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OSCILLATION CRITERIA FOR SECOND ORDER DAMPED DYNAMIC EQUATIONS

QIAOSHUN YANG [1,2] AND ZHITING XU [1,*]

[1]School of Mathematical Sciences, South China Normal University,
Guangzhou, 510631, P. R. China
[2]Department of Mathematics and Computer Science, Normal College,
Jishou University, Jishou, Hunan, 416000, P. R. China

ABSTRACT. In this paper, we establish oscillation criteria for second order damped dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x^{\Delta}(t) + q(t)f(x^{\sigma}(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}$$

on a time scale $\mathbb{T}$. In particular, Our results obtained here can be applied to arbitrary time scales $\mathbb{T}$ and drop the restriction $p(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$ in the literature. Some applications and examples are given to illustrate the main results.

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1. INTRODUCTION

The theory of time scales, which has recently received much attention, was introduced by Hilger [8] in order to unify continuous and discrete analysis. For completeness, we recall the following concepts related to the notion of time scales, see [3, 4] for more details. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since the oscillation of solutions near infinity is our primary concern, throughout this paper we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. On any time scale $\mathbb{T}$ we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T}, s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$; here $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ with $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function μ for the time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$, the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. A function $g : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is said to be rd-continuous provided g is continuous at right-dense points in $[t_0, \infty)_{\mathbb{T}}$ and

* Corresponding author. *E-mail address:* yqs244@163.com (Q. Yang), xuzhit@126.com (Z. Xu).
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g has finite left-hand limits at the left-dense points in $[t_0, \infty)_{\mathbb{T}}$, The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$, and the set of functions $g : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ which are rd-continuous and $1 + \mu(t)g(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ denoted by $g \in \mathcal{R}^+$. If $p \in \mathcal{R}^+$, then we can define the exponential function by

$$e_p(t, s) = \exp\Big(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\Big)$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is defined by

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Alternately, for $p \in \mathcal{R}^+$, one can define the exponential function $e_p(., t_0)$ to be the unique solution of the IVP: $x^\Delta(t) = p(t)x(t)$ with $x(t_0) = 1$.

In this paper, we consider the second order damped nonlinear dynamic equation

$$(r(t)x^\Delta(t))^\Delta + p(t)x^\Delta(t) + q(t)f(x^\sigma(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

on a time scale $\mathbb{T}$. Throughout this paper, we always assume that

(A1) $r(t)$, $q(t)$ are positive rd-continuous functions on $[t_0, \infty)_{\mathbb{T}}$;
(A2) $p(t)$ is a rd-continuous function on $[t_0, \infty)_{\mathbb{T}}$, and $-p/r \in \mathcal{R}^+$, i.e., $1+\mu(t)\big(\frac{-p(t)}{r(t)}\big) > 0$;
(A3) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $xf(x) > 0$ and $f(x) \geq Kx$ for $x \neq 0$ and some $K > 0$.

As will be seen later, in order to discuss the oscillatory properties of (1.1), it is necessary to consider both two cases:

$$\int_{t_0}^{\infty} \frac{1}{r(t)} e_{-p/r}(t, t_0)\Delta t = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(t)} e_{-p/r}(t, t_0)\Delta t < \infty. \tag{1.3}$$

By a solution of (1.1) we mean a nontrivial real-valued function $x \in C^1_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$, $T_x \in [t_0, \infty)_{\mathbb{T}}$, which has the property that $rx^\Delta \in C^1_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ and satisfies Eq. (1.1) on $[T_x, \infty)_{\mathbb{T}}$. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is said to be nonoscillatory.

During the last decades, there has been an increasing interest in studying the oscillation of solution of second order damped dynamic equations on time scale which attempts to harmonize and the oscillation theory for the continuous and the discrete. We refer the reader to the papers [2, 5, 6, 7, 9, 11, 12] and the references cited therein.

Recently, in [11], the authors considered the dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x^{\Delta\sigma}(t) + q(t)f(x^{\sigma}(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.4}$$

under the following conditions:

(H1) $r(t), p(t), q(t)$ are positive *rd*-continuous functions on $[t_0, \infty)_{\mathbb{T}}$;
(H2) $f : \mathbb{R} \to \mathbb{R}$ is such that $xf(x) > 0$;
(H3) $f : \mathbb{R} \to \mathbb{R}$ is such that $f(x) \geq Kx$ for $x \neq 0$ and some $K > 0$,

and they established the following results:

Theorem 1.1 (see, [11, Theorem 2.1]). *Assume that* (H1)-(H3) *and* (1.2) *hold. Furthermore, assume that there exists a positive* rd*-continuous differentiable function* $\rho(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[K\rho(s)q(s) - \frac{r(s)\psi^2(s)}{4\rho(s)}\Big]\Delta s = \infty,$$

where

$$\psi(t) = \frac{r^{\sigma}(t)\rho^{\Delta}(t) - \rho(t)p(t)}{r^{\sigma}(t)}.$$

Then Eq. (1.4) *is oscillatory.*

Theorem 1.2 (see, [11, Theorem 2.2]). *Let* (H1)-(H3) *and* (1.2) *hold. Let* $\rho(t)$ *be as defined in Theorem* 2.1 *and let* $H : \mathbb{D} \to \mathbb{R}$ *be* rd*-continuous such that* H *belongs to the class* $\mathfrak{R}$ *and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[KH(t,s)\rho(s)q(s) - \frac{(\rho^{\sigma}(s))^2 r(s)A^2(t,s)}{4\rho(s)H(t,s)}\Big]\Delta s = \infty,$$

where

$$A(t,s) = H(t,s)\frac{\psi(s)}{\rho^{\sigma}(s)} + H^{\Delta_s}(t,s),$$

and the function set $\mathfrak{R}$ *is defined in* [11]. *Then Eq.* (1.4) *is oscillatory.*

Here, we note that Theorems 1.1 and 1.2 (see also, [11, Theorems 2.1, 2.2]) are valid only when $p(t) > 0$ which is a restrictive condition. The aim of this paper is to obtain some new oscillation criteria dropping the restriction $p(t) > 0$. Our results obtained here can be applied to arbitrary time scales $\mathbb{T}$. Finally, some applications and examples are given to illustrate the main results.

2. MAIN RESULTS

For simplicity, define, for $t \in [t_0, \infty)_{\mathbb{T}}$,

$$R(t) = \frac{r(t)}{e_{-p/r}(t,t_0)} \int_{t_0}^{t} \frac{1}{r(s)} e_{-p/r}(s,t_0)\Delta s, \quad \alpha(t) = \frac{R(t)}{R(t)+\mu(t)},$$

and

$$\beta(t) = \begin{cases} \alpha(t)p(t), & p(t) \geq 0, \\ p(t), & p(t) < 0. \end{cases}$$

We begin with the following lemma.

Lemma 2.1. *Let* (1.2) *hold. Assume that Eq.* (1.1) *has a positive solution* $x(t)$ *on* $[t_0,\infty)_{\mathbb{T}}$. *Then*

$$x^{\Delta}(t)>0,\quad \frac{x(t)}{x^{\sigma}(t)}\ge\alpha(t),\quad t\in[t_0,\infty)_{\mathbb{T}}. \tag{2.1}$$

Proof. From (1.1), and noting that $x(t)>0$ on $[t_0,\infty)_{\mathbb{T}}$, we have

$$(r(t)x^{\Delta}(t))^{\Delta}+p(t)x^{\Delta}(t)<0.$$

Hence, by [3, Theorem 1.20],

$$\begin{aligned}\Big(\frac{r(t)x^{\Delta}(t)}{e_{-p/r}(t,t_0)}\Big)^{\Delta}=&\frac{(r(t)x^{\Delta}(t))^{\Delta}e_{-p/r}(t,t_0)-e^{\Delta}_{-p/r}(t,t_0)r(t)x^{\Delta}(t)}{e_{-p/r}(t,t_0)e^{\sigma}_{-p/r}(t,t_0)}\\ =&\frac{(r(t)x^{\Delta}(t))^{\Delta}+p(t)x^{\Delta}(t)}{e^{\sigma}_{-p/r}(t,t_0)}<0.\end{aligned}$$

Then, $\frac{r(t)x^{\Delta}(t)}{e_{-p/r}(t,t_0)}$ is strictly decreasing on $[t_0,\infty)_{\mathbb{T}}$. We now claim that $x^{\Delta}(t)>0$ on $t\in[t_0,\infty)_{\mathbb{T}}$. If not, then there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x^{\Delta}(t_1)<0$. It follows that

$$\frac{r(t)x^{\Delta}(t)}{e_{-p/r}(t,t_0)}\le\frac{r(t_1)x^{\Delta}(t_1)}{e_{-p/r}(t_1,t_0)}:=c<0,\quad t\ge t_1,$$

i.e.,

$$x^{\Delta}(t)\le\frac{c}{r(t)}e_{-p/r}(t,t_0). \tag{2.2}$$

Integrating (2.2) from t_1 to t, we find from (1.2) that

$$x(t)\le x(t_1)+c\int_{t_1}^{t}\frac{1}{r(s)}\,e_{-p/r}(s,t_0)\Delta s\to-\infty \text{ as } t\to\infty,$$

this implies that $x(t)$ is eventually negative, which is a contradiction to $x(t)>0$ on $[t_0,\infty)_{\mathbb{T}}$. Thus, $x^{\Delta}(t)>0$ on $[t_0,\infty)_{\mathbb{T}}$. Therefore,

$$\begin{aligned}x(t)>x(t)-x(t_0)=&\int_{t_0}^{t}\frac{r(s)x^{\Delta}(s)}{e_{-p/r}(s,t_0)}\,\frac{e_{-p/r}(s,t_0)}{r(s)}\Delta s\\ >&\frac{r(t)x^{\Delta}(t)}{e_{-p/r}(t,t_0)}\int_{t_0}^{t}\frac{1}{r(s)}\,e_{-p/r}(s,t_0)\Delta s,\end{aligned}$$

which yields

$$x(t)>R(t)x^{\Delta}(t).$$

Consequently,

$$\frac{x(t)}{x^{\sigma}(t)}=\frac{x(t)}{x(t)+\mu(t)x^{\Delta}(t)}\ge\frac{R(t)}{R(t)+\mu(t)}=\alpha(t).$$

This completes the proof. ∎

We are now in a position to state and prove our main results.

Theorem 2.1. *Let* (1.2) *hold. Assume that there exists a positive* Δ*-differentiable function* $\delta(t)$ *such that*

$$\lim_{t\to\infty}\int_{t_0}^{t}\Big(Kq(s)\delta^\sigma(s)-\frac{\big(r(s)\delta^\Delta(s)-\beta(s)\delta^\sigma(s)\big)^2}{4\alpha(s)r(s)\delta^\sigma(s)}\Big)\Delta s=\infty. \tag{2.3}$$

Then Eq. (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary and assume that Eq. (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we assume that there exists a $T\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ for $t\in[T,\infty)_{\mathbb{T}}$. Let

$$w(t)=\delta(t)\frac{r(t)x^\Delta(t)}{x(t)},\quad t\in[T,\infty)_{\mathbb{T}}.$$

By [3, Theorem 1.20] and (1.1), we have

$$\begin{aligned}
w^\Delta&=\delta^\Delta\frac{rx^\Delta}{x}+\delta^\sigma\Big(\frac{rx^\Delta}{x}\Big)^\Delta\\
&=\frac{\delta^\Delta}{\delta}w+\delta^\sigma\Big[\frac{(rx^\Delta)^\Delta}{x^\sigma}-\frac{r(x^\Delta)^2}{xx^\sigma}\Big]\\
&=\frac{\delta^\Delta}{\delta}w+\delta^\sigma\Big[-Kq-\frac{px^\Delta}{x^\sigma}-\frac{r(x^\Delta)^2}{xx^\sigma}\Big].
\end{aligned}$$

If $p(t)\ge 0$, then, by the fact $x(t)/x^\sigma(t)>\alpha(t)$ (in view of Lemma 2.1), we get

$$\begin{aligned}
w^\Delta&=\frac{\delta^\Delta}{\delta}w+\delta^\sigma\Big[-Kq-p\frac{x^\Delta}{x}\frac{x}{x^\sigma}-\frac{x}{rx^\sigma}\Big(\frac{rx^\Delta}{x}\Big)^2\Big]\\
&<\frac{\delta^\Delta}{\delta}w+\delta^\sigma\Big[-Kq-\frac{\alpha p}{r\delta}w-\frac{\alpha}{r}\Big(\frac{w}{\delta}\Big)^2\Big]\\
&=-Kq\delta^\sigma+\Big(\frac{\delta^\Delta}{\delta}-\frac{\alpha p\delta^\sigma}{r\delta}\Big)w-\frac{\alpha\delta^\sigma}{r\delta^2}w^2. \qquad (2.4)
\end{aligned}$$

If $p(t)<0$, then, noting that $x^\Delta(t)>0$, we have

$$\begin{aligned}
w^\Delta&<\frac{\delta^\Delta}{\delta}w-\delta^\sigma\Big[Kq+\frac{px^\Delta}{x}+\frac{r(x^\Delta)^2}{xx^\sigma}\Big]\\
&<-Kq\delta^\sigma+\Big(\frac{\delta^\Delta}{\delta}-\frac{p\delta^\sigma}{r\delta}\Big)w-\frac{\alpha\delta^\sigma}{r\delta^2}w^2. \qquad (2.5)
\end{aligned}$$

Hence, by (2.4), (2.5) and the definition of $\beta(t)$, we get

$$\begin{aligned}
w^\Delta&<-Kq\delta^\sigma+\Big(\frac{\delta^\Delta}{\delta}-\frac{\beta\delta^\sigma}{r\delta}\Big)w-\frac{\alpha\delta^\sigma}{r\delta^2}w^2\\
&=-Kq\delta^\sigma-\frac{\alpha\delta^\sigma}{r\delta^2}\Big[w-\frac{\delta(r\delta^\Delta-\beta\delta^\sigma)}{2\alpha\delta^\sigma}\Big]^2+\frac{(r\delta^\Delta-\beta\delta^\sigma)^2}{4\alpha r\delta^\sigma}\\
&\le -Kq\delta^\sigma+\frac{(r\delta^\Delta-\beta\delta^\sigma)^2}{4\alpha r\delta^\sigma}. \qquad (2.6)
\end{aligned}$$

Integrating (2.6) from T to t, we obtain

$$\int_T^t\Big(Kq(s)\delta^\sigma(s)-\frac{(r(s)\delta^\Delta(s)-\beta(s)\delta^\sigma(s))^2}{4\alpha(s)r(s)\delta^\sigma(s)}\Big)\Delta s<w(T)-w(t)\le w(T),$$

which contradicts (2.3). This completes the proof. $\square$

We next establish Philos-type oscillation criterion [10] for Eq. (1.1). First, let us introduce the class of functions $\mathcal{H}$ which will be extensively used in the sequel.

Let $\mathbb{D}_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \geq t_0\}$ and $\mathbb{D} \equiv \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. We say the function H is belonged to the class $\mathcal{H}$, denoted by $H \in \mathcal{H}$, if $H : [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}$ and satisfies

(i) $H(t,t) = 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$, $H(t,s) > 0$ on $\mathbb{D}_0$;
(ii) $H^{\Delta_s}(t,s) \leq 0$ on $\mathbb{D}$, and for each fixed t, $H(t,s)$ is rd-continuous function with respect to s.

Theorem 2.2. *Let* (1.2) *hold. Assume that there exist a positive* Δ*-differentiable function* $\delta(t)$ *and* $H \in \mathcal{H}$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,s)\Big(Kq(s)\delta^\sigma(s) - \frac{\big(r(s)\delta^\Delta(s) - \beta(s)\delta^\sigma(s)\big)^2}{4\alpha(s)r(s)\delta^\sigma(s)}\Big)\Delta s = \infty, \tag{2.7}$$

for sufficiently large T. *Then Eq.* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary and assume that $x(t)$ is a nonoscillatory solution of Eq. (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality we suppose that $x(t) > 0$ for $t \in [T,\infty)_{\mathbb{T}} \subseteq [t_0,\infty)_{\mathbb{T}}$. Proceeding as the proof of Theorem 2.1, we get (2.6) hold. i.e.,

$$Kq(t)\delta^\sigma(t) - \frac{(r(t)\delta^\Delta(t) - \beta(t)\delta^\sigma(t))^2}{4\alpha(t)r(t)\delta^\sigma(t)} < -w^\Delta(t). \tag{2.8}$$

Multiplying (2.8)(with t replaced by s) by $H(t,s)$, and integrating with respect to s from T to t, we get

$$\int_T^t H(t,s)\Big(Kq(s)\delta^\sigma(s) - \frac{(r(s)\delta^\Delta(s) - \beta(s)\delta^\sigma(s))^2}{4\alpha(s)r(s)\delta^\sigma(s)}\Big)\Delta s < -\int_T^t H(t,s)w^\Delta(s)\Delta s.$$

Integrating by parts, for $t \in [T,\infty)_{\mathbb{T}}$, gives

$$\begin{aligned}&\int_T^t H(t,s)\Big(Kq(s)\delta^\sigma(s) - \frac{(r(s)\delta^\Delta(s) - \beta(s)\delta^\sigma(s))^2}{4\alpha(s)r(s)\delta^\sigma(s)}\Big)\Delta s\\ &< H(t,T)w(T) + \int_T^t H^{\Delta_s}(t,s)w^\sigma(s)\Delta s\\ &\leq H(t,T)w(T).\end{aligned}$$

Note that $H^{\Delta_s}(t,s) \leq 0$ on $\mathbb{D}$ and $w(t) > 0$, then

$$\frac{1}{H(t,T)}\int_T^t H(t,s)\Big(Kq(s)\delta^\sigma(s) - \frac{(r(s)\delta^\Delta(s) - \beta(s)\delta^\sigma(s))^2}{4\alpha(s)r(s)\delta^\sigma(s)}\Big)\Delta s < w(T),$$

which contradicts (2.7). This completes the proof. $\square$

As an immediate consequence of Theorem 2.2, we have

Corollary 2.1. *Let assumption* (2.7) *in Theorem* 2.2 *be replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)q(s)\delta^\sigma(s)\Delta s=\infty,$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \frac{H(t,s)\big(r(s)\delta^\Delta(s)-\beta(s)\delta^\sigma(s)\big)^2}{4\alpha(s)r(s)\delta^\sigma(s)}\Delta s<\infty.$$

Then Eq. (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

The results in Theorem 2.2 are very general, with appropriate choices of the function $H(t,s)$ in Theorem 2.2, we can obtain different conditions for oscillation of (1.1). For instance, define $H(t,s)$ by

$$H(t,s)=(t-s)^m,\quad m\ge 1,\quad (t,s)\in\mathbb{D}_0,$$

we have the following oscillation result.

Corollary 2.2. *Let* (1.2) *hold. Assume that there exists a positive* Δ*-differentiable function* $\delta(t)$ *such that*

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^t (t-s)^m\Big(Kq(s)\delta^\sigma(s)-\frac{\big(r(s)\delta^\Delta(s)-\beta(s)\delta^\sigma(s)\big)^2}{4\alpha(s)r(s)\delta^\sigma(s)}\Big)\Delta s=\infty,$$

where $m\ge 1$. *Then Eq.* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

In the following, we consider the case when (1.3) holds.

Theorem 2.3. *Let* (1.3) *hold. Assume that there exists a positive* Δ*-differentiable function* $\delta(t)$ *such that* (2.3) *holds, and*

$$\int_{t_0}^\infty \frac{1}{r(t)}\int_{t_0}^t q(s)\,e_{-p/r}(t,\sigma(s))\Delta s\Delta t=\infty. \tag{2.9}$$

Then Eq. (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or converges to zero as* $t\to\infty$.

Proof. Suppose to the contrary and assume that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we assume that there exists a $T\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ for $t\in[T,\infty)_T$. Proceeding as the proof of Lemma 2.1, we know that $\frac{r(t)x^\Delta(t)}{e_{-p/r}(t,t_0)}$ is decreasing. Hence $x^\Delta(t)$ is either eventually positive or eventually negative. Thus, we shall distinguish the following two case:

(*i*) $x^\Delta(t)>0$ for $t\in[T,\infty)_{\mathbb{T}}$;
(*ii*) $x^\Delta(t)<0$ for $t\in[T,\infty)_{\mathbb{T}}$.

Case (*i*). Since the proof when $x^\Delta(t)$ eventually positive is similar to that of Theorem 2.1, we omit the details.

Case (ii). It follows from the fact $x^{\Delta}(t) < 0$ and $x(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$ that $\lim_{t\to\infty} x(t) = b \geq 0$. We will show that $b = 0$. If not, then, by (A2), there exists a $t_1 \in [T, \infty)_{\mathbb{T}}$ so that $f(x^{\sigma}(t)) \geq \frac{b}{2}K$ for $t \in [t_1, \infty)_{\mathbb{T}} \subseteq [T, \infty)_{\mathbb{T}}$. From (1.1), we have

$$-(r(t)x^{\Delta}(t))^{\Delta} = p(t)x^{\Delta}(t) + q(t)f(x^{\sigma}(t)) > p(t)x^{\Delta}(t) + \frac{b}{2}Kq(t). \tag{2.10}$$

Let $u(t) = r(t)x^{\Delta}(t)$. It follows from (2.10) that

$$u^{\Delta}(t) < -\frac{p(t)}{r(t)}u(t) - \frac{b}{2}Kq(t).$$

Hence, by [3, Theorem 6.1], we obtain

$$\begin{aligned} r(t)x^{\Delta}(t) = u(t) &< u(t_1)e_{-p/r}(t, t_1) - \frac{b}{2}K\int_{t_1}^{t} e_{-p/r}(t, \sigma(s))\, q(s)\Delta s \\ &< -\frac{b}{2}K\int_{t_1}^{t} q(s)\, e_{-p/r}(t, \sigma(s))\Delta s, \end{aligned}$$

since $u(t_1) = r(t_1)x^{\Delta}(t_1) < 0$. Dividing both sides by $r(t)$ and integrating from t_1 to t, we have

$$\int_{t_1}^{t} x^{\Delta}(s)\Delta s < -\frac{b}{2}K\int_{t_1}^{t}\frac{1}{r(s)}\int_{t_1}^{s} q(\tau)\, e_{-p/r}(s, \sigma(\tau))\Delta\tau\Delta s,$$

hence, by (2.9),

$$x(t) < x(t_1) - \frac{b}{2}K\int_{t_1}^{t}\frac{1}{r(s)}\int_{t_1}^{s} q(\tau)\, e_{-p/r}(s, \sigma(\tau))\Delta\tau\Delta s \to -\infty \text{ as } t \to \infty,$$

which contradicts the fact $x(t) > 0$, Thus $b = 0$. This completes the proof. □

Using the same ideas as the proof of Theorems 2.2 and 2.3, we obtain.

Theorem 2.4. *Let* (1.3)*and* (2.9) *hold. Assume that there exist a positive* Δ*-differentiable function* $\delta(t)$ *and* $H \in \mathcal{H}$ *such that* (2.7) *holds. Then Eq.* (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$ *or converges to zero as* $t \to \infty$.

Remark 2.1. Theorems 2.1 and 2.2 improve and extend [11, Theorems 2.1, 2.2], respectively.

3. APPLICATION AND EXAMPLES

In this section, we apply our main results to the time scales $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0}$ and establish some oscillation criteria for Eq. (1.1). Finally, we give two examples to illustrate our main results.

When $\mathbb{T} = \mathbb{N}$, Eq. (1.1) reduces to the second order difference equation

$$\Delta(r(n)\Delta x(n)) + p(n)\Delta x(n) + q(n)f(x(n+1)) = 0,\ n \geq n_0. \tag{3.1}$$

Hence, for Eq. (3.1), (1.2) and (1.3) become

$$\sum_{i=n_0}^{\infty} \frac{1}{r(i)}\, e_{-p/r}(i, n_0) = \infty, \tag{3.2}$$

and

$$\sum_{i=n_0}^{\infty} \frac{1}{r(i)}\, e_{-p/r}(i, n_0) < \infty, \tag{3.3}$$

respectively. Note that, for all $n \geq n_0$,

$$R(n) = \frac{r(n)}{e_{-p/r}(n, n_0)} \sum_{i=n_0}^{n-1} \frac{e_{-p/r}(i, n_0)}{r(i)}, \quad \alpha(n) = \frac{R(n)}{R(n)+1},$$

and

$$\beta(n) = \begin{cases} \alpha(n)p(n), & p(n) \geq 0, \\ p(n), & p(n) < 0. \end{cases}$$

By Theorems 2.1-2.4, we have

Theorem 3.1. *Let* (3.2) *hold. Assume that there exists a positive sequence* $\delta(n)$ *such that*

$$\lim_{n\to\infty} \sum_{i=n_0}^{n-1} \Big[Kq(i)\delta(i+1) - \frac{\big[r(i)\Delta\delta(i) - \beta(i)\delta(i+1)\big]^2}{4\alpha(i)r(i)\delta(i+1)}\Big] = \infty. \tag{3.4}$$

Then Eq. (3.1) *is oscillatory on* $\mathbb{N}$.

We say a double sequence $H \in \mathcal{H}_1$ if H satisfies

($B1$) $H(m, m) = 0$ for $m \geq m_0 > 0$, $H(m, n) > 0$ for $m > n \geq n_0 > 0$;
($B2$) $\Delta_2 H(m, n) = H(m, n+1) - H(m, n) \leq 0$ for $m > n \geq n_0 > 0$.

Theorem 3.2. *Let* (3.2) *hold. Assume that there exist a positive sequence* $\delta(n)$ *and* $H \in \mathcal{H}_1$ *such that*

$$\limsup_{n\to\infty} \frac{1}{H(n, n_0)} \sum_{i=n_0}^{n-1} H(n, i)\Big[Kq(i)\delta(i+1) - \frac{\big[r(i)\Delta\delta(i) - \beta(i)\delta(i+1)\big]^2}{4\alpha(i)r(i)\delta(i+1)}\Big] = \infty. \tag{3.5}$$

Then Eq. (3.1) *is oscillatory on* $\mathbb{N}$.

Theorem 3.3. *Let* (3.3) *hold. Assume that there exists a positive sequence* $\delta(n)$ *such that* (3.4) *and*

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} \sum_{i=n_0}^{n-1} q(i)\, e_{-p/r}(n, i+1) = \infty \tag{3.6}$$

hold. Then Eq. (3.1) *is oscillatory on* $\mathbb{N}$ *or converges to zero as* $n \to \infty$.

Theorem 3.4. *Let* (3.3) *hold. Assume that there exist a positive sequence* $\delta(n)$ *and* $H \in \mathcal{H}_1$ *such that* (3.5) *and* (3.6) *hold. Then every solution of Eq.* (3.1) *is oscillatory on* $\mathbb{N}$ *or converges to zero as* $n \to \infty$.

When $\mathbb{T} = q^{\mathbb{N}_0} = \{t : t = q^k, k \in \mathbb{N} \cup \{0\}, q > 1\}$, then Eq. (1.1) reduces to

$$\Delta_q(r(q^k)\Delta_q x(q^k)) + p(q^k)\Delta_q x(q^k) + q(q^k)f(x(q^{k+1})) = 0,\ k \geq k_0. \tag{3.7}$$

Hence, for Eq. (3.7), (1.2) and (1.3) become

$$\sum_{i=k_0}^{\infty} \frac{e_{-p/r}(q^i, q^{k_0})}{r(q^i)} = \infty, \tag{3.8}$$

and

$$\sum_{i=k_0}^{\infty} \frac{e_{-p/r}(q^i, q^{k_0})}{r(q^i)} < \infty, \tag{3.9}$$

respectively. It is easy to show that

$$R(q^k) = \frac{r(q^k)}{e_{-p/r}(q^k, q^{k_0})} \sum_{i=k_0}^{k-1} \frac{e_{-p/r}(q^i, q^{k_0})}{r(q^i)} \mu(q^i), \quad \alpha(q^k) = \frac{R(q^k)}{R(q^k) + q^k(q-1)},$$

and

$$\beta(q^k) = \begin{cases} \alpha(q^k)p(q^k), & p(q^k) \geq 0, \\ p(q^k), & p(q^k) < 0. \end{cases}$$

By Theorems 2.1–2.4, we have

Theorem 3.5. *Let* (3.8) *hold. Assume that there exists a positive sequence* $\delta(n)$ *such that*

$$\lim_{k\to\infty} \sum_{i=k_0}^{k-1} \Big[Kq(q^i)\delta(q^{i+1}) - \frac{\big[r(q^i)\Delta\delta(q^i) - \beta(q^i)\delta(q^{i+1})\big]^2}{4\alpha(q^i)r(q^i)\delta(q^{i+1})}\Big] q^i = \infty. \tag{3.10}$$

Then Eq. (3.7) *is oscillatory on* $q^{\mathbb{N}_0}$.

We say $H \in \mathcal{H}_2$ if H satisfies

$(C1)$ $H(q^k, q^k) = 0,\ k \geq k_0 > 0, H(q^k, q^s) > 0, k > s \geq k_0 > 0.$
$(C2)$ $H^{\Delta_s}(q^k, q^s) = \frac{1}{(q-1)q^s}\big[H(q^k,\ q^{s+1}) - H(q^k,\ q^s)\big] \leq 0,\ k > s \geq k_0 > 0.$

Theorem 3.6. *Let* (3.8) *hold. Assume that there exist a positive sequence* $\delta(n)$ *and* $H \in \mathcal{H}_2$ *such that*

$$\limsup_{k\to\infty} \frac{1}{H(q^k, q^{k_0})} \sum_{i=k_0}^{k-1} H(q^k, q^i)\Big[Kq(q^i)\delta(q^{i+1}) - \frac{\big[r(q^i)\Delta\delta(q^i) - \beta(q^i)\delta(q^{i+1})\big]^2}{4\alpha(q^i)r(q^i)\delta(q^{i+1})}\Big] q^i = \infty. \tag{3.11}$$

Then Eq. (3.7) *is oscillatory on* $q^{\mathbb{N}_0}$.

Theorem 3.7. *Let* (3.9) *hold. Assume that there exists a positive sequence* $\delta(n)$ *such that* (3.10) *holds, and*

$$\sum_{k=k_0}^{\infty} \frac{1}{r(q^k)} \Big[\sum_{i=k_0}^{k-1} e_{-p/r}(q^k, q^{i+1}) q(q^i) q^i \Big] q^k = \infty. \tag{3.12}$$

Then Eq. (3.7) *is oscillatory on* $q^{\mathbb{N}_0}$ *or converges to zero as* $n \to \infty$.

Theorem 3.8. *Let* (3.9) *hold. Assume that there exist a positive sequence* $\delta(n)$ *and* $H \in \mathcal{H}_2$ *such that* (3.11) *and* (3.12) *hold. Then Eq.* (3.7) *is oscillatory on* $q^{\mathbb{N}_0}$ *or converges to zero as* $n \to \infty$.

Finally, we give two examples to illustrate our main results.

Example 3.1. Consider Eq. (3.1) with $K = 1$ and $n \geq 2$,

$$r(n) = 1,\ p(n) = \frac{1}{n^2},\ q(n) = 1 + \frac{1}{n^4}. \tag{3.13}$$

Here,

$$1 - \mu(n)\frac{p(n)}{r(n)} = 1 - \frac{1}{n^2} > 0.$$

By [1, Lemma 2], we have

$$e_{-p/r}(n,2) \geq 1 - \sum_{i=2}^{n-1} \frac{p(i)}{r(i)} = 1 - \sum_{i=2}^{n-1} \frac{1}{i^2} > 1 - \sum_{i=2}^{n-1} \frac{1}{(i-1)i} = \frac{1}{n-1},$$

and

$$e_{-p/r}(n,2) \leq \exp\Big(- \sum_{i=2}^{n-1} p(i) \Big) = \exp\Big(- \sum_{i=2}^{n-1} \frac{1}{i^2} \Big) \leq 1,$$

for all $n \geq 2$. Then

$$\sum_{i=2}^{\infty} \frac{1}{r(i)} e_{-p/r}(i,2) > \sum_{i=2}^{n-1} \frac{1}{i-1} = \sum_{i=1}^{n-2} \frac{1}{i} \to \infty \ \text{ as } \ n \to \infty.$$

Hence, (3.2) is satisfied. Also, we have

$$R(n) = \frac{1}{e_{-p/r}(n,2)} \sum_{i=2}^{n-1} e_{-p/r}(i,2) \geq \sum_{i=2}^{n-1} e_{-p/r}(i,2) > \sum_{i=1}^{n-2} \frac{1}{i},$$

and

$$\alpha(n) = \frac{R(n)}{R(n)+1} > \frac{R(n)}{2R(n)} = \frac{1}{2}$$

for sufficiently large n. Let $\delta(n) = 1$, we get

$$\lim_{n\to\infty} \sum_{i=2}^{n-1} \Big[q(i) - \frac{\alpha(i) p^2(i)}{4} \Big] > \lim_{n\to\infty} \sum_{i=2}^{n-1} \Big[q(i) - p^2(i) \Big]$$

$$= \lim_{n\to\infty} \sum_{i=2}^{n-1} \Big(1 + \frac{1}{i^4} - \frac{1}{i^4} \Big) = \lim_{n\to\infty} \Big(n - 2 \Big) = \infty,$$

i.e., (3.4) holds. Hence, by Theorem 3.1, Eq. (3.13) is oscillatory on $\mathbb{N}$.

Example 3.2. Consider Eq. (3.7) with $q = 2$, $K = 1$ and $k \geq 0$,

$$r(2^k) = e^{2^k-1}, \ \ p(2^k) = \frac{1}{2^k}(1-2^{2^k})e^{2^k-1}, \tag{3.14}$$

$$q(2^k) = 1 + \frac{1}{2^{\,2k}}(1-2^{2^k})^2 e^{2^k-1}.$$

Here,

$$1 - \mu(2^k)\frac{p(2^k)}{r(2^k)} = 1 - 2^k \ \frac{1}{e^{2^k-1}} \ \frac{1}{2^k}(1-2^{2^k})e^{2^k-1} = 2^{2^k} > 0,$$

and

$$\begin{aligned} e_{-p/r}(2^k, 1) &= \exp\Big(\sum_{i=0}^{k-1} \frac{\log_2\Big[1-\mu(2^i)\frac{p(2^i)}{r(2^i)}\Big]}{\mu(2^i)}\mu(2^i)\Big) \\ &= \exp\Big(\sum_{i=0}^{k-1} 2^i\Big) = e^{2^k-1}. \end{aligned}$$

Thus,

$$\sum_{i=0}^{\infty} \frac{e_{-p/r}(2^i, 1)}{r(2^i)}\mu(2^i) = \sum_{i=0}^{\infty} 2^i = \infty,$$

i.e., (3.8) is satisfied. Also, we have

$$R(2^k) = \frac{r(2^k)}{e_{-p/r}(2^k, 1)} \sum_{i=0}^{k-1} \frac{e_{-p/r}(2^i, 1)}{r(2^i)} 2^i = \sum_{i=0}^{k-1} 2^i = 2^k - 1.$$

Then

$$\alpha(2^k) = \frac{R(2^k)}{R(2^k)+\mu(2^k)} = \frac{2^k-1}{2^k-1+2^k} = \frac{2^k-1}{2^{k+1}-1}.$$

Clearly, $1/3 < \alpha(2^k) < 1$ for $k \geq 2$. Let $\delta(2^k) = 1$, we get

$$\begin{aligned} &\lim_{k\to\infty} \sum_{i=0}^{k-1} \Big[q(2^i) - \frac{p^2(2^i)}{4\alpha(2^i)r(2^i)}\Big]2^i \\ &> \lim_{k\to\infty} \sum_{i=0}^{k-1} \Big[q(2^i) - \frac{3}{4}\frac{p^2(2^i)}{r(2^i)}\Big]2^i \\ &> \lim_{k\to\infty} \sum_{i=0}^{k-1} \Big[q(2^i) - \frac{p^2(2^i)}{r(2^i)}\Big]2^i \\ &\geq \lim_{k\to\infty} \sum_{i=0}^{k-1} \Big[1 + \frac{(1-2^{2^i})^2 e^{2^i-1}}{2^{2i}} - \frac{(1-2^{2^i})^2 e^{2^i-1}}{2^{2i}}\Big]2^i \\ &= \lim_{k\to\infty} \Big(2^k - 1\Big) = \infty, \end{aligned}$$

i.e., (3.10) holds. Hence, by Theorem 3.5, Eq. (3.14) is oscillatory on $2^{\mathbb{N}_0}$.

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