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UNIFORM DECAY IN MILDLY DAMPED TIMOSHENKO SYSTEMS WITH NON-EQUAL WAVE SPEED PROPAGATION

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ABSTRACT. In this paper we consider a one-dimensional linear Timoshenko system in a bounded interval. The dissipation in the system is in the rotation-angle equation through heat conduction. We show that in the non-equal wave speed situation, the decay rate is polynomial.

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1. INTRODUCTION

In [13], Rivera and Racke studied a Timoshenko-type system of the form

$$
\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, & \text{in } (0,L) \times \mathbb{R}_+ \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0,L) \times \mathbb{R}_+ \\ \rho_3\theta_t - \kappa\theta_{xx} + \gamma\psi_{xt} = 0, & \text{in } (0,L) \times \mathbb{R}_+, \end{cases} \tag{1.1}
$$

where φ, ψ, and θ are functions of (x,t), denote the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature, respectively, and $\rho_i, b, k, \gamma, \kappa, L$ are positive constants. Under the equal-speed wave-propagation condition $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$, they established exponential decay results for (1.1) together with initial conditions and boundary conditions of the form

$$
\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0 \tag{1.2}
$$

or

$$
\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = \theta(0,t) = \theta(L,t) = 0. \tag{1.3}
$$

In addition, they showed that the equal-speed wave-propagation condition is necessary for the exponential stability of (1.1), (1.3). However, in this case of non-equal speed, no rate of decay has been discussed.

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In the isothermal case, (1.1) reduces to

$$\left\{\begin{array}{ll}\rho_1\varphi_{tt}-k(\varphi_x+\psi)_x=0, & \text{in } (0,L)\times\mathbb{R}_+\\ \rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi)=0, & \text{in } (0,L)\times\mathbb{R}_+.\end{array}\right. \tag{1.4}$$

This system is conservative and one would be interest in adding some kind of damping that may help in stabilizing such a system. In this regard, different types of dampings have been introduced to system (1.1) and several uniform stability results have been obtained. Kim and Renardy [7] considered (1.4) together with two boundary controls of the form

$$\begin{aligned} k\varphi(L,t)-ku_x(L,t) &= \alpha u_t(L,t), \quad \text{on } \mathbb{R}_+\\ EI\varphi_x(L,t) &= -\beta\varphi_t(L,t), \quad \text{on } \mathbb{R}_+\end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (1.4). Raposo *et al.* [16] studied the following system

$$\left\{\begin{array}{l}\rho_1 u_{tt}-k(u_x-\varphi)_x+u_t=0, \ \text{in } (0,L)\times\mathbb{R}_+\\ \rho_2\varphi_{tt}-b\varphi_{xx}+k(u_x-\varphi)+\varphi_t=0, \ \text{in } (0,L)\times\mathbb{R}_+\\ u(0,L)=u(L,t)=\varphi(0,t)=\varphi(L,t)=0, \quad \text{on } \mathbb{R}_+\end{array}\right.$$

and proved that the energy decays exponentially. Soufyane and Wehbe [17] considered

$$\left\{\begin{array}{l}\rho u_{tt}=(k(u_x-\varphi))_x, \ \text{in } (0,L)\times\mathbb{R}_+\\ I_\rho\varphi_{tt}=(EI\varphi_x)_x+k(u_x-\varphi)-b\varphi_t, \ \text{in } (0,L)\times\mathbb{R}_+\\ u(0,t)=u(L,t)=\varphi(0,t)=\varphi(L,t)=0, \quad \text{on } \mathbb{R}_+,\end{array}\right. \tag{1.5}$$

where b is a positive and continuous function, which satisfies

$$b(x)\geq b_0>0, \quad \forall\ x\ \in\ [a_0,a_1]\subset[0,L]$$

and proved that the uniform stability of (1.5) holds if and only if the wave speeds are equal $\left(\frac{k}{\rho}=\frac{EI}{I_\rho}\right)$; otherwise only the asymptotic stability has been proved. This result has been recently improved by Rivera and Racke [15], where an exponential decay of the solution energy of (1.5) has been established, allowing b to be with an indefinite sign. Also, Rivera and Racke [14] considered the following nonlinear Timoshenko system

$$\left\{\begin{array}{ll}\rho_1\varphi_{tt}-\sigma(\varphi_x,\psi)_x=0, & \text{in } (0,L)\times\mathbb{R}_+\\ \rho_2\psi_{tt}-b\psi_{xx}+k(\varphi_x+\psi)+d\psi_t=0, & \text{in } (0,L)\times\mathbb{R}_+,\end{array}\right.$$

with homogeneous boundary conditions and proved that the system is exponentially stable if and only if $\frac{K}{\rho_1}=\frac{b}{\rho_2}$. Otherwise, only the polynomial stability holds. Alabau-Boussouira [1] extended the results of [14] to the case of nonlinear feedback $\alpha(\psi_t)$, instead of $d\psi_t$, where α is a globally Lipchitz function satisfying some growth conditions at the origin.

Ammar-Khodja *et al.* [2] considered a linear Timoshenko-type system with memory of the form

$$(1.6)\qquad \begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0\,, & \text{in } (0,L)\times \mathbb{R}_+ \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^t g(t-s)\psi_{xx}(s)ds + k(\varphi_x+\psi) = 0, & \text{in } (0,L)\times \mathbb{R}_+ \end{cases}$$

together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{k}{\rho_1}=\frac{b}{\rho_2}\right)$ and g decays uniformly. Precisely, they proved an exponential decay if g decays in an exponential rate and polynomially if g decays in a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their result. Guesmia and Messaoudi [4] obtained the same uniform decay result under weaker conditions on the regularity and the growth of the relaxation function. Recently, Messaoudi and Mustafa [11] treated (1.6) for a wider class of relaxation functions and established a more general decay estimate, from which the usual exponential and polynomial decay results are only special cases. This latter result has been improved by Guesmia and Messaoudi [5] to accommodate systems, where frictional and viscoelastic dampings are cooperating.

Fernández Sare and Rivera [3], considered a Timoshenko-type system with a past history of the form

$$(1.7)\qquad \begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x = 0\,, & \text{in } (0,L)\times \mathbb{R}_+ \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^\infty g(s)\psi_{xx}(t-s,.)ds + k(\varphi_x+\psi) = 0, & \text{in } (0,L)\times \mathbb{R}_+ \end{cases}$$

and showed that if g is of exponential decay, the dissipation given by the history term is strong enough to stabilize the system exponentially if and only if the wave speeds are equal. They also proved that the solution decays polynomially for the case of different wave speeds. Messaoudi and Said-Houari [12] considered the case of polynomially decaying relaxation functions and improved the result of [3]. For more results, we refer the reader to [9] and [10].

In the present work we are concerned with (1.1) with the boundary conditions (1.3) or with boundary conditions of the form

$$(1.8)\qquad \varphi_x(0,t) = \varphi_x(L,t) = \psi(0,t) = \psi(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0.$$

We discuss the situation of different speed wave propagation and show that the rate of decay is polynomial provided that the initial data are regular enough. We also establish a non-exponential decay result for case when (1.1) is supplemented with (1.8).

The paper is organized as follow. In section 2, we discuss system (1.1) with the boundary conditions (1.3) and establish a polynomial decay result in the case of non-equal wave speed propagation. A similar result is obtained for system (1.1) supplemented with (1.8). Section 4 is devoted to the proof of the non-exponential decay for system (1.1) with (1.8).

2. BOUNDARY CONDITIONS $\varphi = \theta = \psi_x = 0$ at $x = 0, 1$

In this section, we consider (1.1), (1.3), for $L = 1$, and the initial conditions

$$\begin{aligned} &\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad 0 < x < 1 \\ &\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad 0 < x < 1 \\ &\theta(x,0) = \theta_0(x), \quad 0 < x < 1. \end{aligned} \tag{2.1}$$

Our goal is to establish a uniform decay of solutions. To do so, we introduce a new dependent variable. Namely,

$$\tilde{\psi}(x,t) = \psi(x,t) - \left(\int_0^1 \psi_0(x)dx\right) \cos\sqrt{\frac{k}{\rho_2}}t - \sqrt{\frac{\rho_2}{k}}\left(\int_0^1 \psi_1(x)dx\right) \sin\sqrt{\frac{k}{\rho_2}}t.$$

It is straight forward to check that $(\varphi, \tilde{\psi}, \theta)$ satisfies system (1.1), $\tilde{\psi}_x(0,t) = \tilde{\psi}_x(1,t) = 0$, and more importantly, we have

$$\int_0^1 \tilde{\psi}(x,t)dx = 0, \, \forall \, t \geq 0.$$

Remark 2.1. With this change of dependent variable, we do not need to assume that

$$\int_0^1 \psi_0(x)dx = 0, \qquad \int_0^1 \psi_1(x)dx = 0$$

as in [13].

In what follows, we work with $\tilde{\psi}$ but we use ψ for simplicity. The well-posedness is standard. It can be established by adopting the method mentioned in section 5 of [13]. See also [6].

The first-order energy functional is then given by

$$E(t) = E_1(\varphi, \psi, \theta) = \frac{1}{2}\int_0^1 \left[(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + b\psi_x^2 + k(\varphi_x + \psi)^2 + \rho_3\theta^2\right]. \tag{2.2}$$

In addition, for any H^2-regular solution, we define $E_2(t) = E_1(\varphi_t, \psi_t, \theta_t)$. Standard computations yield

$$E'(t) = E_1'(t) = -\kappa\int_0^1 \theta_x^2, \qquad E_2'(t) = -\kappa\int_0^1 \theta_{xt}^2. \tag{2.3}$$

Theorem 2.1. *Let* (φ, ψ, θ) *be any* H^2*-regular solution of* (1.1), (1.3), (2.1). *Then there exist two positive constants* λ, c_0, *for which the energy functional* E *satisfies,* $\forall \, t > 0$,

$$E(t) \leq c_0 e^{-\lambda t} \quad \text{if } \frac{k}{\rho_1} = \frac{b}{\rho_2} \tag{2.4}$$

$$E(t) \leq \frac{c_0}{t} \quad \text{if } \frac{k}{\rho_1} \neq \frac{b}{\rho_2}. \tag{2.5}$$

In order to prove our main result, we introduce, similarly to [13], several functionals and establish several estimates. We start with

$$I_1(t) = \rho_2 \int_0^1 \psi\psi_t dx - \rho_1 \int_0^1 \varphi_t \left(\int_0^x \psi(y,t)dy \right) dx. \tag{2.6}$$

Lemma 2.2. *Let* (φ, ψ, θ) *be the solution of* (1.1), (1.3). *Then* I_1 *satisfies, for any* $\varepsilon_1 > 0$,

$$I_1'(t) \le -\frac{b}{2} \int_0^1 \psi_x^2 + \left(\rho_2 - \frac{c}{4\varepsilon_1} \right) \int_0^1 \psi_t^2 + \varepsilon_1 \int_0^1 \varphi_t^2 + c \int_0^1 \theta_x^2. \tag{2.7}$$

Proof. A simple differentiation of I_1, using (1.1), (1.3), gives

$$\begin{aligned} I_1'(t) &= \rho_2 \int_0^1 \psi_t^2 dx + \int_0^1 \psi[b\psi_{xx} - k(\varphi_x + \psi) - \gamma\theta_x]dx \\ &\quad -\rho_1 \int_0^1 \varphi_t \left(\int_0^x \psi_t(y,t)dy \right) dx - k \int_0^1 (\varphi_x + \psi)dx - \gamma \int_0^1 \psi\theta_x dx \\ &= \rho_2 \int_0^1 \psi_t^2 dx - b \int_0^1 \psi_x^2 dx - \gamma \int_0^1 \psi\theta_x dx - \rho_1 \int_0^1 \varphi_t \left(\int_0^x \psi_t(y,t)dy \right) dx. \end{aligned}$$

By integrating by parts and using the fact $\int_0^1 \psi(y,t)dy = 0$ (remember that we are working with $\tilde{\psi}$), we obtain

$$I_1'(t) = \rho_2 \int_0^1 \psi_t^2 dx - b \int_0^1 \psi_x^2 dx - \gamma \int_0^1 \psi\theta_x dx - \rho_1 \int_0^1 \varphi_t \left(\int_0^x \psi_t(y,t) \right) dy.$$

By exploiting the inequality

$$\left(\int_0^x \psi_t(y,t)dy \right)^2 \le \left(\int_0^1 |\psi_t(y,t)|dy \right)^2 \le \int_0^1 \psi_t^2(y,t)dy$$

and using Poincaré's and Young's inequalities, the desired result (2.7) is established. □

Next, we define

$$I_2(t) = -\rho_2\rho_3 \int_0^1 \theta \left(\int_0^x \psi_t(y,t)dy \right) dx. \tag{2.8}$$

Lemma 2.3. *The functional* I_2 *satisfies, along solutions of* (1.1), (1.3), *and for any* $\varepsilon_2 > 0$,

$$I_2'(t) \le -\frac{\gamma\rho_2}{2} \int_0^1 \psi_t^2 + \varepsilon_2 \int_0^1 (\varphi_x^2 + \psi_x^2) + \frac{c}{\varepsilon_2} \int_0^1 \theta_x^2. \tag{2.9}$$

Proof. By using equations (1.1), a simple integration leads to

$$\begin{aligned} I_2'(t) &= \rho_2 \int_0^1 \psi_t(k\theta_x - \gamma\psi_t) - \rho_3 \int_0^1 \theta(b\psi_x - k\varphi - \gamma\theta) \\ &\quad + \rho_3 k \int_0^1 \theta \left(\int_0^x \psi(y,t)dy \right) \\ &= -\gamma\rho_2 \int_0^1 \psi_t^2 + k\rho_2 \int_0^1 \psi_t\theta_x - \rho_3 b \int_0^1 \theta\psi_x \\ &\quad + \rho_3\gamma \int_0^1 \theta^2 + k\rho_3 \int_0^1 \theta\varphi - \rho_3 k \int_0^1 \theta_x \psi. \end{aligned}$$

By recalling Young's and Poincaré's inequalities, (2.9) is established. □

For N and N_2 large enough, we set

$$I_3 = NE + I_1 + N_2 I_2.$$

Direct calculations, using (2.3), (2.7), and (2.9), yield

$$\begin{aligned} I_3'(t) &\le -\left(N\kappa - N_2 \frac{c}{\varepsilon_2} \right) \int_0^1 \theta_x^2 - \left(\frac{b}{2} - \varepsilon_2 N_2 \right) \int_0^1 \psi_x^2 \\ &\quad - \left(N_2 \frac{\gamma\rho_2}{2} - \rho_2 - \frac{c}{4\varepsilon_1} \right) \int_0^1 \psi_t^2 + \varepsilon_1 \int_0^1 \varphi_t^2 + N_2\varepsilon_2 \int_0^1 \varphi_x^2. \end{aligned} \tag{2.10}$$

To obtain a negative term of $\int_0^1 (\varphi_x + \psi)^2$, we introduce the functional

$$I_4(t) = \rho_2 \int_0^1 \psi_t(\varphi_x + \psi) + \frac{\rho_1 b}{k} \int_0^1 \varphi_t \psi_x.$$

Similar Calculations, using equations (1.1), lead to

$$I_4'(t) \le -\frac{k}{2} \int_0^1 (\varphi_x + \psi)^2 + \rho_2 \int_0^1 \psi_t^2 + c \int_0^1 \theta_x^2 + \left(\frac{\rho_1 b}{k} - \rho_2 \right) \int_0^1 \varphi_t \psi_{xt}. \tag{2.11}$$

The following functional allows us to obtain negative terms involving $\int_0^1 (\varphi_t^2 + \psi_t^2)$:

$$I_5(t) = -\rho_1 \int_0^1 \varphi\varphi_t - \rho_2 \int_0^1 \psi\psi_t.$$

Again, similar computations, using (1.1), yield

$$I_5'(t) \le -\rho_1 \int_0^1 \varphi_t^2 - \rho_2 \int_0^1 \psi_t^2 + k \int_0^1 (\varphi_x + \psi)^2 + \tilde{c} \int_0^1 (\psi_x^2 + \theta_x^2). \tag{2.12}$$

Proof of Theorem 2.1. We introduce the functional $\mathcal{L} = I_3 + \mu I_4 + \tau I_5$, for μ and τ to be carefully chosen. It is, then, easy to see that

$$\begin{aligned} \mathcal{L}'(t) &\le -\left[N\kappa - N_2 \frac{c}{\varepsilon_2} - \mu\tilde{c} - \tau\tilde{c} \right] \int_0^1 \theta_x^2 - (\mu\rho_1 - \varepsilon_1) \int_0^1 \varphi_t^2 \\ &\quad - \left(\frac{b}{2} - 2N_2\varepsilon_2 - \tau\tilde{c} \right) \int_0^1 \psi_x^2 - \left(N_2 \frac{\gamma\rho_2}{2} - \mu\rho_2 - \frac{c}{4\varepsilon_1} + \tau\rho_2 \right) \int_0^1 \psi_t^2 \\ &\quad - \left(\mu\frac{k}{2} - \tau k - 2N_2\varepsilon_2 \right) \int_0^1 (\varphi_x + \psi)^2 + \mu \left(\frac{\rho_1 b}{k} - \rho_2 \right) \int_0^1 \varphi_t \psi_{xt}, \end{aligned} \tag{2.13}$$

where we have used

$$\varphi_x^2 \le 2(\varphi_x + \psi)^2 + 2\psi^2.$$

Now, we choose our constants. Let $\tau > 0$ be small enough so that

$$\frac{b}{2} - \tau\tilde{c} > 0 \text{ and } \mu = 3\tau.$$

We then pick $\varepsilon_1 = \frac{1}{2}\mu\rho_1$ and N_2 large enough so that

$$\frac{\gamma\rho_2}{2}N_2 - \mu\rho_2 + \tau\rho_2 - \frac{c}{4\varepsilon_1} > 0.$$

Once N_2 is fixed, we pick ε_2 so small that

$$\frac{b}{2} - \tau\tilde{c} - 2N_2\varepsilon_2 > 0$$

and

$$\frac{\mu k}{2} - \tau k - 2N_2\varepsilon_2 = \frac{3\tau k}{2} - 2N_2\varepsilon_2 > 0.$$

Finally, we choose N large enough so that

$$\mathcal{L} \sim E \tag{2.14}$$

and

$$N\kappa - \frac{N_2 c}{\varepsilon_2} - \mu\tilde{c} - \tau\tilde{c} > 0.$$

Therefore, we arrive at

$$\mathcal{L}'(t) \le -\beta_0 E(t) + \mu\left(\frac{\rho_1 b}{k} - \rho_2\right)\int_0^1 \varphi_t\psi_{xt}, \tag{2.15}$$

for some constant $\beta_0 > 0$. We distinguish two cases:

Case 1. $\frac{k}{\rho_1} = \frac{b}{\rho_2}$

In this case, we use (2.14) and (2.15), to get

$$\mathcal{L}'(t) \le -\lambda\mathcal{L}(t), \quad \forall\, t \ge 0.$$

A simple integration then leads to

$$\mathcal{L}(t) \le \mathcal{L}(0)e^{-\lambda t}, \quad \forall\, t \ge 0.$$

Again, the use of (2.14) yields (2.4).

Case 2. $\frac{k}{\rho_1} \ne \frac{b}{\rho_2}$

In this case, we require that our solutions are H^2 regular and define the functional

$$L = N(E_1 + E_2) + I_1 + N_2 I_2 + \mu I_4 + \tau I_5. \tag{2.16}$$

Note here that (2.14) does not hold. Again, a similar choice of the constants leads to

$$L'(t) \le -\beta_1 E(t) + \mu\left(\frac{\rho_1 b}{k} - \rho_2\right)\int_0^1 \varphi_t\psi_{xt} - cN\int_0^1 \theta_{xt}^2, \tag{2.17}$$

where $\beta_1 > 0$ and c is a generic positive constant.

By exploiting equation $(1.1)_3$ and the boundary conditions (1.3), we easily see that

$$\begin{aligned} \int_0^1 \varphi_t \psi_{xt} &= \frac{\kappa}{\gamma}\int_0^1 \varphi_t \theta_{xx} - \frac{\rho_3}{\gamma}\int_0^1 \varphi_t\theta_t \\ &= -\frac{\kappa}{\gamma}\int_0^1 \varphi_{xt}\theta_x - \frac{\rho_3}{\gamma}\int_0^1 \varphi_t\theta_t \\ &= -\frac{\kappa}{\gamma}\frac{d}{dt}\int_0^1 \varphi_x\theta_x + \frac{\kappa}{\gamma}\int_0^1 \varphi_x\theta_{xt} - \frac{\rho_3}{\gamma}\int_0^1 \varphi_t\theta_t \end{aligned} \tag{2.18}$$

Consequently, if

$$F(t) = L(t) + \mu\frac{\kappa}{\gamma}\left(\frac{\rho_1 b}{k} - \rho_2\right)\int_0^1 \varphi_x\theta_x$$

then

$$F'(t) \le -\beta_1 E(t) + \mu\left(\frac{\rho_1 b}{k} - \rho_2\right)\left(\frac{\kappa}{\gamma}\int_0^1 \varphi_x\theta_{xt} - \frac{\rho_3}{\gamma}\int_0^1 \varphi_t\theta_t\right) - cN\int_0^1 \theta_{xt}^2$$

Therefore, exploiting Young's inequality and Poincaré's inequality, we get, for some constant $\tilde{c} > 0$,

$$F'(t) \le -\beta_1 E(t) + \varepsilon\int_0^1 \left(\varphi_x^2 + \varphi_t^2\right) - \left(cN - \frac{\tilde{c}}{\varepsilon}\right)\int_0^1 \theta_{xt}^2, \qquad \forall \varepsilon > 0.$$

By using (2.2) and the the fact that

$$\begin{aligned} \int_0^1 \left(\varphi_x^2 + \varphi_t^2\right) &\le \int_0^1 \varphi_t^2 + 2\int_0^1 (\varphi_x + \psi)^2 + 2\int_0^1 \psi^2 \\ &\le \int_0^1 \varphi_t^2 + 2\int_0^1 (\varphi_x + \psi)^2 + 2\int_0^1 \psi_x^2 \\ &\le \left(\frac{2}{\rho_1} + \frac{4}{k} + \frac{4}{b}\right)E(t), \end{aligned}$$

one can easily deduce that

$$F'(t) \le -\left(\beta_1 - \frac{2\varepsilon}{\rho_1} - \frac{4\varepsilon}{k} - \frac{4\varepsilon}{b}\right)E(t) - \left(cN - \frac{\tilde{c}}{\varepsilon}\right)\int_0^1 \theta_{xt}^2.$$

Choosing ε small enough and N even larger (if needed), we get

$$F'(t) \le -\frac{\beta_1}{2}E(t), \qquad \forall\, t > 0. \tag{2.19}$$

By integrating (2.19) over $(0, t)$ we arrive at

$$\int_0^t E(s)\,ds \le \frac{2}{\beta_1}\left(F(0) - F(t)\right) \le \frac{2}{\beta_1}F(0) = \beta_2 F(0).$$

We then use the fact that E is non-increasing to obtain

$$tE(t) \le \beta_2 F(0) \le \beta_3(E(0) + E_2(0)),$$

which yields, in turn,

$$E(t) \le \frac{c_0}{t},\ \forall\, t > 0.$$

This completes the proof of Theorem 2.1. □

Remark 2.2. In the case of equal speed, the exponential decay result can be established for H^1-solutions. See [13].

3. BOUNDARY CONDITIONS $\varphi_x = \psi = \theta_x = 0$ at $x = 0, 1$

In this section, we consider system (1.1) together with the initial conditions (2.1), and the boundary conditions (1.8); namely,

$$\varphi_x(0,t) = \varphi_x(1,t) = \theta_x(0,t) = \theta_x(1,t) = \psi(0,t) = \psi(1,t) = 0. \tag{3.1}$$

From equation $(1.1)_3$ and the boundary conditions (3.1), we easily verify that

$$\frac{d}{dt}\int_0^1 \theta(x,t)dx = 0.$$

So, we set

$$\bar{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)dx$$

to conclude that $(\varphi, \psi, \bar{\theta})$ satisfies system (1.1), $\bar{\theta}_x(0,t) = \bar{\theta}_x(1,t) = 0$ and more importantly

$$\int_0^1 \bar{\theta}(x,t)dx = 0\,,\ \forall\, t \geq 0.$$

Similarly to Section 2, we work with $\bar{\theta}$ but we use θ for simplicity.

Our main result in this section is

Theorem 3.1. *Let (φ, ψ, θ) be any H^2-regular solution of* (1.1), (2.1), (3.1). *Then there exist two positive constants λ, c_0, for which the energy functional E satisfies, $\forall\, t > 0$,*

$$E(t) \leq c_0 e^{-\lambda t} \qquad \text{if } \frac{k}{\rho_1} = \frac{b}{\rho_2} \tag{3.2}$$

$$E(t) \leq \frac{c_0}{t} \qquad \text{if } \frac{k}{\rho_1} \neq \frac{b}{\rho_2}. \tag{3.3}$$

In order to prove this result, we define

$$J_2(t) = \rho_2\rho_3 \int_0^1 \psi_t \int_0^x \theta(y,t)dy. \tag{3.4}$$

Lemma 3.2. *The functional J_2 satisfies, along solutions of* (1.1), (3.1) *and for any $\varepsilon_2 > 0$,*

$$J_2'(t) \leq -\frac{\delta\rho_2}{2}\int_0^1 \psi_t^2 + \varepsilon_2 \int_0^1 (\varphi_x^2 + \psi_x^2) + \frac{c}{\varepsilon_2}\int_0^1 \theta_x^2. \tag{3.5}$$

Proof. By taking in account equations (1.1) and $\int_0^1 \theta(x,t)dx = 0$, we get

$$\begin{aligned} J_2'(t) &= \rho_2 \int_0^1 \psi_t(k\theta_x - \gamma\psi_t) - \rho_3 b \int_0^1 \theta\psi_x + k\int_0^1 \theta\varphi \\ &\quad + \rho_3 k \int_0^1 \psi \left(\int_0^x \theta(y,t)dy\right) - \gamma\rho_3 \int_0^1 \theta^2. \end{aligned} \tag{3.6}$$

By using Young's and Poincaré's inequality, (3.6) yields the desired result. □

For N and N_2, positive constants, we define

$$J_3 = NE + I_1 + N_2 J_2.$$

It is straight forward to see that

$$\begin{aligned} J_3'(t) &\leq -\left(N\kappa - \frac{c}{\varepsilon_1} - \frac{c}{\varepsilon_2}\right)\int_0^1 \theta_x^2 - \left(\frac{N_2\gamma\rho_2}{2} - \frac{c}{\varepsilon_1}\right)\int_0^1 \psi_t^2 \\ &\quad - \left(\frac{b}{2} - N_2\varepsilon_2\right)\int_0^1 \psi_x^2 + \varepsilon_1 \int_0^1 \varphi_t^2 + N_2\varepsilon_2 \int_0^1 \varphi_x^2. \end{aligned} \tag{3.7}$$

At this point, we introduce our functional $\mathcal{L} = J_3 + \mu I_4 + \tau I_5$, where I_4 and I_5 are given in Section 2. Direct computations, using (2.11), (2.12) and (3.7), lead to

$$\begin{aligned} \mathcal{L}'(t) \leq\; & -\left(N\kappa - \frac{c}{\varepsilon_1} - \frac{c}{\varepsilon_2} - c(\mu + c)\right)\int_0^1 \theta_x^2 \\ & -\left(\frac{N_2\gamma\rho_2}{2} - \frac{c}{\varepsilon_1} - \mu\rho_2 + \tau\rho_2\right)\int_0^1 \psi_t^2 \\ & -\left(\frac{b}{2} - 3N_2\varepsilon_2 - \tau c\right)\int_0^1 \psi_x^2 + (\tau\rho_2 - \varepsilon_1)\int_0^1 \varphi_t^2 \\ & -\left[\frac{k}{2}(\mu - 2\tau) - 2N_2\varepsilon_2\right]\int_0^1 (\varphi_x + \psi)^2 + \mu\left(\frac{\rho_1 b}{K} - \rho_2\right)\int_0^1 \varphi_t\psi_{xt}, \end{aligned}$$

where we have used again

$$\varphi_x^2 \leq 2(\varphi_x + \psi)^2 + 2\psi^2.$$

A similar choice of the constants, then leads to

$$\mathcal{L}'(t) \leq -\alpha_0 E(t) + \mu\left(\frac{\rho_1 b}{k} - \rho_2\right)\int_0^1 \varphi_t\psi_{xt},$$

for some $\alpha_0 > 0$. The rest of the proof goes exactly as in the previous section.

4. NON-EXPONENTIAL DECAY IF $\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$

For the problem discussed in Section 3, we will show that the equal-wave speed propagation is necessary for the exponential decay.

Theorem 4.1. *If*

$$\frac{k}{\rho_1} \neq \frac{b}{\rho_2}.$$

Then any solution of problem (1.1), (2.1), (3.1) *is not exponentially stable.*

Proof. As in [13], let $V = (\varphi, \varphi_t, \psi, \psi_t, \theta)^T$, then V satisfies, formally, the problem

$$V_t = \mathcal{A}V, \qquad V(0) = V_0,$$

where $V_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0)^T$ and $\mathcal{A}$ is the "formal" differential operator

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{k}{\rho_1}\partial_x^2 & 0 & \frac{k}{\rho_1}\partial_x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{k}{\rho_2}\partial_x & 0 & \frac{b}{\rho_2}\partial_x^2 - \frac{k}{\rho_2} & 0 & -\frac{\gamma}{\rho_2}\partial_x \\ 0 & 0 & 0 & -\frac{\gamma}{\rho_3}\partial_x & \frac{\kappa}{\rho_3}\partial_x^2 \end{pmatrix}.$$

Let

$$\mathcal{H} = H^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L_*^2(0,1) \times L^2(0,1)$$

be the Hilbert space with

$$L_*^2(0,1) = \left\{ w \in L^2(0,1) \ / \ \int_0^1 w(x)ds = 0 \right\}$$

and the norm given by

$$\begin{aligned} ||V||_{\mathcal{H}}^2 &= ||(V^1, V^2, V^3, V^4, V^5)^T||_{\mathcal{H}}^2 \\ &= k||V_x^1 + V^3||_{L^2}^2 + b||V_x^3||_{L^2}^2 + \rho_2||V^4||_{L^2}^2 + \rho_3||V^5||_{L^2}^2. \end{aligned}$$

The domain of $\mathcal{A}$ is then

$$\begin{aligned} D(\mathcal{A}) &= \big\{ V \in \mathcal{H} \ / \ V^1 \in H^2(0,1), \ V_x^1, V^2, V^4, V_x^5 \in H_0^1(0,1), \\ &\qquad V^3 \in H^2(0,1) \cap H_0^1(0,1), V^5 \in H_*^2(0,1) \cap L_*^2(0,1) \big\} \end{aligned}$$

with

$$H_*^2(0,1) = \left\{ w \in H^2(0,1) \ / \ \int_0^1 w(x)ds = 0 \right\}.$$

To prove the non-exponential decay, we use the same approach as in [13]. From Theorem 1.3.2 in [8], it suffices to show the existence of sequences $(\lambda_n) \subset \mathbb{R}$ with $\lim_{n\to+\infty} |\lambda_n| = +\infty$, $(V_n) \subset D(\mathcal{A})$, and $(F_n) \subset \mathcal{H}$, such that $(i\lambda_n I - \mathcal{A})V_n = F_n$ is bounded and $\lim_{n\to+\infty} ||V_n|| = +\infty$.

We choose $F_n = F = (0, 0, f_3, f_4, f_5) \in \mathcal{H}$ with

$$f_3(x) = \cos(\delta\lambda x), \quad f_4(x) = \sin(\delta\lambda x), \quad f_5(x) = \cos(\delta\lambda x),$$

where

$$\delta = \sqrt{\frac{\rho_1}{k}}, \quad \lambda = \frac{n\pi}{\delta}, \qquad \forall n \in \mathbb{N}.$$

We solve the following system of equations

$$i\lambda\varphi - u = 0 \tag{4.1}$$

$$i\lambda\psi - v = 0 \tag{4.2}$$

$$i\lambda u - \frac{k}{\rho_1}(\varphi_{xx} + \psi_x) = f_3 \tag{4.3}$$

$$i\lambda v - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x + \psi) = f_4 \tag{4.4}$$

$$i\lambda\theta - \frac{\kappa}{\rho_3}\theta_{xx} + i\lambda\frac{\gamma}{\rho_3}\psi_x = f_5. \tag{4.5}$$

Eliminating u, v in (4.1)-(4.4), we obtain

$$-\lambda^2\varphi - \frac{k}{\rho_1}(\varphi_{xx} + \psi_x) = f_3 \tag{4.6}$$

$$-\lambda^2\psi - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x + \psi) = f_4 \tag{4.7}$$

$$i\lambda\theta - \frac{\kappa}{\rho_3}\theta_{xx} + i\lambda\frac{\gamma}{\rho_3}\psi_x = f_5. \tag{4.8}$$

This can be solved by

$$\varphi(x) = A\cos(\delta\lambda x), \quad \psi(x) = B\sin(\delta\lambda x), \quad \theta(x) = C\cos(\delta\lambda x), \tag{4.9}$$

where A, B and C are constants depending on λ to be determined. Inserting (4.9) in (4.6)–(4.8) we get

$$\begin{aligned}
(-\lambda^2 + \frac{k}{\rho_1}\delta^2\lambda^2)A + \frac{k}{\rho_1}\delta\lambda B &= 1\\
-\frac{k}{\rho_2}\delta\lambda A + [(-\lambda^2 + \frac{b}{\rho_2}\delta^2\lambda^2) + \alpha]B &= 1\\
(i\lambda + \delta^2\lambda^2\frac{\kappa}{\rho_3})C + i\delta\lambda^2\frac{\gamma}{\rho_3}B &= 1.
\end{aligned}$$

Letting $\alpha = \frac{\kappa\rho_1}{k\rho_3}$ and $\beta = \frac{\gamma\rho_1}{k\rho_3}$ and using the fact $\frac{k}{\rho_1}\delta^2 - 1 = 0$, we arrive at

$$A = \frac{\rho_2\rho_1}{k^2}\left(\frac{k}{\rho_1} - \frac{b}{\rho_2}\right) - \frac{\rho_2}{k\delta\lambda} + \frac{1}{\lambda^2}, \quad B = \frac{\rho_1}{k\delta\lambda}, \quad C = \frac{\alpha-\beta}{\alpha^2\lambda^2+1} - i\frac{\alpha\beta\lambda^2}{\alpha^2\lambda^3+\lambda}. \tag{4.10}$$

Now, let $V_n = (\varphi, \psi, i\lambda\varphi, i\lambda\psi, \theta)^T$, where φ, ψ and θ are given by (4.9) and (4.10). It is easy to check that

$$||V_n||_{\mathcal{H}} \geq \rho_1\lambda^2A^2\int_0^L \cos^2(\delta\lambda x)dx = \frac{1}{2}\rho_1\lambda^2A^2 \to +\infty, \text{ as } |\lambda| \to +\infty.$$

On the other hand, using (4.1)–(4.8) we deduce that

$$||(i\lambda I - A)V_n||_{\mathcal{H}} = \rho_1||f_3||^2_{L^2(0,L)} + \rho_2||f_4 + \frac{\gamma}{\rho_3}\theta_x||^2_{L^2(0,L)} + \rho_3||f_5||^2_{L^2(0,L)},$$

which implies that

$$||(i\lambda I - A)V_n||_{\mathcal{H}} \leq \frac{1}{2}\left[(\rho_1+\rho_2+\rho_3)+\frac{\rho_2\delta^2\gamma^2}{\rho_3^2}\lambda^2|C|^2\right].$$

Recalling (4.10), we easily see that $||(i\lambda I - A)V_n||_{\mathcal{H}}$ remains bounded as $|\lambda|$ goes to $+\infty$. The proof is thus complete. □

Remark. By repeating exactly the computations of Rivera and Racke [13] and estimating the term $\int_0^1 \varphi_t\psi_{xt}$ as in (2.18), Theorem 3.1 also holds for problem (1.1), (1.2).

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