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QUASILINEARIZATION FOR HYBRID CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we develop the method of Quasilinearization for hybrid Caputo fractional differential equations which are Caputo fractional differential equations with fixed moments of impulse. In order to prove this result we use the weakened assumption of C^q-continuity in place of local Hölder continuity.

Keywords and Phrases. Hybrid Caputo fractional differential equations, Quasilinearization, Existence

AMS Subject Classification. 34K07, 34A08

1. INTRODUCTION

In the recent years, there has been a significant amount of work done in the theory of fractional differential equations and many researchers are delving into this area due to its immense potential in applications such as Fluid Flow, Rheology, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport Akin to Diffusion, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and Signal Processing, and so on. The works of Kilbas et al [2], Podlubny [1], Lakshmikantham et al [3] and then references [4–9] bear testimony to the continued interest in this area.

Another field which has a lot of scope is the theory of hybrid systems or impulsive differential systems [10]. This is due to the fact that many evolution processes are characterized by the fact that they experience a change of state abruptly, that is, in a very short duration of time. This abrupt change can be considered as short term perturbations whose duration is negligible. Thus we assume that these perturbations act instantaneously in the form of impulses. Thus it is obvious that hybrid systems form a better model to represent physical phenomena.

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Combining these two areas of interest, we consider hybrid fractional differential equations and propose to study existence of solutions. As the method of Quasilinearization [11] is a flexible mechanism that gives a sequence of iterations that converges quadratically to a solution, we propose to study it. In [12] Quasilinearization for IVP of fractional differential equations has been studied and in [13] generalized Quasilinearization has been developed for IVP of fractional differential equations. In this paper, we develop the method of Quasilinearization for hybrid Caputo fractional differential equations.

It is observed in [14] that the results in fractional differential equations can be studied with the weakened hypothesis of C_p or C^q continuity. We propose to use weakened hypothesis in this paper.

2. PRELIMINARIES

The basic results that are needed to prove our main result are presented in this section. We begin with the definition of C_p-continuity, $R-L$ fractional derivative, Caputo fractional derivative and proceed to state a lemma with the weakened hypothesis of C_p-continuity. This lemma is essential in proving the basic differential inequality results. All these results are from [14].

As observed above, the comparison theorems [3], in fractional differential equations set-up require Hölder continuity. Although this requirement is used to develop iterative techniques such as the monotone iterative technique and the method of quasilinearization, there is no feasible way to check whether the functions involved are Hölder continuous. To avoid this situation, it has been shown in [14] that comparison results can be proved under the weaker condition of C_p-continuity. Lemma 2.3.1 in [3] is essential in establishing the comparison theorems, a detailed proof of this result under the weaker hypothesis was given in [14]. The basic differential inequality theorem, required comparison theorems and the lemma which are proved in [14] all are stated below.

We begin with the definition of the class $C_p[[t_0,T],\mathbb{R}]$.

Definition 2.1. m is said to be C_p continuous if $m \in C_p[[t_0,T],\mathbb{R}]$ that is $m \in C[(t_0,T],\mathbb{R}]$ and $(t-t_0)^p m(t) \in C[[t_0,T],\mathbb{R}]$ with $p+q=1$.

Definition 2.2. For $m \in C_p[[t_0,T],\mathbb{R}]$, the Riemann-Liouville derivative of $m(t)$ is defined as

$$D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{p-1}\, m(s) ds. \tag{2.1}$$

We next state a lemma that is vital for our main result.

Lemma 2.3. *Let $m \in C_p[[t_0, T], \mathbb{R}]$. Suppose that for any $t_1 \in [t_0, T]$, we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \le t < t_1$, then it follows that*

$$D^q m(t_1) \ge 0. \tag{2.2}$$

We next state the fundamental fractional differential inequality result in the set up of Riemann-Liouville fractional derivative, with a weaker hypothesis from [14].

Theorem 2.4. *Let $v, w \in C_p[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and*

$$(i)\ D^q v(t) \le f(t, v(t))$$

and

$$(ii)\ D^q w(t) \ge f(t, w(t)),$$

$t_0 < t \le T$, with one of the inequalities (i) or (ii) being strict. Then $v^0 < w^0$, where $v^0 = v(t)(t - t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t - t_0)^{1-q}|_{t=t_0}$ implies that

$$v(t) < w(t), \quad t_0 \le t \le T. \tag{2.3}$$

The next result deals with the inequality theorem for non strict inequalities.

Theorem 2.5. *Let $v, w \in C_p[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and*

$$(i)\ D^q v(t) \le f(t, v(t))$$

and

$$(ii)\ D^q w(t) \ge f(t, w(t)),$$

$t_0 < t \le T$. Assume f satisfies the Lipschitz condition

$$f(t, x) - f(t, y) \le L(x - y), \quad x \ge y, \quad L > 0. \tag{2.4}$$

Then, $v^0 < w^0$, where $v^0 = v(t)(t - t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t - t_0)^{1-q}|_{t=t_0}$, implies $v(t) \le w(t)$, $t \in [t_0, T]$.

We now define a C^q-continuous function.

Definition 2.6. u is said to be C^q continuous that is $u \in C^q[[t_0, T], \mathbb{R}]$ iff the Caputo derivative of u denoted by ${}^cD^q u$ exists and satisfies

$$ {}^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t - s)^{-q} u'(s) ds. \tag{2.5}$$

We note that the Caputo and Riemann-Liouville derivatives are related as follows:

$$ {}^cD^q x(t) = D^q[x(t) - x(t_0)]. \tag{2.6}$$

We choose to work with the Caputo fractional derivative, since the initial conditions for fractional differential equations are of the same form as those of ordinary differential equations. Further, the Caputo fractional derivative of a constant is zero, which is useful in our work. Consider the IVP for the Caputo fractional differential equation given by

$$ {}^{c}D^{q}x = f(t,x), \quad x(t_0) = x_0, \tag{2.7} $$

for $0 < q < 1$, $f \in C^q[[t_0, T] \times \mathbb{R}^n, \mathbb{R}^n]$.

If $x \in C^q[[t_0, T], \mathbb{R}^n]$ satisfies (2.7), then it also satisfies the Volterra fractional integral

$$ x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s,\, x(s))ds, \tag{2.8} $$

for $t_0 \le t \le T$.

We now state the comparison theorem for the Caputo fractional differential equation using the same weaker hypothesis. As the proof is similar to that of Theorem 2.4.3 in [3], we omit it.

Theorem 2.7. *Assume that $m \in C^q[[t_0, T], \mathbb{R}]$ and*

$$ {}^{c}D^{q}m(t) \le g(t, m(t)), \quad t_0 \le t \le T, $$

where $g \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$. Let $r(t)$ be the maximal solution of the IVP

$$ {}^{c}D^{q}u = g(t,u), \quad u(t_0) = u_0, \tag{2.9} $$

existing on $[t_0, T]$ such that $m(t_0) \le u_0$. Then we have $m(t) \le r(t)$, $t_0 \le t \le T$.

3. IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

In this section, we begin with the basic definitions given in [15], where in the existence and stability results for hybrid Caputo fractional differential equation with fixed moments of impulse are studied.

Definition 3.1. Let $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $t_k \to \infty$ as $k \to \infty$. Then we say that $h \in PC_p[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ if $h : (t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ is C_p-continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$ and for any $x \in \mathbb{R}^n$

$$ \lim_{(t,y) \to (t_k^+, x)} h(t,y) = h(t_k^+, x) $$

exists for $k = 1, 2, \ldots, n-1$.

Definition 3.2. Let $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $t_k \to \infty$ as $k \to \infty$. Then we say that $h \in PC^q[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ if $h : (t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ is C^q-continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$ and for any $x \in \mathbb{R}^n$

$$ \lim_{(t,y) \to (t_k^+, x)} h(t,y) = h(t_k^+, x) $$

exists for $k = 1, 2, \ldots, n-1$.

Consider the hybrid Caputo fractional differential system defined by

$$
\begin{cases} {}^cD^q x = f(t,x), \ t \neq t_k, \\ x(t_k^+) = I_k(x(t_k)), \ k = 1,2,3,\ldots,n-1, \\ x(t_0) = x_0, \end{cases} \tag{3.1}
$$

where $f \in PC[I \times \mathbb{R}^n, \mathbb{R}^n]$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $t \in I = [t_0, T]$, $k = 1, 2, \ldots, n-1$.

Definition 3.3. By a solution of the system (3.1), we mean a PC^q continuous function $x \in PC^q[[t_0,T], R^n]$ such that

$$
x(t) = \begin{cases} x_0(t,t_0,x_0), \ t_0 \leq t \leq t_1, \\ x_1(t,t_1,x_1^+), \ t_1 < t \leq t_2, \\ \vdots \\ x_k(t,t_k,x_k^+), \ t_k < t \leq t_{k+1}, \\ \vdots \\ x_{n-1}(t,t_{n-1},x_{n-1}^+), \ t_{n-1} < t \leq T, \end{cases} \tag{3.2}
$$

where $0 \leq t_0 < t_1 < t_2 < \cdots < t_{n-1} \leq T$ and $x_k(t,t_k,x_k^+)$ is the solution of the IVP of the fractional differential equation

$$
\begin{cases} {}^cD^q x = f(t,x), \\ x_k^+ = x(t_k^+) = I_k(x(t_k)). \end{cases}
$$

Now we state the basic differential inequality result in this set up from [15].

Theorem 3.4. *Let $u,w \in PC^q[[t_0,T], \mathbb{R}]$ with*

$$
\begin{cases} {}^cD^q v(t) \leq f(t,v(t)), & t \neq t_k, \\ v(t_k^+) \leq I_k(v(t_k)), & k = 1,2,3,\ldots,n-1, \\ v(t_0) \leq x_0, \end{cases}
$$

and

$$
\begin{cases} {}^cD^q w(t) \geq f(t,w(t)), & t \neq t_k, \\ w(t_k^+) \geq I_k(w(t_k)), & k = 1,2,3,\ldots,n-1, \\ w(t_0) \geq x_0, \end{cases}
$$

where $f \in PC[I \times \mathbb{R}^n, \mathbb{R}^n]$ and f satisfies the hypothesis

$$
f(t,x) - f(t,y) \leq L(x-y), \quad x \geq y, \ L > 0
$$

and I_k is a monotonically nondecreasing function of x. Then $v_0 < w_0$ implies that $v(t) \leq w(t)$, $t \in [t_0,T]$.

Lemma 3.5. *The linear non-homogeneous hybrid Caputo fractional differential equation*

$$\begin{cases} {}^cD^q x = -M(x-y) + f(t,y), & t \neq t_k, \\ x(t_k^+) = I_k(x(t_k)), & k = 1,2,3,\dots,n-1, \\ x(t_0) = x_0, \end{cases}$$

has a unique solution on the interval $[t_0, T]$.

Proof. We proceed to prove the theorem in each subinterval. Let $t \in [t_0, t_1]$ and consider the Caputo fractional differential equation

$$\begin{cases} {}^cD^q x = -M(x-y) + f(t,y), \\ x(t_0) = x_0. \end{cases}$$

Then from [2], we have that $x(t,t_0,x_0) = x(t) = x_0\ E_q(M(t-t_0)^q) + \int_{t_0}^{t} (t-s)^{q-1}\, E_{q,q}(M(t-s)^q)\ f(s,y(s))ds,\ t \in [t_0,t_1]$ is the unique solution. Then we have $x(t_1,t_0,x_0) = x(t_1) = x_0\ E_q(M(t_1-t_0)^q) + \int_{t_0}^{t_1} (t_1-s)^{q-1}\, E_{q,q}(M(t_1-s)^q)\ f(s,y(s))ds$ and $x(t_1^+) = I_1(x(t_1)) = x_1^+$ (say). Now we consider the interval $(t_1, t_2]$ and the Caputo fractional differential equation

$$\begin{cases} {}^cD^q x = -M(x-y) + f(t,y), \\ x(t_1^+) = x_1^+. \end{cases}$$

Thus as earlier, the unique solution is given by,

$$\begin{aligned} x(t,t_1,x_1^+) &= x(t) \\ &= x_1^+\ E_q(M(t-t_1)^q) \\ &\quad + \int_{t_1}^{t} (t-s)^{q-1}\, E_{q,q}(M(t-s)^q)\ f(s,y(s))ds,\ t \in (t_1,t_2]. \end{aligned}$$

Then

$$\begin{aligned} x(t_2,t_1,x_1^+) &= x(t_2) \\ &= x_1^+\ E_q(M(t_2-t_1)^q) \\ &\quad + \int_{t_1}^{t_2} (t_2-s)^{q-1}\, E_{q,q}(M(t_2-s)^q)\ f(s,y(s))ds,\ t \in (t_1,t_2]. \end{aligned}$$

and

$$x(t_2^+) = I_2(x(t_2)) = x_2^+.$$

Then proceeding as earlier, we obtain the unique solution for the linear non-homogeneous hybrid Caputo fractional differential equation as

$$x(t) = \begin{cases} x_0(t, t_0, x_0), & t_0 \le t \le t_1, \\ x_1(t, t_1, x_1^+), & t_1 < t \le t_2, \\ \vdots & \\ x_{n-1}(t, t_{n-1}, x_{n-1}^+), & t_{n-1} < t \le T. \end{cases} \tag{3.3}$$

We begin with the definition of lower and upper solutions for the hybrid Caputo fractional differential equation given by

$$\begin{cases} {}^cD^q x = f(t, x),\ t \ne t_k, x(t_k^+) = I_k(x(t_k)),\ k = 1, 2, \ldots, n-1, x(t_0) = x_0, \end{cases} \tag{3.4}$$

where $f \in PC[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, I_k; $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $k = 1, 2, 3, \ldots, n-1$ and $t \in [t_0, T]$. $\square$

Definition 3.6. $\alpha, \beta \in PC^q[[t_0, T], \mathbb{R}^n]$ are said to be lower and upper solutions of equation (3.4), if and only if they satisfy the following inequalities

$$\begin{cases} {}^cD^q \alpha \le f(t, \alpha),\ t \ne t_k, \\ \alpha(t_k^+) \le I_k(\alpha(t_k)),\ k = 1, 2, 3, \ldots, n-1, \\ \alpha(t_0) \le x_0, \end{cases} \tag{3.5}$$

and

$$\begin{cases} {}^cD^q \beta \ge f(t, \beta),\ t \ne t_k, \\ \beta(t_k^+) \ge I_k(\beta(t_k)),\ k = 1, 2, 3, \ldots, n-1, \\ \beta(t_0) \ge x_0, \end{cases} \tag{3.6}$$

respectively.

Lemma 3.7. *Suppose that*

(i) $v_0(t)$ *is the lower solution of the hybrid Caputo fractional differential equation* (3.4).

(ii) *Let* $v_1(t)$ *be the unique solution of the linear non-homogeneous hybrid Caputo fractional differential equation*

$$\begin{cases} {}^cD^q v_1 = f(t, v_0) + f_x(t, v_0)\ (v_1 - v_0), & t \ne t_k, \\ v_1(t_k^+) = I_k(v_1(t_k)), & k = 1, 2, 3, \ldots, n-1, \\ v_1(t_0) = x_0. & \end{cases} \tag{3.7}$$

(iii) I_k *is a nondecreasing function in* x, *for each* $k = 1, 2, 3, \ldots, n-1$.

(iv) f_x *is continuous and Lipschitz on* $[t_0, T]$.

Then $v_0(t) \le v_1(t)$, $t \in [t_0, T]$.

Proof. Suppose that $v_0(t)$ is a lower solution of (3.4) and $v_1(t)$ be the unique solution of (3.7), set $p(t) = v_0(t) - v_1(t)$, $t \in [t_0, t_1]$.

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v_0(t) - {}^cD^q v_1(t) \\ &\leq f(t, v_0) - [f(t, v_0) + f_x(t, v_0)(v_1 - v_0)] \\ &\leq Mp(t), \end{aligned}$$

where $| f_x(t, v_0) | \leq M$, by assumption (iv). Then

$$ {}^cD^q p(t) \leq Mp(t) \tag{3.8}$$

and

$$p(t_0) \leq 0.$$

Thus from the solution of the linear non-homogeneous hybrid Caputo fractional differential equation, we get

$$p(t) \leq p(t_0)E_q(M(t - t_0)^q), \quad t \in [t_0, t_1],$$

which yields

$$p(t) \leq 0, \quad t \in [t_0, t_1].$$

Thus we have

$$v_0(t) \leq v_1(t), \quad t \in [t_0, t_1],$$

and therefore we get

$$v_0(t_1) \leq v_1(t_1).$$

From the assumption (iii) we obtain that

$$v_0^+ = v_0(t_1^+) = I_1(v_0(t_1)) \leq I_1(v_1(t_1)) = v_1(t_1^+) = v_1^+.$$

For $t \in (t_1, t_2]$ and consider the Caputo fractional differential equation

$$\begin{cases} {}^cD^q v_1 = f(t, v_0) + f_x(t, v_0)\ (v_1 - v_0), \\ v_1(t_1^+) = v_1^+. \end{cases}$$

Again setting $p(t) = v_0(t) - v_1(t)$, we get ${}^cD^q p(t) = {}^cD^q v_0(t) - {}^cD^q v_1(t)$ that is, ${}^cD^q p(t) \leq Mp(t)$ and $p(t_1^+) \leq 0$. Working as earlier, we get that $p(t) \leq 0$, $t \in (t_1, t_2]$. From which we can conclude that

$$v_0(t) \leq v_1(t),\ t \in (t_1, t_2].$$

Proceeding in a similar fashion over each subinterval $(t_k, t_{k+1}]$, we can show that

$$v_0(t) \leq v_1(t),\ t \in [t_0, T].$$

□

Lemma 3.8. *Suppose that in Lemma* 3.7*, the assumption (i) and (ii) are replaced by (i)* $w_0(t)$ *be the upper solution of the hybrid Caputo fractional differential equation* (3.4) *and (ii)* $w_1(t)$ *be the unique solution of the linear non-homogeneous hybrid Caputo fractional differential equation*

$$\begin{cases} {}^cD^q w_1 = f(t,w_0) + f_x(t,w_0)\ (w_1 - w_0), & t \neq t_k, \\ w_1(t_k^+) = I_k(w_1(t_k)), & k = 1,2,3,\ldots,n-1, \\ w_1(t_0) = x_0, \end{cases}$$

and the assumptions (iii) and (iv) of Lemma 3.7 *hold. Then* $w_1(t) \leq w_0(t),\ t \in [t_0, T]$.

The proof of Lemma is similar to the proof of Lemma 3.7 and hence we omit it.

4. QUASILINEARIZATION

The method of Quasilinearization is an useful Iterative technique to obtain the solutions of hybrid Caputo fractional differential equation, where the iterations converge quadratically. In this section, we proceed to develop this technique.

The main result of this paper is as follows.

Theorem 4.1. *Suppose that*

(i) α_0, β_0 *be lower and upper solutions of equation* (3.4) *such that* $\alpha_0 \leq \beta_0$ *on* $[t_0, T]$.
(ii) $f \in PC[[t_0,T] \times \mathbb{R}, \mathbb{R}]$ *and* $f(t,x) \geq f(t,y) + f_x(t,y)(x-y)$ *for* $\alpha_0 \leq y \leq x \leq \beta_0$;
(iii) I_k *is continuous and nondecreasing in* x, $k = 1,2,3\ldots,n-1$.
(iv) f_x *is continuous and Lipschitz on* $[t_0, T]$.

Then there exist monotone sequences $\{\alpha_n\},\{\beta_n\}$ *such that* $\alpha_n \to \rho, \beta_n \to r$ $n \to \infty$ *uniformly and monotonically to the unique solution* $\rho = r = x$ *of IVP* (3.4) *on* $[t_0, T]$ *and the convergence is quadratic.*

Proof. For any $\eta \in PC^q([t_0,T],\mathbb{R})$ such that $\alpha_0 \leq \eta \leq \beta_0$. Consider the linear non-homogeneous hybrid Caputo fractional differential equation

$$\begin{cases} {}^cD^q x = f(t,\eta) + f_x(t,\eta)\ (x - \eta), & t \neq t_k, \\ x(t_k^+) = I_k(x(t_k)), & k = 1,2,3,\ldots,n-1, \\ x(t_0) = x_0, \end{cases} \tag{4.1}$$

then from Lemma 3.5, we obtain that equation (4.1) has a unique solution on the interval $[t_0, T]$.

Now, replacing η, x with α_0, α_1. We get the hybrid Caputo fractional differential equation

$$(4.2) \qquad \begin{cases} {}^cD^q\alpha_1 = f(t,\alpha_0) + f_x(t,\alpha_0)\ (\alpha_1 - \alpha_0), & t \neq t_k, \\ \alpha_1(t_k^+) = I_k(\alpha_1(t_k)), & k = 1,2,3,\ldots,n-1, \\ \alpha_1(t_0) = x_0. \end{cases}$$

Since $\alpha_0(t)$ is the lower solution of the hybrid Caputo fractional differential equation (3.4), $\alpha_1(t)$ is the unique solution of equation (4.2) and I_k, f_x satisfy the hypothesis of Lemma 3.7 we conclude that $\alpha_0(t) \leq \alpha_1(t)$, $t \in [t_0, T]$. Similarly, replacing η, x with β_0, β_1. We get

$$(4.3) \qquad \begin{cases} {}^cD^q\beta_1 = f(t,\beta_0) + f_x(t,\alpha_0)\ (\beta_1 - \beta_0), & t \neq t_k, \\ \beta_1(t_k^+) = I_k(\beta_1(t_k)), & k = 1,2,3,\ldots,n-1, \\ \beta_1(t_0) = x_0. \end{cases}$$

Clearly β_1 is a unique solution of equation (4.3), β_0 is an upper solution of (3.4) and I_k, f_x satisfy the hypothesis of Lemma 3.8, we have that $\beta_0 \geq \beta_1$. To show $\alpha_1 \leq \beta_1$ we set $p(t) = \alpha_1(t) - \beta_1(t)$ for $t \in [t_0, T]$, using the hypothesis(i), we get,

$$\begin{aligned} {}^cD^q p &= {}^cD^q\alpha_1 - {}^cD^q\beta_1 \\ &= f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_1 - \alpha_0) - [f(t,\beta_0) + f_x(t,\alpha_0)(\beta_1 - \beta_0)] \\ &\leq f_x(t,\alpha_0)(\alpha_0 - \beta_0) + f_x(t,\alpha_0)\ [\alpha_1 - \alpha_0 - \beta_1 + \beta_0] \\ &\leq f_x(t,\alpha_0)p \leq Mp, \end{aligned}$$

where $\mid f_x(t,\alpha_0) \mid \leq M$. Thus

$${}^cD^q p(t) \leq M\ p(t),\ t \neq t_k.$$

Now for $t \in [t_0, t_1]$

$${}^cD^q p(t) \leq M\ p(t)$$

and

$$p(t_0) \leq 0.$$

Hence using Corollary 2.3.1 from [3], we get relation $p(t) \leq 0$ for $t \in [t_0, t_1]$. Using the non-decreasing nature of I_1, and the fact that $\alpha_1(t_1) \leq \beta_1(t_1)$, we obtain that $\alpha_1(t_1^+) \leq \beta_1(t_1^+)$, $t = t_1$, which yields that $p(t_1^+) \leq 0$.

Thus repeating the same process over the interval $(t_1, t_2]$ and continuing over each subinterval $(t_i, t_{i+1}]$ for $i = 3, \ldots, n-1$, we get that $\alpha_1 \leq \beta_1$ on $[t_0, T]$.

Hence $\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0$ on $[t_0, T]$.

Suppose now that for some $k > 1$ $\alpha_0 \le \alpha_{k-1} \le \alpha_k \le \beta_k \le \beta_{k-1} \le \beta_0$ on $[t_0, T]$. For $t \in [t_0, T]$, we claim that

$$\alpha_k \le \alpha_{k+1} \le \beta_{k+1} \le \beta_k. \tag{4.4}$$

Since we know that α_k is a solution of hybrid Caputo fractional differential equation (4.1) when $\eta = \alpha_{k-1}$. Now using hypothesis (i), we obtain that α_k is a lower solution of equation (3.4) and α_{k+1} is the solution of the linear non-homogeneous hybrid Caputo fractional differential equation

$$\begin{cases} {}^cD^q\alpha_{k+1} = f(t,\alpha_k) + f_x(t,\alpha_k)\ (\alpha_{k+1} - \alpha_k), & t \ne t_k, \\ \alpha_{k+1}(t_k^+) = I_k(\alpha_{k+1}(t_k)), & k = 1,2,3,\dots,n-1, \\ \alpha_{k+1}(t_0) = x_0. \end{cases} \tag{4.5}$$

On applying Lemma 3.7, we conclude that $\alpha_k(t) \le \alpha_{k+1}(t)$, $t \in [t_0, T]$.

Similarly β_k is the upper solution of equation (3.4), β_{k+1} is the solution of the linear non-homogeneous hybrid Caputo fractional differential equation

$$\begin{cases} {}^cD^q\beta_{k+1} = f(t,\beta_k) + f_x(t,\alpha_k)\ (\beta_{k+1} - \beta_k), & t \ne t_k, \\ \beta_{k+1}(t_k^+) = I_k(\beta_{k+1}(t_k)), & k = 1,2,3,\dots,n-1, \\ \beta_{k+1}(t_0) = x_0, \end{cases} \tag{4.6}$$

and I_k, f_x satisfy the hypothesis of Lemma 3.7, imply that $\beta_{k+1} \le \beta_k$ on $[t_0, T]$.

Next to prove $\alpha_{k+1} \le \beta_{k+1}$ we set $p(t) = \alpha_{k+1}(t) - \beta_{k+1}(t)$ using the fact that $\alpha_k \le \beta_k$ and the relation in equations (4.5) and (4.6), we have

$$\begin{aligned} {}^cD^q p &\le f_x(t,\alpha_k)[\alpha_k - \beta_k + \alpha_{k+1} - \alpha_k] - f_x(t,\alpha_k)\ (\beta_{k+1} - \beta_k) \\ &= f_x(t,\alpha_k)p \\ &\le Mp \end{aligned}$$

and $p(t_0) = 0$. For $t \in [t_0, t_1]$ which gives ${}^cD^q p(t) \le Mp(t)$ and $p(t_0) \le 0$.

Again applying the Corollary 2.3.1 from [3] we have $p(t) \le 0$ for $t \in [t_0, t_1]$, next the non-decreasing nature of I_1, $\alpha_{k+1}(t_1^+) \le \beta_{k+1}(t_1^+)$ we obtain that $p(t_1^+) \le 0$. Working in a similar fashion, we can show that

$$\alpha_{k+1}(t) \le \beta_{k+1}(t),\ t \in [t_0, T].$$

Hence, by the Principle of mathematical induction (4.4) holds for all k. Thus we have the sequences of functions $\{\alpha_n\}, \{\beta_n\}$ which are piece-wise continuous functions satisfying (4.4), (4.5) respectively and satisfying the relation $\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \beta_n \le \beta_{n+1} \cdots \le \beta_1 \le \beta_0$ on $[t_0, T]$.

These sequences are uniformly bounded in each subinterval $(t_k, t_{k+1}]$. Now using the Lemma 2.3.2 in [3] and the relation between the solutions of Caputo fractional differential equation and Riemann-Liouville fractional differential equation in [15], we can conclude that $\{x_{\in}(t)\}$ is equicontinuous on each subinterval $(t_k, t_{k+1}]$. Hence by using Arzela-Ascoli's theorem on each subinterval $(t_k, t_{k+1}]$, we show that the entire sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ converges uniformly and monotonically to a unique solution $x(t)$ of the IVP (3.1) on $(t_k, t_{k+1}]$, as f is Lipschitz. Since I_k is a continuous function for each $k = 1, 2, \ldots, n-1$, we have

$$\lim_{n\to\infty} \alpha_n(t_k^+) = \lim_{n\to\infty} I_k(\alpha_{n-1}(t_k))$$

and

$$\rho(t_k^+) = I_k(\rho(t_k))$$

similarly

$$r(t_k^+) = I_k(r(t_k)).$$

It is easy to show that ρ and r are the solutions corresponding Volterra's integral equations

$$\alpha_{n+1}(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, \alpha_n(s)) ds \tag{4.7}$$

and

$$\beta_{n+1}(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, \beta_n(s)) ds, \tag{4.8}$$

where $g(s, \alpha_n(s)) = f(t, \alpha_n(s)) - M(\alpha_{n+1} - \alpha_n)$ and $g(s, \beta_n(s)) = f(t, \beta_n(s)) - M(\beta_{n+1} - \beta_n)$. Now by taking the limits as $n \to \infty$ and using the uniform continuity of f and the uniform convergence of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ on each sub-interval $(t_k, t_{k+1}]$. For $t \in [t_0, T]$, we get

$$\begin{gathered} {}^cD^q\rho = f(t, \rho), \quad t \neq t_k, \\ \rho(t_k^+) = I_k(\rho(t_k)), \quad k = 1, 2, 3, \ldots, n-1, \\ \rho(t_0) = x_0, \end{gathered}$$

and

$$\begin{gathered} {}^cD^q r = f(t, r), \quad t \neq t_k, \\ r(t_k^+) = I_k(r(t_k)), \quad k = 1, 2, 3, \ldots, n-1, \\ r(t_0) = x_0. \end{gathered}$$

Further $\alpha_0 \leq \rho \leq r \leq \beta_0$ on $t \in [t_0, T]$. Since the solution is unique $\rho = x = r$ on $[t_0, T]$.

To prove the quadratic convergence of $\{\alpha_n\}, \{\beta_n\}$ to the solution, we consider

$$p_{n+1} = x - \alpha_{n+1}$$

so that $p_{n+1}(t_0) = 0$ then using the hypothesis (ii) f_x is Lipschitz, we get

$$\begin{aligned}
{}^cD^q p_{n+1} &= f(t,x) - [f(t,\alpha_n) + f_x(t,\alpha_n)\ (\alpha_{n+1} - \alpha_n)] \\
&= f_x(t,\eta)p_n - f_x(t,\alpha_n)(p_n - p_{n+1}) \\
&\leq L \mid \eta - \alpha_n \mid p_n + \mid f_x(t,\alpha_n) \mid p_{n+1} \\
&\leq L \mid x - \alpha_n \mid p_n + Mp_{n+1} \\
&\leq L \mid p_n \mid_0^2 + Mp_{n+1}
\end{aligned}$$

further since I_k is Lipschitz, we obtain

$$\begin{aligned}
p_{n+1}(t_k^+) &= x(t_k^+) - \alpha_{n+1}(t_k^+) \\
&= I_k(x(t_k)) - I_k(\alpha_{n+1}(t_k)) \\
&\leq K\ p_{n+1}(t_k)
\end{aligned}$$

Therefore $p_{n+1}(t_k^+) \leq Kp_{n+1}(t_k)$, $t = t_k$. Since $p_{n+1}(0) = 0$, we arrive at the hybrid Caputo fractional differential equation

$${}^cD^q p_{n+1} = L \mid p_n \mid_0^2 + Mp_{n+1}, \quad t \neq t_k,$$

$$p_{n+1}(t_k^+) = K_k p_{n+1}(t_k), \quad k = 1,2,3\ldots,n-1,$$

$$p_{n+1}(0) = 0.$$

Now using the solution of the linear non homogeneous fractional differential equation on each subinterval, we get $t \in (t_k, t_{k+1}]$. For $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned}
p_{n+1}(t) &= K_k \cdots K_3K_2K_1\ \frac{L \mid p_n \mid_0^2}{\Gamma(q+1)}(t_1 - t_0)^q\ E_{q,q}(M(t_1-t_0)^q)E_q(M(t_2-t_1)^q) \\
&\quad E_q(M(t_3-t_2)^q)\cdots E_q(M(t_{k+1}-t_k)^q) + K_k\cdots K_3K_2 \\
&\quad \frac{L \mid p_n \mid_0^2}{\Gamma(q+1)}(t_2 - t_1)^q\ E_{q,q}(M(t_2-t_1)^q)\ E_q(M(t_3-t_2)^q)\cdots E_q(M(t_{k+1}-t_k)^q) \\
&\quad + K_k\cdots K_3\ \frac{L \mid p_n \mid_0^2}{\Gamma(q+1)}(t_3-t_2)^q\ E_{q,q}(M(t_3-t_2)^q)\cdots E_q(M(t_{k+1}-t_k)^q) + \cdots \\
&\quad + K_k\ \frac{L \mid p_n \mid_0^2}{\Gamma(q+1)}(t_k - t_{k-1})^q E_{q,q}(M(t_k-t_{k-1})^q)\ E_q(M(t_{k+1}-t_k)^q) \\
&\quad + \frac{L \mid p_n \mid_0^2}{\Gamma(q+1)}(t_{k+1}-t_k)^q\ E_{q,q}(M(t_{k+1}-t_k)^q) \\
&\leq \frac{L \mid p_n \mid_0^2}{\Gamma(q+1)} \sum_{j=1}^{k+1}(t_j - t_{j-1})^q E_{q,q}(M(t_j-t_{j-1})^q) \prod_{i=j}^{k} K_i E_q(M(t_{i+1}-t_i)^q) \\
&\leq \frac{L \mid p_n \mid_0^2}{\Gamma(q+1)} \sum_{j=1}^{k+1} l^q E_{q,q}[M(l)^q] \prod_{i=j}^{k} K_i E_q(M(l)^q)\ (since\ t_j - t_{j-1} = l)
\end{aligned}$$

$$\leq \frac{L \mid p_n \mid_0^2}{\Gamma(q+1)} \sum_{j=1}^{k+1} l^q K_j, K_{j+1}, \ldots, K_k [E_q(M(l^q))]^{k-j} \; E_{q,q}(M(l)^q)$$

$$\leq \frac{L \mid p_n \mid_0^2}{\Gamma(q+1)} \sum_{j=1}^{k+1} l^q \tilde{K} [E_q(M(l^q))]^{k-j} E_{q,q}(M(l)^q)$$

$$\leq \frac{L\tilde{K}}{\Gamma(q+1)} E_{q,q}(Ml^q) \Omega \mid p_n \mid_0^2$$

where

$$\tilde{K} = K_1 \cdots K_N$$

and

$$\Omega = \sum_{j=1}^{N} l^q [E_q(Ml^q)]^{k-j}$$

Thus

$$|p_{n+1}(t)| \leq \frac{L\tilde{K}}{\Gamma(q+1)} \; \Omega \; E_{q,q}(Ml^q) |p_n|_0^2$$

This implies the quadratic convergence of the sequence $[\alpha_n(t)]$. Similarly, we can prove the quadratic convergence of the sequence $\{\beta_n(t)\}$ to the solution $x(t)$ of IVP (3.1). □

Remark: It can be observed that if we set $I_k \equiv 0$ for all k, then IVP (3.1) reduces to Caputo fractional differential equation and quasilinearization for there equations has been studied in [6]. Thus these results hold with the weakened hypothesis of C^q-continuity.

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