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INTEGRAL BOUNDARY VALUE PROBLEMS FOR FRACTIONAL ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we consider the existence of generalized solutions for fractional order integro-differential equations with integral boundary value conditions. We firstly build a new comparison theorem. Then by utilizing the monotone iterative technique and the method of lower and upper generalized solutions, we obtain the existence of extremal generalized solutions or generalized quasi-solutions.

1. INTRODUCTION

In this paper, we deal with the integral boundary value problems for fractional order integro-differential equations

$$(1.1)\qquad \begin{cases} D^\alpha x(t) = f(t, x(t), (Tx)(t), (Sx)(t)), n-1 < \alpha \leq n, t \in J = [0,1], \\ x^{(k)}(0) = a_k, k = 1, 2, \dots, n-1, \\ x(0) = \lambda_1 x(\tau) + \lambda_2 \int_0^1 \omega(s, x(s))ds + b, \end{cases}$$

where $f \in C(J \times R^3, R)$, $\omega \in C(J \times R, R)$, $\tau \in J$, λ_1, λ_2, $b \in R$, $\lambda_2 \geq 0$, D^α is Caputo fractional derivative of order $n-1 < \alpha \leq n$ and

$$(Tx)(t) = \int_0^t k(t,s)x(s)ds, \quad Sx)(t) = \int_0^1 h(t,s)x(s)ds$$

$k \in C(D, R^+)$, $D = \{(t,s) \in J \times J : t \geq s\}$, $h \in C(J \times J, R^+)$, $R^+ = [0, \infty)$.

Differential equations with fractional order are generalization of ordinary differential equations to non-integer order. This generalization is not mere mathematical curiosities but rather has interesting applications in many areas of science and engineering such as electrochemistry, control, porous media, electromagnetic, etc.(see [1]–[3]). There has been a significant development in fractional differential equations in recent years(see [4]–[20]).

In [9], A. Anguraj, P. Karthikeyan and J. J. Trujillo have investigated the existence and uniqueness theorem for the nonlinear fractional mixed Volterra- Fredholm

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integro-differential equation with nonlocal initial condition

$$\begin{cases} \frac{d^\alpha x(t)}{dt^\alpha} = f(t, x(t), \int_0^t k(t,s,x(s))ds, \int_0^1 h(t,s,x(s))ds), \\ x(0) = \int_0^1 g(s)x(s)ds, \end{cases} \tag{1.2}$$

where $t \in J = [0,1]$, $0 < \alpha < 1$, $g(t) \in (0,1]$, $g(t) \in L^1([0,1], R_+)$, $x \in C(J,E)$ is a continuous function on J with values in the Banach space E and $\| x \|_C = \max_{t\in J} \| x(t) \|_E$, and $f : J \times E \times E \times E \to E, k : D \times E \to E$, and $h : D_0 \times E \to E$ are continuous E-valued functions. Here $D = \{(t,s) \in R^2 : 0 \le s \le t \le 1\}$, and $D_0 = J \times J$. The operator d^α/dt^α denotes the Caputo fractional derivative of order α. By means of Krasnoselkii theorem, some results on the existence of solutions are obtained for the above fractional boundary value problems.

It is well known that the monotone iterative technique is quite useful in the theory of differential equations and partial differential equations [21, 22, 23]. The method combining with the upper and lower solutions has also been used to solve the problems for nonlinear fractional differential equations in [13]-[16].

Motivated by [9], we will investigate the existence of generalized extremal solutions of the higher order fractional differential equations (1.1) by means of the method of lower and upper generalized solutions combined with the monotone iterative technique.

This paper is organized as follows: In section 2, we establish a new comparison principle and consider the linear problem of (1.1). In section 3, under the condition of $\lambda_1 \ge 0$, we obtain the existence of extremal generalized solutions for (1.1) by utilizing the monotone iterative technique and the method of lower and upper generalized solutions. In section 4, if $\lambda_1 < 0$, we obtain the existence of a coupled generalized quasi-solution of (1.1).

2. PRELIMINARIES AND COMPARISON PRINCIPLE

In this section, we introduce some preliminary facts which will be used throughout this paper.

Let $C(J,R)$ is Banach space with a norm $\| x \|_C = \max\{|x(t)| : t \in J\}$.

Definition 2.1 ([1]). The fractional (arbitrary) order integral of the function $y \in L^1([0,1], R)$ of order $\alpha > 0$ is defined by

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds.$$

Definition 2.2 ([1]). The Caputo fractional derivative of order $\alpha > 0$ of a function $y \in C^n([0,1], R)$ is given by

$$D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)ds,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of the number α.

Lemma 2.3 ([1]). *If $f(t) \in C^n([0,1], R)$ and $n - 1 < \alpha \leq n$, then*

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

Now we consider a linear initial value problem, which is important to us in obtaining the existence of generalized solutions for problem (1.1).

$$\begin{cases} D^\alpha u(t) - Mu(t) - N(Tu)(t) - N_1(Su)(t) = \rho(t), \quad t \in J, \\ u^{(k)}(0) = a_k, \ a_k \in R, \ k = 0, 1, \ldots, n-1, \end{cases} \tag{2.1}$$

where $n - 1 < \alpha \leq n, M, N, N_1,\ a_k \in R$ $(k = 1, \ldots, n-1)$ and $\rho(t) \in C(J, R)$.

We give a definition of generalized solutions for fractional differential equations as follows

Definition 2.4. We say that $u(t) \in C(J, R)$ is a generalized solution of fractional differential equations (2.1) if $u(t)$ can be represented by

$$u(t) = \sum_{k=0}^{n-1} \frac{a_k}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\rho(s) + Mu(s) + N(Tu)(s) + N_1(Su)(s)] ds. \tag{2.2}$$

Remark 2.5. If $u(t)$ is a solution of (2.1), i.e., $u \in C^n(J, R)$ and (2.1) holds, we easily get $u(t)$ is a generalized solution of (2.1) by Lemma 2.3, i.e., $u \in C(J, R)$ and (2.2) holds. However, by the following simple example, a generalized solution of (2.1) is not a solutions of (2.1) in general. But that $u \in C(J, R)$ is a generalized solution of (2.1) implies $u \in C^{n-1}(J, R)$ and $u^{(k)}(0) = a_k, k = 0, \ldots, n-1$ by a simple calculation.

Example 2.6. In (2.1), we let $\rho(t) = a$ (a is a constant), $M = N = N_1 = 0$, $\alpha = n - 1 + \frac{1}{2}$ (n is any natural number). According to (2.2), we get

$$u(t) = \sum_{k=0}^{n-1} \frac{a_k}{k!} t^k + \frac{a}{\Gamma(n - 1 + \frac{1}{2})} \int_0^t (t-s)^{n-2+\frac{1}{2}} ds = \sum_{k=0}^{n-1} \frac{a_k}{k!} t^k + \frac{a}{\Gamma(n + \frac{1}{2})} t^{n-1+\frac{1}{2}}, t \in [0,1],$$

which imply that $u(t) \notin C^n([0,1])$. According to the definition of Caupto derivative, we could not define Caupto derivative of order α for the generalized solution u of (2.1).

Lemma 2.7. *Assume that $M, N, N_1 \geq 0$ are constants and the following inequality holds*

$$\frac{M + Nk_0 + N_1 h_0}{\Gamma(\alpha + 1)} < 1, \quad n = 1, 2, \ldots, \tag{2.3}$$

then (2.1) *has a unique generalized solution.*

Proof. We firstly define an operator $A : C(J,R) \to C(J,R)$ by

$$Au = \sum_{k=0}^{n-1} \frac{a_k}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[\rho(s) + Mu(s) + N(Tu)(s) + N_1(Su)(s)]ds.$$

Then, we have

$$\begin{aligned}
\| Au - Av \|_C &= \frac{1}{\Gamma(\alpha)} \| \int_0^t (t-s)^{\alpha-1}[M(u(s)-v(s)) \\
&\quad + N((Tu)(s) - (Tv)(s)) + N_1((Su)(s) - (Sv)(s))]ds \|_C \\
&= \frac{1}{\Gamma(\alpha)} \| \int_0^t (t-s)^{\alpha-1}[M(u(s)-v(s)) \\
&\quad + N\int_0^s k(s,r)(u(r)-v(r))dr + N_1 \int_0^1 h(s,r)(u(r)-v(r))dr]ds \|_C \\
&\le \frac{M + Nk_0 + N_1 h_0}{\Gamma(\alpha)} \max_{t\in[0,1]} \int_0^t (t-s)^{\alpha-1} ds \| u - v \|_C \\
&\le \frac{M + Nk_0 + N_1 h_0}{\Gamma(\alpha+1)} \| u - v \|_C, \qquad n = 1, 2, \ldots .
\end{aligned}$$

From (2.3) and the Banach fixed point theorem, (2.1) has a unique generalized solution $u(t) \in C(J,R)$. □

Lemma 2.8. *Suppose $u(t) \in C^{n-1}(J,R)$ satisfies the following inequalities*

$$\begin{cases} u(t) \le \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) ds, \\ u^{(k)}(0) \le 0, \quad k = 0, \ldots, n-1, \end{cases} \tag{2.4}$$

where $M, N, N_1 \ge 0$ are constants,

$$q(t) = Mu(t) + N(Tu)(t) + N_1(Su)(t),$$

and the inequality (2.3) *holds. Then $u(t) \le 0$ for all $t \in J$.*

Proof. Suppose, to the contrary, that there exists a $t^* \in J$ such that $u(t^*) > 0$. Let $u(t^*) = \max\{u(t) : t \in J\} = \lambda, \lambda > 0$. We obtain that

$$\begin{aligned}
u(t) &\le \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[Mu(s) + N(Tu)(s) + N_1(Su)(s)]ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[Mu(s) + N(Tu)(s) + N_1(Su)(s)]ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[Mu(s) + N\int_0^s k(s,r)u(r)dr + N_1 \int_0^1 h(s,r)u(r)dr]ds \\
&\le \lambda \frac{M + Nk_0 + N_1 h_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\
&= \lambda \frac{M + Nk_0 + N_1 h_0}{\Gamma(\alpha+1)} t^\alpha
\end{aligned}$$

$$\leq \lambda \frac{M + Nk_0 + N_1 h_0}{\Gamma(\alpha + 1)}.$$

Let $t = t^*$. We have

$$\lambda \leq \lambda \frac{M + Nk_0 + N_1 h_0}{\Gamma(\alpha + 1)}.$$

So

$$\frac{M + Nk_0 + N_1 h_0}{\Gamma(\alpha + 1)} \geq 1,$$

which is a contradiction. Hence $u(t) \leq 0$ for all $t \in J$. □

3. EXTREMAL GENERALIZED SOLUTIONS OF PROBLEM (1.1)

In this section, We consider the condition of $\lambda_1 \geq 0$. we shall establish the existence of extremal generalized solutions of problem (1.1).

Definition 3.1. A function $\varphi \in C^{n-1}(J, R)$ is called a lower generalized solution of (1.1) if

$$\begin{cases} \varphi(t) \leq \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \varphi(s), (T\varphi)(s), (S\varphi)(s)) ds, \ t \in J, \\ \varphi^{(k)}(0) \leq a_k, \quad k = 1, \ldots, n-1, \\ \varphi(0) \leq \lambda_1 \varphi(\tau) + \lambda_2 \int_0^1 \omega(s, \varphi(s)) ds + b. \end{cases}$$

Analogously, $\phi \in C^{n-1}(J, R)$ is called an upper generalized solution of (1.1) if

$$\begin{cases} \phi(t) \geq \sum_{k=0}^{n-1} \frac{\phi^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \phi(s), (T\phi)(s), (S\phi)(s)) ds, \ t \in J, \\ \phi^{(k)}(0) \geq a_k, \quad k = 1, \ldots, n-1, \\ \phi(0) \geq \lambda_1 \phi(\tau) + \lambda_2 \int_0^1 \omega(s, \phi(s)) ds + b. \end{cases}$$

We need the following assumptions.

(H1) φ, ϕ are lower and upper generalized solutions of (1.1), respectively, such that $\varphi \leq \phi$ on J.

(H2) There exists two constants $M, N \geq 0$ such that

$$f(t, x, y, z) - f(t, \bar{x}, \bar{y}, \bar{z}) \geq M(x - \bar{x}) + N(y - \bar{y}) + N_1(z - \bar{z}),$$

wherever $\varphi(t) \leq \bar{x} \leq x \leq \phi(t)$, $(T\varphi)(t) \leq \bar{y} \leq y \leq (T\phi)(t)$, $(S\varphi)(t) \leq \bar{z} \leq z \leq (S\phi)(t)$, $t \in J$.

(H3) There exists $m(t) \in C(J, R^+)$ such that

$$\omega(t, x) - \omega(t, y) \leq m(t)(x - y),$$

if $\varphi(t) \leq x \leq y \leq \phi(t)$.

Let $[\varphi, \phi] = \{x \in C(J, R) : \varphi(t) \leq x(t) \leq \phi(t), t \in J\}$.

Now we are in the position to establish the main results of this paper.

Theorem 3.2. *Let inequality* (2.3) *and* (H1)–(H3) *hold. If* $y, z \in C(J, R)$ *such that*

$$\begin{aligned} y(t) &= \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, \varphi(s), (T\varphi)(s), (S\varphi)(s)) \\ &\quad - M(\varphi(s) - y(s)) - N((T\varphi)(s) - (Ty)(s)) - N_1((S\varphi)(s) - (Sy)(s))] ds, \end{aligned}$$

$$\begin{aligned} z(t) &= \sum_{k=0}^{n-1} \frac{z^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, \phi(s), (T\phi)(s), (S\phi)(s)) \\ &\quad - M(\phi(s) - z(s)) - N((T\phi)(s) - (Tz)(s)) - N_1((S\phi)(s) - (Sz)(s))] ds, \end{aligned}$$

$$y(0) = \lambda_1 \varphi(\tau) + \lambda_2 \int_0^1 \omega(s, \varphi(s)) ds + b, z(0) = \lambda_1 \phi(\tau) + \lambda_2 \int_0^1 \omega(s, \phi(s)) ds + b,$$

$$y^{(k)}(0) = z^{(k)}(0) = a_k, \quad k = 1, \ldots, n-1,$$

then $\varphi(t) \le y(t) \le z(t) \le \phi(t),\ t \in J,$ *and* $y(t), z(t)$ *are lower and upper generalized solutions of* (1.1), *respectively.*

Proof. Let $m(t) = \varphi(t) - y(t)$, then

$$\begin{aligned} m(t) &\le \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(0)}{k!} t^k - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [M(\varphi(s) - y(s)) \\ &\quad + N((T\varphi)(s) - (Ty)(s)) + N_1((S\varphi)(s) - (Sy)(s))] ds \\ &= \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Mm(s) + N(Tm)(s) + N_1(Sm)(s)] ds, \end{aligned}$$

$$m^{(k)}(0) = \varphi^{(k)}(0) - y^{(k)}(0) \le a_k - a_k = 0,\ k = 1, \ldots, n-1,$$

$$\begin{aligned} m(0) &= \varphi(0) - y(0) \\ &\le \lambda_1 \varphi(\tau) + \lambda_2 \int_0^1 \omega(s, \varphi(s)) ds + b - (\lambda_1 \varphi(\tau) + \lambda_2 \int_0^1 \omega(s, \varphi(s)) ds + b) = 0. \end{aligned}$$

By Lemma 2.8, we get that $m(t) \le 0$ on J. That is, $\varphi(t) \le y(t)$. Similarly, we can prove that $z(t) \le \phi(t)$.

Next we suppose that $m(t) = y(t) - z(t)$, then

$$\begin{aligned} m(t) &= \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [(f(s, \varphi(s), (T\varphi)(s), (S\varphi)(s)) \\ &\quad - M\varphi(s) - N(T\varphi)(s) - N_1(S\varphi)(s)) \\ &\quad - (f(s, \phi(s), (T\phi)(s), (S\phi)(s)) - M\phi(s) - N(T\phi)(s) - N_1(S\phi)(s)) \\ &\quad + M(y(s) - z(s)) + N((Ty)(s) - (Tz)(s)) + N_1((Sy)(s) - (Sz)(s))] \\ &\le \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Mm(s) + N(Tm)(s) + N_1(Sm)(s)] ds, \end{aligned}$$

$$m^{(k)}(0) = y^{(k)}(0) - z^{(k)}(0) = a_k - a_k = 0, \qquad k = 1, \ldots, n-1,$$

$$
\begin{aligned}
m(0) &= y(0) - z(0) \\
&= (\lambda_1\varphi(\tau) + \lambda_2 \int_0^1 \omega(s,\varphi(s))ds + b) - (\lambda_1\phi(\tau) + \lambda_2 \int_0^1 \omega(s,\phi(s))ds + b) \\
&\le \lambda_1\varphi(\tau) - \lambda_1\phi(\tau) + \lambda_2 \int_0^1 m(s)(\varphi(s)ds - \phi(s))ds \\
&\le 0.
\end{aligned}
$$

By Lemma 2.8, we get that $m(t) \le 0$ on J. That is $y(t) \le z(t)$. So $\varphi(t) \le y(t) \le z(t) \le \phi(t), t \in J$.

In the following ,we need to prove $y(t)$ is a lower generalized solution of (1.1).

$$
\begin{aligned}
y(t) &= \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!}t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[f(s,\varphi(s),(T\varphi)(s),(S\varphi)(s)) \\
&\quad -M(\varphi(s)-y(s)) - N((T\varphi)(s)-(Ty)(s)) - N_1((S\varphi)(s)-(Sy)(s))]ds \\
&\le \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!}t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s),(Ty)(s),(Sy)(s))ds, t \in J,
\end{aligned}
$$

$$y^{(k)}(0) = a_k, k = 1, \dots, n-1,$$

$$
\begin{aligned}
y(0) &= \lambda_1\varphi(\tau) + \lambda_2 \int_0^1 \omega(s,\varphi(s))ds + b \\
&\le \lambda_1 y(\tau) + \lambda_2 \int_0^1 [\omega(s,y(s)) + m(s)(\varphi(s)-y(s))]ds + b \\
&\le \lambda_1 y(\tau) + \lambda_2 \int_0^1 \omega(s,y(s))ds + b.
\end{aligned}
$$

So $y(t)$ is a lower generalized solution of (1.1). Similarly, we can prove that $z(t)$ is an upper generalized solution of (1.1). □

Theorem 3.3. *Let inequality* (2.3) *and* (H1)–(H3) *hold. Then there exist monotone sequences* $\{\varphi_n\}$, $\{\phi_n\} \subset [\varphi,\phi]$ *which converge uniformly to the extremal generalized solutions of* (1.1) *in* $[\varphi,\phi]$, *respectively.*

Proof. For $i = 1, 2, \dots,$ we suppose that

$$
(3.1) \qquad \begin{cases} D^\alpha \varphi_i(t) - M\varphi_i(t) - N(T\varphi_i)(t) - N_1(S\varphi_i)(t) = \rho_{i-1}(t), t \in J, \\ \varphi_i^{(k)}(0) = a_k, k = 1, \dots, n-1, \\ \varphi_i(0) = \lambda_1\varphi_{i-1}(\tau) + \lambda_2 \int_0^1 \omega(s,\varphi_{i-1}(s))ds + b, \end{cases}
$$

and

$$
(3.2) \qquad \begin{cases} D^\alpha \phi_i(t) - M\phi_i(t) - N(T\phi_i)(t) - N_1(S\phi_i)(t) = \bar\rho_{i-1}(t), t \in J, \\ \phi_i^{(k)}(0) = a_k, k = 1, \dots, n-1, \\ \phi_i(0) = \lambda_1\phi_{i-1}(\tau) + \lambda_2 \int_0^1 \omega(s,\phi_{i-1}(s))ds + b. \end{cases}
$$

where

$$\begin{aligned}\rho_{i-1}(t) &= f(t,\varphi_{i-1}(t),(T\varphi_{i-1})(t),(S\varphi_{i-1})(t))\\ &\quad -M\varphi_{i-1}(t)-N(T\varphi_{i-1})(t)-N_1(S\varphi_{i-1})(t),\end{aligned}$$

$$\begin{aligned}\bar{\rho}_{i-1}(t) &= f(t,\phi_{i-1}(t),(T\phi_{i-1})(t),(S\phi_{i-1})(t))\\ &\quad -M\phi_{i-1}(t)-N(T\phi_{i-1})(t)-N_1(S\phi_{i-1})(t).\end{aligned}$$

Obviously, Eqs. (3.1) and (3.2) have the following generalized solutions, respectively.

$$\begin{aligned}\varphi_i(t) &= \sum_{k=0}^{n-1}\frac{\varphi_i^{(k)}(0)}{k!}t^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,\varphi_{i-1}(s),(T\varphi_{i-1})(s),(S\varphi_{i-1})(s))\\ &\quad -M(\varphi_{i-1}(s)-\varphi_i(s))-N((T\varphi_{i-1})(s)-(T\varphi_i)(s))\\ &\quad -N_1((S\varphi_{i-1})(s)-(S\varphi_i)(s))]ds,\ t\in J,\ k=0,\dots,n-1,\end{aligned}$$

$$\begin{aligned}\phi_i(t) &= \sum_{k=0}^{n-1}\frac{\phi_i^{(k)}(0)}{k!}t^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,\phi_{i-1}(s),(T\phi_{i-1})(s),(S\phi_{i-1})(s))\\ &\quad -M(\phi_{i-1}(s)-\phi_i(s))-N((T\phi_{i-1})(s)-(T\phi_i)(s))\\ &\quad -N_1((S\phi_{i-1})(s)-(S\phi_i)(s))]ds,\ t\in J,\ k=0,\dots,n-1.\end{aligned}$$

In view of Theorem 3.2, we have that

$$\varphi=\varphi_0\le\varphi_1\le\cdots\varphi_i\le\cdots\le\phi_i\le\cdots\le\phi_1\le\phi_0=\phi$$

and each $\varphi_i,\phi_i\in[\varphi,\phi]$ $(i=1,2\dots)$.

Obviously the sequences $\{\varphi_i\},\{\phi_i\}$ are uniformly bounded and equicontinuous, one can employ the standard arguments, namely the Ascoli-Arzela criterion to conclude that the sequences $\{\varphi_i\},\{\phi_i\}$ converge uniformly on J with $\lim_{i\to\infty}\varphi_i=x_*$, $\lim_{i\to\infty}\phi_i=x^*$ uniformly on J. Moreover, x_*,x^* are generalized solutions of (1.1) in $[\varphi,\phi]$.

To prove that $x_*(t),x^*(t)$ are extremal generalized solutions of (1.1), let $x\in[\varphi,\phi]$ be any generalized solution of (1.1). That is,

$$x(t)=\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s),(Tx)(s),(Sx)(s))ds,k=0,\dots,n-1,$$

where $x^{(k)}(0)=a_k,k=1,\dots,n-1,x(0)=\lambda_1x(\tau)+\lambda_2\int_0^1\omega(s,x(s))ds+b$.

Suppose that there exists a positive integer i such that $\varphi_i(t)\le x(t)\le\phi_i(t)$ on J. Let $m(t)=\varphi_{i+1}(t)-x(t)$, we have

$$m(t) = \sum_{k=0}^{n-1}\frac{m^{(k)}(0)}{k!}t^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,\varphi_i(s),(T\varphi_i)(s),(S\varphi_i)(s))$$

$$
\begin{aligned}
&-M\varphi_i(s) - N(T\varphi_i)(s) - N_1(S\varphi_i)(s) \\
&-(f(s,x(s),(Tx)(s),(Sx)(s)) - Mx(s) - N(Tx)(s) - N_1(Sx)(s)) \\
&+M(\varphi_{i+1}(s) - x(s)) + N((T\varphi_i)(s) - (Tx)(s)) \\
&+N_1((S\varphi_i)(s) - (Sx)(s))]ds \\
\leq\ & \sum_{k=0}^{n-1} \frac{m^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Mm(s) + N(Tm)(s) + N_1(Sm)(s)]ds,
\end{aligned}
$$

$$
m^{(k)}(0) = \varphi_{i+1}^{(k)}(0) - x^{(k)}(0) = a_k - a_k = 0, k = 1, \dots, n-1,
$$

$$
\begin{aligned}
m(0) &= \varphi_{i+1}(0) - x(0) \\
&= \lambda_1 \varphi_i(\tau) + \lambda_2 \int_0^1 \omega(s,\varphi_i(s))ds + b - (\lambda_1 x(\tau) + \lambda_2 \int_0^1 \omega(s,x(s))ds + b) \\
&\leq \lambda_1(\varphi_i(\tau) - x(\tau)) + \lambda_2 \int_0^1 m(s)(\varphi_i(s) - x(s))ds \\
&\leq 0.
\end{aligned}
$$

By Lemma 2.8, we know that $m(t) \leq 0$ on J, i.e, $\varphi_{i+1}(t) \leq x(t)$ on J. Similarly we obtain that $x(t) \leq \phi_{i+1}(t)$ on J. Since $\varphi_0 \leq x(t) \leq \phi_0$ on J, by induction we get that $\varphi_i(t) \leq x \leq \phi_i(t)$ on J for every i. Therefore, $x_*(t) \leq x(t) \leq x^*(t)$ on J by taking $i \to \infty$. □

4. GENERALIZED QUASI-SOLUTIONS OF PROBLEM (1.1)

In this section, we consider the condition of $\lambda_1 < 0$, we also use monotone iterative technique to obtain the existence of a coupled generalized quasi-solution for (1.1).

Definition 4.1. Functions $\varphi, \phi \in C^{n-1}(J,R)$ are called coupled lower and upper generalized solutions of problem (1.1), respectively, if

$$
\begin{cases}
\varphi(t) \leq \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,\varphi(s),(T\varphi)(s),(S\varphi)(s))ds, t \in J, \\
\varphi^{(k)}(0) \leq a_k, k = 1, \dots, n-1, \\
\varphi(0) \leq \lambda_1 \phi(\tau) + \lambda_2 \int_0^1 \omega(s,\varphi(s))ds + b,
\end{cases}
$$

and

$$
\begin{cases}
\phi(t) \geq \sum_{k=0}^{n-1} \frac{\phi^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,\phi(s),(T\phi)(s),(S\phi)(s))ds, t \in J, \\
\phi^{(k)}(0) \geq a_k, k = 1, \dots, n-1, \\
\phi(0) \geq \lambda_1 \varphi(\tau) + \lambda_2 \int_0^1 \omega(s,\phi(s))ds + b.
\end{cases}
$$

Definition 4.2. A pair $(U,V) \in C(J,R) \times C(J,R)$ is called a coupled generalized quasi-solution of problem (1.1) if

$$
U(t) = \sum_{k=0}^{n-1} \frac{U^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,U(s),(TU)(s),(SU)(s))ds,
$$

$$V(t)=\sum_{k=0}^{n-1}\frac{V^{(k)}(0)}{k!}t^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,V(s),(TV)(s),(SV)(s))ds,$$

where

$$U^{(k)}(0)=V^{(k)}(0)=a_k,\quad k=1,\dots,n-1,$$
$$U(0)=\lambda_1V(\tau)+\lambda_2\int_0^1\omega(s,U(s))ds+b,$$
$$V(0)=\lambda_1U(\tau)+\lambda_2\int_0^1\omega(s,V(s))ds+b.$$

We need the following assumption.

(H4) φ,ϕ are coupled lower and upper generalized solutions of (1.1), respectively, such that $\varphi\le\phi$ on J.

Theorem 4.3. *Let inequality* (2.3) *and* (H2)–(H4) *hold. If*

$$\begin{aligned}y(t)&=\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}t^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,\varphi(s),(T\varphi)(s),(S\varphi)(s))\\&\quad-M(\varphi(s)-y(s))-N((T\varphi)(s)-(Ty)(s))\\&\quad-N_1((S\varphi)(s)-(Sy)(s))]ds,\ t\in J,\quad k=0,\dots,n-1,\end{aligned}$$

$$\begin{aligned}z(t)&=\sum_{k=0}^{n-1}\frac{z^{(k)}(0)}{k!}t^k+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,\phi(s),(T\phi)(s),(S\phi)(s))\\&\quad-M(\phi(s)-z(s))-N((T\phi)(s)-(Tz)(s))\\&\quad-N_1((S\phi)(s)-(Sz)(s))]ds,\quad t\in J,\quad k=0,\dots,n-1,\end{aligned}$$

where

$$y^{(k)}(0)=z^{(k)}(0)=a_k,\quad k=1,\dots,n-1,$$
$$a_0=y(0)=\lambda_1\phi(\tau)+\lambda_2\int_0^1\omega(s,\varphi(s))ds+b,$$
$$a_0^*=z(0)=\lambda_1\varphi(\tau)+\lambda_2\int_0^1\omega(s,\phi(s))ds+b,$$

then $\varphi(t)\le y(t)\le z(t)\le\phi(t)$, $t\in J$, *and* $y(t),z(t)$ *are coupled lower and upper generalized solutions of* (1.1), *respectively.*

The way of proof is similar to the one we used in the proof of Theorem 3.1, so we omit it.

Theorem 4.4. *Let inequality* (2.3) *and* (H2)–(H4) *hold. Then there exist monotone sequences* $\{\varphi_n\}$, $\{\phi_n\}\subset[\varphi,\phi]$ *which converge uniformly to a coupled generalized quasi-solution of* (1.1) *in* $[\varphi,\phi]$.

Proof. For $i = 1, 2, \dots,$ we suppose that

$$\begin{cases} D^\alpha \varphi_i(t) - M\varphi_i(t) - N(T\varphi_i)(t) - N_1(S\varphi_i)(t) = \rho_{i-1}(t), t \in J, \\ \varphi_i^{(k)}(0) = a_k, k = 1, \dots, n-1, \\ \varphi_i(0) = \lambda_1 \phi_{i-1}(\tau) + \lambda_2 \int_0^1 \omega(s, \varphi_{i-1}(s))ds + b, \end{cases} \tag{4.1}$$

and

$$\begin{cases} D^\alpha \phi_i(t) - M\phi_i(t) - N(T\phi_i)(t) - N_1(S\phi_i)(t) = \bar{\rho}_{i-1}(t), t \in J, \\ \phi_i^{(k)}(0) = a_k, k = 1, \dots, n-1, \\ \phi_i(0) = \lambda_1 \varphi_{i-1}(\tau) + \lambda_2 \int_0^1 \omega(s, \phi_{i-1}(s))ds + b. \end{cases} \tag{4.2}$$

where

$$\begin{aligned} \rho_{i-1}(t) &= f(t, \varphi_{i-1}(t), (T\varphi_{i-1})(t), (S\varphi_{i-1})(t)) \\ &\quad - M\varphi_{i-1}(t) - N(T\varphi_{i-1})(t) - N_1(S\varphi_{i-1})(t), \end{aligned}$$

$$\begin{aligned} \bar{\rho}_{i-1}(t) &= f(t, \phi_{i-1}(t), (T\phi_{i-1})(t), (S\phi_{i-1})(t)) \\ &\quad - M\phi_{i-1}(t) - N(T\phi_{i-1})(t) - N_1(S\phi_{i-1})(t). \end{aligned}$$

Obviously, Eqs. (4.1) and (4.2) have the following generalized solutions, respectively.

$$\begin{aligned} \varphi_i(t) &= \sum_{k=0}^{n-1} \frac{\varphi_i^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, \varphi_{i-1}(s), (T\varphi_{i-1})(s), (S\varphi_{i-1})(s)) \\ &\quad - M(\varphi_{i-1}(s) - \varphi_i(s)) - N((T\varphi_{i-1})(s) - (T\varphi_i)(s)) \\ &\quad - N_1((S\varphi_{i-1})(s) - (S\varphi_i)(s))]ds, \quad t \in J, \quad k = 0, \dots, n-1, \end{aligned}$$

$$\begin{aligned} \phi_i(t) &= \sum_{k=0}^{n-1} \frac{\phi_i^{(k)}(0)}{k!} t^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, \phi_{i-1}(s), (T\phi_{i-1})(s), (S\phi_{i-1})(s)) \\ &\quad - M(\phi_{i-1}(s) - \phi_i(s)) - N((T\phi_{i-1})(s) - (T\phi_i)(s)) \\ &\quad - N_1((S\phi_{i-1})(s) - (S\phi_i)(s))]ds, \quad t \in J, \quad k = 0, \dots, n-1. \end{aligned}$$

In view of Theorem 4.3, we have that

$$\varphi = \varphi_0 \le \varphi_1 \le \cdots \varphi_i \le \cdots \le \phi_i \le \cdots \le \phi_1 \le \phi_0 = \phi$$

and each $\varphi_i, \phi_i \in [\varphi, \phi] (i = 1, 2 \dots)$.

Obviously the sequences $\{\varphi_i\}, \{\phi_i\}$ are uniformly bounded and equicontinuous, one can employ the standard arguments, namely the Ascoli-Arzela criterion to conclude that the sequences $\{\varphi_i\}, \{\phi_i\}$ converge uniformly on J with $\lim_{i\to\infty} \varphi_i = x_*$, $\lim_{i\to\infty} \phi_i = x^*$ uniformly on J. Indeed, (x_*, x^*) is a coupled generalized quasi-solution of problem (1.1) in $[\varphi, \phi]$. $\square$

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