.}$$

If $n = 2$, term containing d_2 vanishes. From the above equality it follows that the function $D^{\beta}_{b-}a_1 - I^{1-\beta}_{b-}b_1$ is absolutely continuous and

$$D^1(D^{\beta}_{b-}a_1 - I^{1-\beta}_{b-}b_1)(t) = -(I^{1-\beta}_{b-}b_2)(t) - \cdots - (I^{(n-2)-\beta}_{b-}b_{n-1})(t) - \frac{d_2}{\Gamma(\alpha-n+3)}\frac{\Gamma(\beta+2)}{\Gamma(2)}(b-t)^0 - \cdots - \frac{d_{n-1}}{\Gamma(\alpha)}\frac{\Gamma(\beta+(n-1))}{\Gamma(n-1)}(n-2)(b-t)^{(n-3)} - (I^{n-2}_{b-}a_0)(t),\ t \in [a,b] \text{ a.e.}$$

If $n = 2$, the proof is completed. If $n \geq 3$, the above equality implies that the function $D^1(D^{\beta}_{b-}a_1 - I^{1-\beta}_{b-}b_1) + I^{1-\beta}_{b-}b_2$ is absolutely continuous and

$$D^1(D^1(D^{\beta}_{b-}a_1 - I^{1-\beta}_{b-}b_1) + I^{1-\beta}_{b-}b_2)(t) = (I^{1-\beta}_{b-}b_3)(t) + \cdots + (I^{(n-3)-\beta}_{b-}b_{n-1})(t) + \frac{d_3}{\Gamma(\alpha-n+4)}\frac{\Gamma(\beta+3)}{\Gamma(3)}2(b-t)^0 + \cdots + \frac{d_{n-1}}{\Gamma(\alpha)}\frac{\Gamma(\beta+(n-1))}{\Gamma(n-1)}(n-2)(n-3)(b-t)^{(n-4)} + (I^{n-3}_{b-}a_0)(t),\ t \in [a,b] \text{ a.e.}$$

Continuing this procedure we obtain

$$D^1(\ldots(D^1(D^1(D^{\alpha-(n-1)}_{b-}a_1 - I^{n-\alpha}_{b-}b_1) + I^{n-\alpha}_{b-}b_2) + \ldots + (-1)^{n-1}I^{n-\alpha}_{b-}b_{n-1}) = (-1)^{n-1}a_0$$

a.e. on $[a,b]$ (the operator D^1 acts $(n-1)$ times). □

REFERENCES

[1] O. P. Agrawal, *Formulation of Euler-Lagrange equations for fractional variational problems*, J. Math. Anal. Appl., vol. 272 (2002), 368–379.

[2] D. Baleanu, S. I. Muslih, *Formulation of Hamiltonian equations for fractional variational problems*, Czech. J. Phys., vol. 55, no. 6 (2005), 633–642.

[3] D. Baleanu, S. I. Muslih, E. M. Rabei, *On fractional Euler-Lagrange and Hamilton equations and the fractional generalization of total time derivative*, Nonlinear Dyn., vol. 53 (2008), 67–74.

[4] G. H. Hardy, J. E. Littlewood, *Some properties of fractional integrals*, Proc. London Math. Soc. Ser. 2, vol. 24 (1925), 37–41.

[5] R. A. El-Nabulusi, D. F. M. Torres, *Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order (α,β)*, Math. Meth. Appl. Sci., vol. 30 (2007), 1931–1939.

[6] D. Idczak, *Fractional fundamental lemma of order $\alpha \in (\frac{1}{2}, 1)$*, submitted for publication.

[7] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[8] F. Riewe, *Nonconservative Lagrangian and Hamiltonian mechanics*, Physical Review E, vo. 53, no. 2 (1996), 1890–1899.

[9] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives - Theory and Applications,* Gordon and Breach: Amsterdam, 1993.

[10] S. Walczak, On some generalization of the fundamental lemma and its application to differential equations, Bull. Soc. Math. Belg. ser. B, vol. 45, no. 3 (1993).