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OSCILLATION OF THIRD-ORDER NONLINEAR NEUTRAL FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. In this paper, we establish some new sufficient conditions for the oscillation of the third order nonlinear neutral functional dynamic equation

$$\left(p(t)\left[(r(t)x^{\Delta}(t))^{\Delta}\right]^{\gamma}\right)^{\Delta}+f(t,y(\delta(t)))=0,\quad \text{for } t\in[t_0,\infty)_{\mathbb{T}},$$

on a time scale $\mathbb{T}$, where $x(t):=y(t)+a(t)y(\tau(t))$, $\gamma>0$ is the quotient of odd positive integers, and a, p, r, τ, and δ are positive rd-continuous function defined on $\mathbb{T}$. Some of the results can be considered as the extensions of the oscillation criteria of Hille and Nehari for second-order differential equations. Examples are included to illustrate the main results.

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1. INTRODUCTION

In this paper, we are concerned with the oscillation of the third order nonlinear neutral functional dynamic equation

$$\left(p(t)\left[\left(r(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta}+f(t,y(\delta(t)))=0,\quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$ that is unbounded above. Throughout the paper, we will use the following notation for the Δ–quasi-derivatives:

$$\begin{cases} x(t):=y(t)+a(t)y(\tau(t)),\\ x^{[1]}(t):=r(t)x^{\Delta}(t),\\ x^{[2]}(t):=p(t)\left[\left(x^{[1]}(t)\right)^{\Delta}\right]^{\gamma},\\ x^{[3]}(t):=\left(x^{[2]}(t)\right)^{\Delta}. \end{cases} \tag{1.2}$$

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [16], has received a lot of attention in recent years. The monograph by

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Bohner and Peterson [5] summarizes and much of the time scale calculus. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [18]), i.e., $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : q > 1 \text{ and } t \in \mathbb{N}_0\}$, respectively. There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [25] discusses several possible applications. In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillatory and asymptotic behavior of solutions of different classes of dynamic equations on time scales. However, there appear to be relatively few papers dealing with oscillation of third order dynamic equations; see, for example, [12–15, 26].

We will make use of the following conditions:

(h_1) $\gamma > 0$ is the quotient of odd positive integers and τ, $\delta : \mathbb{T} \to \mathbb{T}$ satisfy $\tau(t) \le t$ for all $t \in \mathbb{T}$ and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$;

(h_2) a, r, $p : \mathbb{T} \to \mathbb{R}$ are positive real valued rd-continuous functions such that $0 \le a(t) \le a < 1$;

(h_3) $f(t,u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(t,u) > 0$ for all $u \ne 0$ and there exists a positive rd-continuous function $q : \mathbb{T} \to \mathbb{R}$ such that $|f(t,u)| \ge q(t)\,|u^\gamma|$.

When we write "$t \ge T$" we will understand that what we mean is "$t \in [T,\infty)_{\mathbb{T}}$."

Since we are interested in the oscillatory and asymptotic behavior of solutions for large t, we assume that $\sup \mathbb{T} = \infty$ and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ means $f(\sigma(t))$. Let $T^* = \min_{t \ge t_0}\{\tau(t), \delta(t)\}$ and $\hat{T} = t_0 + T^*$. By a *solution* of (1.1) we mean a nontrivial real-valued function y such that $x \in C^1_{rd}[\hat{T},\infty)$, $x^{[1]} \in C^1_{rd}[\hat{T},\infty)$, and y satisfies equation (1.1). Our attention is restricted to those solutions of (1.1) that exist on some half line $[t_y,\infty)$ and satisfy $\sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 \ge t_y$. A solution y of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is called *nonoscillatory*. Equation (1.1) is said to be oscillatory if there exists at least one oscillatory solution, and it is nonoscillatory if all its solutions are nonoscillatory.

Our aim in this paper is to establish some sufficient conditions for the oscillation of (1.1) by employing the Riccati substitution and Pötzsche's chain rule (see [5, Theorem 1.90]). The paper is organized as follows: In Section 2, we prove some preliminary lemmas that will be used in the proof of the main results and we examine the asymptotic behavior of the nonoscillatory solutions. In Section 3, we establish some sufficient conditions guaranteeing that solutions of equation (1.1) are either oscillatory or tend to zero as $t \to \infty$. The results in the Subsection 3.1 cover the case $\delta(t) > t$ and those in Subsection 3.2 are for the case $\delta(t) \leq t$. Some of the results can be considered as the extensions of the well-known oscillation criteria of Hille [17] and Nehari [20] for second-order differential equations. Examples to illustrate the main results are given in Section 4.

2. SOME PRELIMINARY LEMMAS

In this section, we present some lemmas that will be useful in the proof of our main results. We also discuss the asymptotic behavior of the nonoscillatory solutions of equation (1.1). We begin with the following lemma on the Δ-quasi-derivatives of $y(t)$; its proof is obvious.

Lemma 2.1. *If $y(t)$ is a nonoscillatory solution of* (1.1) *and $x(t)$ is defined as in* (1.2)*, then there exists $T > t_0$ such that $x^{[i]}(t) \neq 0$ for $i = 0, 1, 2$ and $t \geq T$.*

We can classify the nonoscillatory solutions of (1.1) according to the signs of their quasi-derivatives. In view of Lemma 2.1, a nonoscillatory solution y of (1.1) is such that x belongs to one of the following classes for $t \geq T$, for a sufficiently large $T > t_0$:

$$\begin{aligned}
C_0 &= \{x : x(t)x^{[1]}(t) < 0 \text{ and } x(t)x^{[2]}(t) > 0\},\\
C_1 &= \{x : x(t)x^{[1]}(t) > 0 \text{ and } x(t)x^{[2]}(t) < 0\},\\
C_2 &= \{x : x(t)x^{[1]}(t) > 0 \text{ and } x(t)x^{[2]}(t) > 0\},\\
C_3 &= \{x : x(t)x^{[1]}(t) < 0 \text{ and } x(t)x^{[2]}(t) < 0\}.
\end{aligned}$$

In the lemmas that follow, we describe the behavior of solutions in these classes. In some cases we give sufficient conditions for these classes to be empty. The following notation will be used:

$$\begin{aligned}
\Gamma(u) &:= \int_{t_0}^{\infty} u(s)\Delta s,\\
\Gamma(u, \upsilon) &:= \int_{t_0}^{\infty} u(t) \int_{t_0}^{t} \upsilon(s)\Delta s\Delta t,\\
\Gamma(u, \upsilon, w) &:= \int_{t_0}^{\infty} u(t) \int_{t_0}^{t} \upsilon(s) \int_{t_0}^{s} w(\tau)\Delta\tau\Delta s\Delta t,
\end{aligned}$$

where u, υ, and w are positive continuous functions. Proofs will only be given in the case where a nonoscillatory solution is positive since the proofs in the negative case are symmetric.

Lemma 2.2. *Assume that (h$_1$)–(h$_3$) hold. Let $y(t)$ be a nonoscillatory solution of* (1.1) *and let $x(t)$ be given as in* (1.2). *If*

$$\Gamma\left(\frac{1}{r}, \left(\frac{1}{p}\right)^{\frac{1}{\gamma}}\right) = \infty, \tag{2.1}$$

then $C_3 = \emptyset$.

Proof. Without loss of generality, we assume that the solution $y(t)$ is eventually positive, say $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ for $t \geqslant t_1$ for some $t_1 \geqslant t_0$. To prove that C_3 is empty, assume for the sake of a contradiction that there exists $T > t_1$ such that $x^{[2]}(t) < 0$ and $x^{[1]}(t) < 0$ for $t \geq T$. Let $p_0 = x^{[2]}(T) < 0$. Then, since $x^{[2]}(t)$ is deceasing, we have $p(t)(x^{[1]}(t))^{\gamma} < p_0$ for $t \geq T$, and integrating, we obtain

$$x^{[1]}(t) < x^{[1]}(T) + p_0^{\frac{1}{\gamma}} \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s.$$

Now, since $x^{[1]}(T) < 0$, another integration from T to t gives

$$x(t) < x(T) + p_0^{\frac{1}{\gamma}} \int_T^t \frac{1}{r(s)} \int_T^s \left(\frac{1}{p(u)}\right)^{\frac{1}{\gamma}} \Delta u \Delta s \to -\infty$$

as $t \to \infty$ by (2.1). This contradicts the positivity of $x(t)$ and completes the proof of the lemma. $\square$

Lemma 2.3. *Assume that (h$_1$)–(h$_3$) hold. If*

$$\Gamma\left(\left(\frac{1}{p}\right)^{\frac{1}{\gamma}}\right) = \Gamma\left(\frac{1}{r}\right) = \infty \tag{2.2}$$

and y is a nonoscillatory solution of (1.1), *then $x \in C_0 \cup C_2$.*

Proof. Let y be a nonoscillatory solution of (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ for $t \geqslant t_1 \geqslant t_0$. Then $x(t) > 0$, and $x^{[1]}(t)$ and $x^{[2]}(t)$ are monotonic and eventually of one sign. To complete the proof, we need to show that $x \notin C_1 \cup C_3$. If $x^{[2]}(t) < 0$, then $p(t)\left[\left(x^{[1]}(t)\right)^{\Delta}\right]^{\gamma} \leq x^{[2]}(T) < 0$ for $t > T \geqslant t_0$ since $x^{[3]}(t) \leqslant 0$. An integration yields

$$x^{[1]}(t) \leqslant x^{[1]}(T) + x^{[2]}(T)^{\frac{1}{\gamma}} \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s \to -\infty$$

as $t \to \infty$ by (2.2). A second integration shows that x eventually becomes negative, which is a contradiction. $\square$

Lemma 2.4. *Assume that (h$_1$)–(h$_3$) hold. If*

$$\Gamma\left(\frac{1}{r}\right) < \infty \quad \textit{and} \quad \Gamma\left(\frac{1}{r}, \left(\frac{1}{p}\right)^{\frac{1}{\gamma}}\right) = \infty, \tag{2.3}$$

then $C_1 = \emptyset$.

Proof. Assume that $x(t) \in C_1$, say $x(t) > 0$, $x^{[1]}(t) > 0$, and $x^{[2]}(t) < 0$ for $t \geq T_1 \geq t_0$. Since $x^{[2]}(t)$ is decreasing, $x^{[2]}(t) < x^{[2]}(T_1) = c_1 < 0$ for $t \geq T_1$, so $\left(x^{[1]}(t)\right)^{\Delta} < (c_1)^{\frac{1}{\gamma}} p^{\frac{-1}{\gamma}}(t)$. Integrating from T_1 to t, we have

$$x^{[1]}(t) < c_2 + (c_1)^{\frac{1}{\gamma}} \int_{T_1}^{t} (1/p(s))^{\frac{1}{\gamma}} \Delta s,$$

where $c_2 = x^{[1]}(T_1) > 0$. Hence,

$$(x(t))^{\Delta} < c_2 \frac{1}{r(t)} + (c_1)^{\frac{1}{\gamma}} \frac{1}{r(t)} \int_{T_1}^{t} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s,$$

so

$$x(t) < x(T_1) + c_2 \int_{T_1}^{t} \frac{1}{r(s)} \Delta s + (c_1)^{\frac{1}{\gamma}} \int_{T_1}^{t} \frac{1}{r(s)} \int_{T_1}^{s} \left(\frac{1}{p(\theta)}\right)^{\frac{1}{\gamma}} \Delta\theta \Delta s \to -\infty$$

as $t \to \infty$. This contradicts the positivity of $x(t)$ and completes the proof of the lemma. □

Lemma 2.5. *Assume that* (h_1)*–*(h_3) *and* (2.2) *hold. If* $y(t)$ *is a nonoscillatory solution of* (1.1) *such that* $x(t) \in C_2$, *then:*

(a) $\lim_{t\to\infty} x(t) = \infty$;
(b) *If* $\lim_{t\to\infty} x^{[2]}(t) \neq 0$, *then* $\lim_{t\to\infty} x^{[1]}(t) = \infty$.

Proof. If $x(t) \in C_2$, we may assume that there exists $T \geq t_0$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, and $x^{[2]}(t) > 0$ for $t \geq T$. Now $x^{[1]}(t) > 0$ and is increasing, so $x^{[1]}(t) > x^{[1]}(T)$ for $t \geq T$. Integrating, we obtain

$$x(t) > x(T) + x^{[1]}(T) \int_{T}^{t} (1/r(s)) \Delta s \to \infty$$

as $t \to \infty$ by (2.2). This proves (a).

Since $x^{[2]}(t) = p(t) \left(\left(x^{[1]}(t)\right)^{\Delta}\right)^{\gamma}$, integrating from T to t implies

$$x^{[1]}(t) = x^{[1]}(T) + \int_{T}^{t} \left(\frac{1}{p(s)} x^{[2]}(s)\right)^{\frac{1}{\gamma}} \Delta s.$$

Since $x^{[2]}(t)$ is positive and decreasing, we have

$$x^{[1]}(t) \geq x^{[1]}(T) + \left(x^{[2]}(t)\right)^{\frac{1}{\gamma}} \int_{T}^{t} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s.$$

Conclusion (b) follows from (2.2). □

The following inequality will prove to be quite useful.

Lemma 2.6. *Assume that* (h_1)*–*(h_3) *hold. If* $y(t)$ *is a nonoscillatory solution of* (1.1) *such that* $x(t) \in C_2$, *then there exists* $T \geqslant t_0$ *such that*

(2.4) $$y(t) \geqslant (1 - a(t))x(t) \text{ for } t \geqslant T.$$

Proof. Assume that $y(t)$ is an eventually positive solution of (1.1); then there exists $T > t_0$ such that $x(t) > 0$ for $t > T$. Since $x(t) \in C_2$, there exists $T \geq t_0$ such that $x^{[1]}(t) > 0$ and $x(t)$ is increasing for $t \geq T$. This implies

$$y(t) = x(t) - a(t)y(\tau(t)) \geq x(t) - a(t)x(\tau(t)) \geq (1 - a(t))x(t)$$

for $t \geqslant T$, completing the proof. □

Remark 1. Notice that (2.4) also holds if $y(t)$ is a solution of (1.1) with $x(t) \in C_1$.

As a consequence of Lemma 2.6, we have the following result.

Lemma 2.7. *Assume that* (h_1)–(h_3) *hold. If* $y(t)$ *is a nonoscillatory solution of* (1.1) *such that* $x(t) \in C_2$, *then* $x(t)$ *satisfies*

$$\left(p(t)\left[\left(x^{[1]}(t)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta} + P(t)x^{\gamma}(\delta(t)) \leq 0, \quad t \geq T, \tag{2.5}$$

where $P(t) = q(t)(1 - a(\delta(t))^{\gamma}$.

Lemma 2.8. *Assume that* (h_1)–(h_3) *hold. If* $y(t)$ *is a nonoscillatory solution of* (1.1) *such that* $x(t) \in C_0$ *and*

$$(h_4) \qquad \int_{t_0}^{\infty} \frac{1}{r(t)} \int_{t}^{\infty} \left(\frac{1}{p(u)} \int_{u}^{\infty} q(s)\Delta s\right)^{\frac{1}{\gamma}} \Delta u \Delta t = \infty,$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$.

Proof. If $y(t)$ is a positive solution of (1.1) with $x(t) \in C_0$, then

$$x(t) > 0, \quad x^{\Delta}(t) < 0, \quad \text{and} \quad (r(t)x^{\Delta}(t))^{\Delta} > 0 \quad \text{for} \quad t \geq T \geq t_0.$$

It suffices to show that $\lim_{t\to\infty} x(t) = 0$. We know that $\lim_{t\to\infty} x(t) = L$, where $0 \leq L < \infty$. We will prove that $L = 0$. Assume that $L > 0$; then for any $\epsilon > 0$, we have $L + \epsilon > x(t) > L - \epsilon$, eventually. Choose $0 < \epsilon < \frac{L(1-a)}{1+a}$; then

$$\begin{aligned} y(t) &= x(t) - a(t)y(\tau(t)) > L - \epsilon - ax(\tau(t)) \\ &> L - \epsilon - a(L + \epsilon) = k(L + \epsilon) > kx(t), \end{aligned}$$

where $k = \frac{L-\epsilon-a(L+\epsilon)}{(L+\epsilon)} > 0$. Using this in (1.1), we have

$$\left(p(t)\left[\left(r(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta} \leq -q(t)k^{\gamma}x^{\gamma}(\delta(t)). \tag{2.6}$$

Integrating and using the fact that $(r(t)x^{\Delta}(t))^{\Delta} > 0$, we obtain

$$-p(t)\left[\left(r(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\gamma} + k^{\gamma}\int_{t}^{\infty} q(u)x^{\gamma}(\delta(u))\Delta u \leq 0, \quad t \geq T.$$

Hence,

$$k\left[\frac{1}{p(t)}\int_{t}^{\infty} q(u)x^{\gamma}(\delta(u))\Delta u\right]^{\frac{1}{\gamma}} \leq (r(t)x^{\Delta}(t))^{\Delta}, \quad t \geq T,$$

and integrating again from s to t $(t \geq s \geq t_1)$ gives

$$k\int_s^t \left[\frac{1}{p(v)}\int_v^\infty q(u)x^\gamma(\delta(u))\Delta u\right]^{\frac{1}{\gamma}}\Delta v \leq r(t)x^\Delta(t) - r(s)x^\Delta(s).$$

Since $x^\Delta(t) < 0$, we have

$$k\frac{1}{r(s)}\int_s^t \left[\frac{1}{p(v)}\int_v^\infty q(u)x^\gamma(\delta(u))\Delta u\right]^{\frac{1}{\gamma}}\Delta v \leq -x^\Delta(s).$$

Letting $t \to \infty$ and then integrating from t_1 to t, we obtain

$$k\int_{t_1}^t \frac{1}{r(s)}\int_s^\infty \left[\frac{1}{p(v)}\int_v^\infty q(u)x^\gamma(\delta(u))\Delta u\right]^{\frac{1}{\gamma}}\Delta v\Delta s \leq -x(t) + x(t_1) \leq x(t_1).$$

Since $x(t)$ is decreasing to L, we have $x^\gamma(\delta(u)) \geq L^\gamma$ and

$$Lk\int_{t_1}^t \frac{1}{r(s)}\int_s^\infty \left[\frac{1}{p(v)}\int_v^\infty q(u)\Delta u\right]^{\frac{1}{\gamma}}\Delta v\Delta s \leq x(t_1).$$

This contradicts (h_4) and so $L = 0$. Finally, the inequality $0 \leq y(t) \leq x(t)$ implies that $\lim_{t\to\infty} y(t) = 0$, and this completes the proof of the lemma. □

The proof of the following lemma is similar to the proof of Lemma 2.2 in [13] using the inequality (2.5); we omit the details.

Lemma 2.9. *Assume that (h_1)–(h_3) hold. If $y(t)$ is a nonoscillatory solution of* (1.1) *such that $x(t) \in C_2$, then there exists $T > t_0$ such that for $t \geq T$,*

$$x^{[1]}(t) \geq P(t,T)\left(x^{[2]}\right)^{\frac{1}{\gamma}}, \quad \textit{where} \quad P(t,T) := \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}}\Delta s.$$

The following lemma will be used to prove our main results for the delay case. This lemma involves the Taylor monomials $\{h_n(t,s)\}_{n=0}^\infty$ that are defined recursively by

$$h_0(t,s) = 1 \quad \text{and} \quad h_{n+1}(t,s) = \int_s^t h_n(u,s)\Delta u, \quad n \geq 1. \tag{2.7}$$

If $n = 1$, this is simply $h_1(t,s) = t - s$, but in general this expression does not hold for $n \geq 2$. However, if $\mathbb{T} = \mathbb{R}$, then $h_n(t,s) = (t-s)^n/n!$; if $\mathbb{T} = \mathbb{N}_0$, then $h_n(t,s) = (t-s)^{\underline{n}}/n!$, where $t^{\underline{n}} = t(t-1)\cdots(t-n+1)$ is the so-called falling function (cf. Kelley and Peterson [19]); and if $\mathbb{T} = q^{\mathbb{N}_0}$, then $h_n(t,s) = \prod_{\nu=0}^{n-1}(t-q^\nu s)/\sum_{r=0}^{\nu} q^r$. Note that $h_n(t,s) \leq h_n(t,0)$.

Lemma 2.10. *Assume that*

$$x(t) > 0,\ x^\Delta(t) > 0,\ x^{\Delta\Delta}(t) > 0,\ \textit{and}\ x^{\Delta\Delta\Delta}(t) \leq 0\ \textit{for}\ t \geq t_0. \tag{2.8}$$

Then there exists $T \geq t_0$ such that

$$\frac{x(t)}{x^\Delta(t)} \geq \frac{h_2(t,T)}{t-T} \quad \textit{for} \quad t > T. \tag{2.9}$$

Proof. To prove this inequality, it suffices to show that $(t-T)x(t) \geq h_2(t,T)x^{\Delta}(t)$. To do this, we define the function $G(t)$ by $G(t) := (t-T)x(t) - h_2(t,T)x^{\Delta}(t)$. Then, $G(T)=0$ and

$$\begin{aligned} G^{\Delta}(t) &= (\sigma(t)-T)x^{\Delta}(t) + x(t) - h_2(\sigma(t),T)x^{\Delta\Delta}(t) - (t-T)x^{\Delta}(t) \\ &= \mu(t)x^{\Delta}(t) + x(t) - h_2(\sigma(t),T)x^{\Delta\Delta}(t) \\ &= x^{\sigma}(t) - h_2(\sigma(t),T)x^{\Delta\Delta}(t) \\ &= x^{\sigma}(t) - \left(\int_T^{\sigma(t)} (u-T)\Delta u\right) x^{\Delta\Delta}(t). \end{aligned}$$

By Taylor's Theorem ([5, Theorem 1.113]), we see that

$$\begin{aligned} x^{\sigma}(t) &= x(T) + h_1(\sigma(t),T)x^{\Delta}(T) + \int_T^{\sigma(t)} h_1(\sigma(t),\sigma(u))x^{\Delta\Delta}(u)\Delta u \\ &\geq x(T) + h_1(\sigma(t),T)x^{\Delta}(T) + x^{\Delta\Delta}(t)\int_T^{\sigma(t)} h_1(\sigma(t),\sigma(u))\Delta u, \end{aligned}$$

since $x^{\Delta\Delta}(t)$ is decreasing. We would have that $G^{\Delta}(t) > 0$ on $[T,\infty)_{\mathbb{T}}$ provided we can show

$$\int_T^{\sigma(t)} h_1(\sigma(t),\sigma(u))\Delta u \geq \int_T^{\sigma(t)} (u-T)\Delta u.$$

Integrating by parts ([5, Theorem 1.77]), we obtain

$$\begin{aligned} &\int_T^{\sigma(t)} h_1(\sigma(t),\sigma(u))\Delta u \\ &\quad = \int_T^{\sigma(t)} (\sigma(t)-\sigma(u))\Delta u \\ &\quad = [(\sigma(t)-u)(u-T)]_{u=T}^{u=\sigma(t)} - \int_T^{\sigma(t)} (-1)(u-T)\Delta u \\ &\quad = \int_T^{\sigma(t)} (u-T)\Delta u. \end{aligned}$$

Hence, $G^{\Delta}(t) > 0$ on $[T,\infty)_{\mathbb{T}}$, and since $G(T)=0$, we have $G(t) \geq 0$ on $[T,\infty)_{\mathbb{T}}$. This completes the proof of the lemma. □

Remark 2. Notice that if $r^{\Delta}(t) \geq 0$, $r^{\Delta\Delta}(t) \geq 0$, and $x(t)$ is replaced by $r(t)x(t)$, then the hypotheses of Lemma 2.10 still hold, so $x(t)$ can be replaced in the conclusion by $r(t)x(t)$ to obtain

$$\frac{r(t)x(t)}{(r(t)x(t))^{\Delta}} \geq \frac{h_2(t,T)}{t-T} \quad \text{for} \quad t > T. \tag{2.10}$$

3. MAIN RESULTS

In this section, we establish some sufficient conditions guaranteeing that any solution $y(t)$ of (1.1) oscillates or satisfies $\lim_{t\to\infty} x(t) = 0$. The results in Subsection 3.1 are for the case $\delta(t) > t$, and the case $\delta(t) \leq t$ will be studied in Subsection 3.2.

3.1. **The case $\boldsymbol{\delta(t) > t}$.** Here, we consider the case $\delta(t) > t$. We introduce the notation:

$$Q(t) := P(t)\left(\frac{R(\delta,t)p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\sigma(t)-t}\right)^{\gamma}, \quad P(t,T) := \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}}\Delta s > 0,$$

and

$$R(\delta,t) := \int_t^{\delta(t)} \frac{1}{r(s)}\Delta s,$$

for $T \geq t_0$, where $P(t)$ is as before, i.e., $P(t) = q(t)\,(1-a(\delta(t))^{\gamma}$. Now, we are ready to state and prove the main results in this subsection.

Theorem 3.1. *Assume that* (h_1)–(h_4) *and* (2.2) *hold. If*

$$\int_{t_0}^{\infty} Q(s)\Delta s = \infty, \tag{3.1}$$

then any solution $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

Proof. Assume that $y(t)$ is a nonoscillatory solution of equation (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$, where $T \geq t_0$ is chosen so that Lemma 2.1 holds. Now, since (2.2) holds, by Lemma 2.3, $x(t) \in C_0 \cup C_2$. If $x(t) \in C_0$, then by Lemma 2.8, $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$.

If $x(t) \in C_2$, by Lemma 2.7, $x(t)$ satisfies inequality (2.5). Define the function $w(t)$ by the Riccati substitution

$$w(t) := \frac{x^{[2]}(t)}{(x^{[1]}(t))^{\gamma}}. \tag{3.2}$$

Now, since $x(t) \in C_2$, we see that $w(t) > 0$, and differentiating, we obtain

$$w^{\Delta} = \left(\frac{x^{[2]}}{(x^{[1]})^{\gamma}}\right)^{\Delta} = \frac{\left(x^{[1]}\right)^{\gamma} x^{[3]} - \left(\left(x^{[1]}\right)^{\gamma}\right)^{\Delta} x^{[2]}}{\left(x^{[1]}\right)^{\gamma}\left(\left(x^{[1]}\right)^{\sigma}\right)^{\gamma}}.$$

From (2.5), we have

$$w^{\Delta} \leq -P(t)\frac{\left((x)^{\delta})\right)^{\gamma}}{\left((x^{[1]})^{\sigma}\right)^{\gamma}} - \frac{\left(\left(x^{[1]}\right)^{\gamma}\right)^{\Delta} x^{[2]}}{\left(x^{[1]}\right)^{\gamma}\left(\left(x^{[1]}\right)^{\sigma}\right)^{\gamma}}. \tag{3.3}$$

By Pötzsche's chain rule ([5, Theorem 1.90]), if $f^{\Delta}(t) > 0$ and $\gamma > 1$, we obtain

$$\begin{aligned} (f^{\gamma}(t))^{\Delta} &= \gamma\int_0^1 \left[f(t)+\mu(t)hf^{\Delta}(t)\right]^{\gamma-1} f^{\Delta}(t)\Delta h & (3.4)\\ &= \gamma\int_0^1 \left[(1-h)f(t)+hf^{\sigma}(t)\right]^{\gamma-1} f^{\Delta}(t)\Delta h \\ &\geq \gamma\int_0^1 (f(t))^{\gamma-1}f^{\Delta}(t)\Delta h = \gamma(f(t))^{\gamma-1}f^{\Delta}(t). & (3.5)\end{aligned}$$

Also by Pötzsche's chain rule, if $f^{\Delta}(t) > 0$ and $0 < \gamma \leq 1$, we obtain

$$(3.6)\qquad \begin{aligned} (f^{\gamma}(t))^{\Delta} &= \gamma \int_0^1 \left[f(t) + h\mu(t) f^{\Delta}(t)\right]^{\gamma-1} \Delta h \ f^{\Delta}(t) \\ &= \gamma \int_0^1 \left[(1-h)\, f(t) + h f^{\sigma}(t)\right]^{\gamma-1} \Delta h \ f^{\Delta}(t) \\ &\geq \gamma \int_0^1 (f^{\sigma}(t))^{\gamma-1} \Delta h \ f^{\Delta}(t) = \gamma (f^{\sigma}(t))^{\gamma-1} f^{\Delta}(t). \end{aligned}$$

So from (3.4) and (3.6) with $f(t) = x^{[1]}(t)$ and using that $x^{[1]}$ is increasing and $x^{[2]}$ is decreasing, we have

$$\begin{aligned} \frac{\left(\left(x^{[1]}\right)^{\gamma}\right)^{\Delta} x^{[2]}}{\left(x^{[1]}\right)^{\gamma}\left(\left(x^{[1]}\right)^{\sigma}\right)^{\gamma}} &\geq \frac{\gamma x^{[2]} \left(x^{[2]}\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}\left(x^{[1]}\right)\left(\left(x^{[1]}\right)^{\sigma}\right)^{\gamma}} \\ &\geq \frac{\gamma \left(x^{[2]}\right)^{\sigma} \left(\left(x^{[2]}\right)^{\sigma}\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}\left(\left(x^{[1]}\right)^{\sigma}\right)\left(\left(x^{[1]}\right)^{\sigma}\right)^{\gamma}} \\ &= \gamma p^{-\frac{1}{\gamma}} \left(w^{\sigma}\right)^{\frac{1}{\gamma}+1} \end{aligned}$$

for $\gamma > 1$, and

$$\begin{aligned} \frac{\left(\left(x^{[1]}\right)^{\gamma}\right)^{\Delta} x^{[2]}}{\left(x^{[1]}\right)^{\gamma}\left(\left(x^{[1]}\right)^{\sigma}\right)^{\gamma}} &\geq \frac{\gamma x^{[2]} \left(\left(x^{[1]}\right)^{\sigma}\right)^{\gamma-1} \left(x^{[2]}\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}\left(x^{[1]}\right)^{\gamma}\left(\left(x^{[1]}\right)^{\sigma}\right)^{\gamma}} \\ &= \frac{\gamma x^{[2]} \left(x^{[2]}\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}\left(x^{[1]}\right)^{\gamma}\left(\left(x^{[1]}\right)^{\sigma}\right)} \\ &\geq \frac{\gamma \left(x^{[2]}\right)^{\sigma} \left(\left(x^{[2]}\right)^{\sigma}\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}\left(\left(x^{[1]}\right)^{\sigma}\right)^{\gamma}\left(x^{[1]}\right)^{\sigma}} \\ &= \gamma \frac{1}{p^{\frac{1}{\gamma}}} \left(w^{\sigma}\right)^{1+\frac{1}{\gamma}} \end{aligned}$$

for $0 < \gamma \leq 1$. Thus,

$$(3.7)\qquad \frac{\left(\left(x^{[1]}\right)^{\gamma}\right)^{\Delta} x^{[2]}}{\left(x^{[1]}\right)^{\gamma}\left(\left(x^{[1]}\right)^{\sigma}\right)^{\gamma}} \geq \gamma \frac{1}{p^{\frac{1}{\gamma}}} \left(w^{\sigma}\right)^{1+\frac{1}{\gamma}} \quad \text{for } \ \gamma > 0.$$

Substituting (3.7) into (3.3), we have

$$(3.8)\qquad w^{\Delta} \leq -P(t) \left(\frac{x(\delta)}{\left(x^{[1]}\right)^{\sigma}}\right)^{\gamma} - \gamma \frac{1}{p^{\frac{1}{\gamma}}} \left(w^{\sigma}\right)^{1+\frac{1}{\gamma}} \quad \text{for } \ t \geq T.$$

Next, we consider the coefficient of $P(t)$ in (3.8). Since $(x^{[1]})^{\sigma} = x^{[1]}(t) + \mu(t)(x^{[1]})^{\Delta}$, we have

$$\frac{(x^{[1]})^{\sigma}}{x^{[1]}(t)} = 1 + \mu(t) \frac{(x^{[1]})^{\Delta}}{x^{[1]}(t)} = 1 + \frac{\mu(t)}{p^{\frac{1}{\gamma}}(t)} \frac{\left(x^{[2]}(t)\right)^{\frac{1}{\gamma}}}{x^{[1]}(t)}.$$

Also, since $x^{[2]}(t)$ is decreasing, we have

$$\begin{aligned} x^{[1]}(t) &= x^{[1]}(T) + \int_T^t \left(x^{[2]}(s)\right)^{\frac{1}{\gamma}} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s \\ &\geq x^{[1]}(T) + \left(x^{[2]}(t)\right)^{\frac{1}{\gamma}} \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s \\ &> \left(x^{[2]}(t)\right)^{\frac{1}{\gamma}} \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta \delta. \end{aligned}$$

It follows that

$$\frac{x^{[1]}(t)}{(x^{[2]}(t))^{\frac{1}{\gamma}}} \geq \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s = P(t,T). \tag{3.9}$$

Hence,

$$\begin{aligned} \frac{(x^{[1]})^\sigma}{x^{[1]}(t)} &= 1 + \mu(t)\frac{(x^{[1]})^\Delta}{x^{[1]}(t)} = 1 + \frac{\mu(t)}{p^{\frac{1}{\gamma}}(t)} \frac{\left(x^{[2]}(t)\right)^{\frac{1}{\gamma}}}{x^{[1]}(t)} \\ &\leq \frac{p^{\frac{1}{\gamma}}(t)P(t,T) + \mu(t)}{p^{\frac{1}{\gamma}}(t)P(t,T)}. \end{aligned}$$

Thus, we have

$$\frac{x^{[1]}(t)}{(x^{[1]})^\sigma} \geq \frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T) + \sigma(t) - t}$$

for $t \geq T$, so

$$\frac{x(\delta)}{(x^{[1]})^\sigma} = \frac{x(\delta)}{x^{[1]}} \frac{x^{[1]}}{(x^{[1]})^\sigma} \geq \frac{x(\delta)}{x^{[1]}} \frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T) + \sigma(t) - t} \tag{3.10}$$

for $t \geq T$. Now, since $\delta(t) > t$ and $x^{[1]}$ is increasing, we have

$$x(\delta(t)) > x(\delta(t)) - x(t) = \int_t^{\delta(t)} \frac{x^{[1]}(s)}{r(s)} \Delta s \geq x^{[1]}(t) R(\delta(t), t).$$

This, and (3.10), lead to

$$\frac{x(\delta)}{(x^{[1]})^\sigma} \geq \frac{R(\delta,t)p^{\frac{1}{\gamma}}(t)P(t,T)}{\left(p^{\frac{1}{\gamma}}(t)P(t,T) + \sigma(t) - t\right)} \tag{3.11}$$

for $t \geq T$. Substituting (3.11) into (3.8), we have

$$-w^\Delta(t) \geq Q(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)} (w^\sigma(t))^{\frac{\gamma+1}{\gamma}}, \tag{3.12}$$

for $t \geq T$. From the definition of $x^{[2]}(t)$, we see that $\left(x^{[1]}(t)\right)^\Delta = \left(x^{[2]}(t)/p(t)\right)^{\frac{1}{\gamma}}$, so integrating from T to t, we obtain

$$x^{[1]}(t) = x^{[1]}(T) + \int_T^t \left(\frac{1}{p(s)} x^{[2]}(s)\right)^{\frac{1}{\gamma}} \Delta s.$$

Now $x^{[2]}(t)$ is positive and decreasing, so

$$x^{[1]}(t) \geq x^{[1]}(T) + \left(x^{[2]}(t)\right)^{1/\gamma} \int_T^t (1/p(s))^{1/\gamma}\, \Delta s.$$

It follows that

$$w(t) = \frac{x^{[2]}}{\left(x^{[1]}\right)^{\gamma}} \leq \left(\int_{t_0}^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s\right)^{-\gamma}$$

for $t \geq T$, which, in view of (2.2) implies $\lim_{t\to\infty} w(t) = 0$. Integrating (3.12) from T to ∞, we have $w(T) \geq \int_T^\infty Q(s)\Delta s$, which contradicts (3.1). This completes the proof of the theorem. $\square$

In the following theorems, condition (3.1) may not hold yet we are still able to obtain some oscillation results.

Theorem 3.2. *Assume that (h_1)–(h_4) and* (2.2) *hold and there exist positive rd-continuous $\Delta-$differentiable functions $\alpha(t)$ and $\phi(t)$ such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\alpha(s)\phi(s)P(s) - \frac{(\alpha(\sigma(s)))^{\gamma+1} r^{\gamma}(s) C^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}\alpha^{\gamma}(s)\phi^{\gamma}(s)P^{\gamma}(s,T)}\right] \Delta s = \infty, \tag{3.13}$$

where $T \geq t_0$ and $C(s) := \phi(s)\alpha^{\Delta}(s)/\alpha^{\sigma} + \phi^{\Delta}(s)$. Then any solution $y(t)$ of (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1.1), say $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$, where T is chosen so that Lemma 2.1 holds. Now, since (2.2) holds, we see from Lemma 2.3 that $x(t) \in C_0 \cup C_2$. If $x(t) \in C_0$, then by Lemma 2.8, we have $\lim_{t\to\infty} y(t) = 0$. Next, let $x(t) \in C_2$ and define the function $\omega(t)$ by the Riccati substitution

$$\omega(t) := \alpha(t)\frac{x^{[2]}(t)}{x^{\gamma}(t)}, \qquad \text{for } t \geq T. \tag{3.14}$$

Noting that $\omega(t) > 0$, and differentiating (as a product first), we obtain

$$\begin{aligned}
\omega^{\Delta} &= \left(x^{[2]}\right)^{\sigma} \left[\frac{\alpha}{x^{\gamma}}\right]^{\Delta} + \frac{\alpha}{x^{\gamma}} \left(x^{[2]}\right)^{\Delta} \\
&= \left(x^{[2]}\right)^{\sigma} \left[\frac{\alpha^{\Delta}}{(x^{\sigma})^{\gamma}} - \frac{\alpha (x^{\gamma})^{\Delta}}{x^{\gamma} (x^{\sigma})^{\gamma}}\right] + \frac{\alpha}{x^{\gamma}} \left(x^{[2]}\right)^{\Delta}.
\end{aligned}$$

Since $x(t) \in C_2$, we have $x^{\Delta}(t) > 0$, so $x^{\delta}(t) > x(t)$ and $x^{\sigma}(t) > x(t)$. Proceeding as in the proof of Theorem 3.1 using Lemma 2.6, (3.14), condition (h_3), and the fact that $\delta(t) \geq t$, we obtain

$$\omega^{\Delta} \leq -\alpha P + \frac{\alpha^{\Delta}}{\alpha^{\sigma}}\omega^{\sigma} - \alpha\frac{\left(x^{[2]}\right)^{\sigma} (x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}. \tag{3.15}$$

Pötzsche's chain rule gives

$$\begin{cases} (x^\gamma(t))^\Delta \geq \gamma(x(t))^{\gamma-1}x^\Delta(t), & \text{for } \gamma \geq 1, \\ (x^\gamma(t))^\Delta \geq \gamma(x^\sigma(t))^{\gamma-1}x^\Delta(t), & \text{for } \gamma < 1. \end{cases} \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$\omega^\Delta \leq -\alpha P + \frac{\alpha^\Delta}{\alpha^\sigma}\omega^\sigma - \gamma\left(x^{[2]}\right)^\sigma \frac{\alpha x^\Delta}{(x^\sigma)^{\gamma+1}}$$

for $\gamma > 0$, where we used the fact that $x^\sigma(t) \geq x(t)$. By Lemma 2.9

$$x^\Delta(t) \geq \frac{P(t,T)}{r(t)}\left(x^{[2]}(t)\right)^{\frac{1}{\gamma}},$$

so

$$\omega^\Delta \leq -\alpha P + \frac{\alpha^\Delta}{\alpha^\sigma}\omega^\sigma - \gamma\alpha\left(x^{[2]}\right)^\sigma \frac{P(t,T)}{r(t)}\frac{\left(x^{[2]}(t)\right)^{\frac{1}{\gamma}}}{(x^\sigma)^{\gamma+1}}. \tag{3.17}$$

Also, since $x^{[2]}(t)$ is decreasing, we have $\left(x^{[2]}\right)^{\frac{1}{\gamma}} \geq \left(\left(x^{[2]}\right)^\sigma\right)^{\frac{1}{\gamma}}$, so (3.17) yields

$$\omega^\Delta(t) \leq -\alpha(t)P(t) + \frac{\alpha^\Delta(t)}{\alpha^\sigma(t)}\omega^\sigma(t) - \frac{\gamma\alpha(t)P(t,T)}{(\alpha^\sigma(t))^{\frac{\gamma+1}{\gamma}}r(t)}(\omega^\sigma)^{\frac{\gamma+1}{\gamma}}(t), \tag{3.18}$$

for $t \geq T$. Multiplying (3.18) by $\phi(s)$ and integrating from T to $t \geq T$, we have

$$\begin{aligned} \int_T^t \phi(s)\alpha(s)P(s)\Delta s \quad &\leq \quad -\int_T^t \phi(s)\omega^\Delta(s)\Delta s \\ &\quad + \int_T^t \left[\phi(s)\frac{\alpha^\Delta(t)}{\alpha^\sigma(t)}\omega^\sigma - \frac{\gamma\phi(s)\alpha(s)P(s,T)}{(\alpha^\sigma(s))^{\frac{\gamma+1}{\gamma}}r(s)}\left(\omega^\sigma\right)^{\frac{\gamma+1}{\gamma}}\right]\Delta s. \end{aligned}$$

An integration by parts yields

$$\begin{aligned} \int_T^t \phi(s)\alpha(s)P(s)\Delta s &\leq \omega(T)\phi(T) \\ &+ \int_T^t \left[\frac{\phi(s)\alpha^\Delta(s)}{\alpha^\sigma} + \phi^\Delta(s)\right]\omega^\sigma(s)\Delta s - \int_{t_1}^t \frac{\gamma\phi(s)\alpha(s)P(s,T)}{(\alpha^\sigma(s))^{\frac{\gamma+1}{\gamma}}r(s)}(\omega^\sigma)^{\frac{\gamma+1}{\gamma}}\Delta s. \end{aligned} \tag{3.19}$$

Applying the inequality

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^\gamma}$$

to the right hand side of (3.19) with

$$A = \frac{\gamma\phi(s)\alpha(s)P(s,T)}{(\alpha^\sigma(s))^{\frac{\gamma+1}{\gamma}}r(s)}, \quad B = \frac{\phi(s)\alpha^\Delta(s)}{\alpha^\sigma} + \phi^\Delta(s), \quad \text{and} \quad u = \omega^\sigma(s),$$

gives

$$\int_T^t \left[\phi(s)\alpha(s)P(s) - \frac{1}{(\gamma+1)^{\gamma+1}}\frac{(\alpha^\sigma)^{\gamma+1}(s)r^\gamma(s)C^{\gamma+1}(s)}{(\alpha(s)\phi(s)P(s,T))^\gamma}\right]\Delta s < \omega(T)\phi(T), \tag{3.20}$$

which contradicts (3.13) and completes the proof of the theorem. $\square$

As a special case of Theorem 3.2 with $\alpha(t) = 1$, we have the following result.

Theorem 3.3. *Assume that* (h_1)*–*(h_4) *and* (2.2) *hold and there exists a positive rd-continuous* $\Delta-$*differentiable function* $\phi(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\phi(s)P(s) - \frac{r^{\gamma}(s)(\phi^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)P^{\gamma}(s,T)}\right]\Delta s = \infty. \tag{3.21}$$

Then any solution $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

The following theorems gives Philos-type oscillation criteria for equation (1.1). The proofs are similar to the proof of Theorem 3.3 in [22] using inequality (3.18); the details are omitted. First, we need to introduce the class of functions $\Re$. Let $\mathbb{D}_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \geq t_0\}$ and $\mathbb{D} \equiv \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. The function $H \in C_r(\mathbb{D},\mathbb{R})$ is said to belong to the class $\Re$ if

(i) $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ on $\mathbb{D}_0$,

(ii) H has a continuous $\Delta-$partial derivative $H^{\Delta_s}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable.

Theorem 3.4. *Assume that* (h_1)*–*(h_4) *and* (2.2) *hold and there is a function* $H \in \Re$ *such that for* $t > s$, *we have*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)P(s) - \frac{r^{\gamma}(s)(H^{\Delta_s}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}H^{\gamma}(t,s)P^{\gamma}(s,T)}\right]\Delta s = \infty. \tag{3.22}$$

Then any solution $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

Theorem 3.5. *Assume that* (h_1)*–*(h_4) *and* (2.2) *hold,* $\alpha(t)$ *is a positive rd-continuous* $\Delta-$*differentiable function, and there is a function* $H \in \Re$ *such that for* $t > s$, *we have*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\alpha(s)P(s) - \frac{(\alpha^{\sigma})^{\gamma+1}r^{\gamma}(s)D^{\gamma+1}(t,s)}{(\gamma+1)^{\gamma+1}\alpha^{\gamma}(s)P^{\gamma}(s,T)H^{\gamma}(t,s)}\right]\Delta s = \infty, \tag{3.23}$$

where $T \geq t_0$ *and* $D(t,s) := H(t,s)\alpha^{\Delta}(s)/\alpha^{\sigma} - H^{\Delta_s}(t,s)$. *Then any solution* $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

Theorem 3.6. *Assume that* (h_1)*–*(h_4) *and* (2.2) *hold,* $\phi(t)$ *is a positive rd-continuous* $\Delta-$*differentiable function, and there is a function* $H \in \Re$ *such that for* $t > s$, *we have*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)P(s) - \frac{r^{\gamma}(s)((\phi^{\Delta}(s))^{\gamma+1}(H^{\Delta_s}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}P^{\gamma}(s,T)\phi^{\gamma}(s)H^{\gamma}(t,s)}\right]\Delta s = \infty, \tag{3.24}$$

where $T \geq t_0$. *Then any solution* $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

Remark 3. With an appropriate choice of the functions H, ϕ, and α, it is possible to derive a number of oscillation criteria for equation (1.1) on different types of time scales. Consider, for example the function $H(t,s) = (t-s)^{\lambda}$, $(t,s) \in \mathbb{D}$ with $\lambda > 1$. We see that H belongs to the class $\Re$ and so we can obtain Kamenev-type oscillation criteria. Here we use the falling function (see [19]) $H(t,s) := (t-s)^{\underline{k}}$, where $t^{\underline{k}} := t(t-1)...(t-k+1)$, $t^{\underline{0}} := 1$. In this case,

$$H^{\Delta_s}(t,s) = \left((t-s)^{\underline{k}}\right)^{\Delta_s} = -k(t-s-2k-3)^{\underline{k-1}} \geq -(k)(t-s)^{\underline{k-1}}.$$

Next, we extend some oscillation criteria established by Hille [17] and Nehari [20] for second-order differential equations to the third order nonlinear neutral dynamic equation (1.1) on an arbitrary time scale $\mathbb{T}$ that is unbounded above. We introduce the following notation:

$$\begin{aligned} r_* &:= \liminf_{t\to\infty} \frac{t^{\gamma}\omega^{\sigma}(t)}{p\left(t\right)}, \quad R := \limsup_{t\to\infty} \frac{t^{\gamma}\omega^{\sigma}(t)}{p(t)}, \\ p_* &:= \liminf_{t\to\infty} \frac{t^{\gamma}}{p(t)} \int_{\sigma(t)}^{\infty} Q(s)\Delta s, \\ q_* &:= \liminf_{t\to\infty} \frac{1}{t} \int_{T}^{t} \frac{s^{\gamma+1}}{p(s)} Q(s)\Delta s, \end{aligned}$$

and set $l := \liminf_{t\to\infty} \frac{t}{\sigma(t)}$. From the definition of $\sigma(t)$ it is clear that $0 \leq l \leq 1$.

Theorem 3.7. *Assume that* (h_1)*–*(h_4) *and* (2.2) *hold and* $p^{\Delta}(t) \geq 0$*. If either*

$$p_* > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}, \tag{3.25}$$

or

$$p_* + q_* > \frac{1}{l^{\gamma(\gamma+1)}}, \tag{3.26}$$

then any solution $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$*.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1.1) with $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$, where T is chosen so Lemma 2.1 holds. By Lemma 2.2, $x(t) \in C_0 \cup C_2$. If $x(t) \in C_0$, then $y(t) \to 0$ as $t \to \infty$ by Lemma 2.8. Now let $x(t) \in C_2$ and define the function $w(t)$ by the Riccati substitution (3.2) as in Theorem 3.1. Then, from (3.12), that

$$-w^{\Delta}(t) \geq Q(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)} \left(w^{\sigma}(t)\right)^{\frac{\gamma+1}{\gamma}}, \quad \text{for } t \in [T,\infty)_{\mathbb{T}}. \tag{3.27}$$

From the definition of $x^{[2]}(t)$, we see that

$$\left(x^{[1]}(t)\right)^{\Delta} = \left(\frac{x^{[2]}(t)}{p(t)}\right)^{\frac{1}{\gamma}},$$

and integrating from T to t, we obtain

$$x^{[1]}(t) = x^{[1]}(T) + \int_T^t \left(\frac{1}{p(s)} x^{[2]}(s)\right)^{\frac{1}{\gamma}} \Delta s.$$

Since $x^{[2]}(t)$ is positive and decreasing, we have

$$x^{[1]}(t) \geq x^{[1]}(T) + \left(x^{[2]}(t)\right)^{\frac{1}{\gamma}} \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s.$$

It follows that

$$w(t) = \frac{x^{[2]}}{(x^{[1]})^{\gamma}} \leq \left(\int_{t_0}^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s\right)^{-\gamma}, \quad \text{for} \quad t \in [T, \infty)_{\mathbb{T}}$$

which, in view of (2.2), implies $\lim_{t\to\infty} w(t) = 0$. First, we assume (3.25) holds. Integrating (3.27) from $\sigma(t)$ to ∞ gives

$$w^{\sigma}(t) \geq \int_{\sigma(t)}^{\infty} Q(s)\Delta s + \gamma \int_{\sigma(t)}^{\infty} \frac{1}{p^{\frac{1}{\gamma}}(s)} (w^{\sigma}(s))^{\frac{1}{\gamma}} w^{\sigma}(s)\Delta s. \tag{3.28}$$

From (3.28) it follows that

$$\frac{t^{\gamma} w^{\sigma}(t)}{p(t)} \geq \frac{t^{\gamma}}{p(t)} \int_{\sigma(t)}^{\infty} Q(s)\Delta s + \gamma \frac{t^{\gamma}}{p(t)} \int_{\sigma(t)}^{\infty} \frac{1}{p^{\frac{1}{\gamma}}(s)} (w^{\sigma}(s))^{\frac{1}{\gamma}+1} \Delta s. \tag{3.29}$$

The remainder of the proof is similar to the proof of Theorem 2.1 in [23] and hence we omit the details. □

From Theorem 3.7, we immediately have the following results.

Corollary 3.8. *Assume that (h_1)–(h_4) and* (2.2) *hold and* $p^{\Delta}(t) \geq 0$. *If*

$$\liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{s^{\gamma+1}}{p(s)} P(s)\Delta s > \frac{1}{l^{\gamma(\gamma+1)}}, \tag{3.30}$$

then any solution $y(t)$ of (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

Corollary 3.9. *Assume that (h_1)–(h_4) and* (2.2) *hold and* $p^{\Delta}(t) \geq 0$. *If*

$$\liminf_{t\to\infty} \frac{t^{\gamma}}{p(t)} \int_{\sigma(t)}^{\infty} P(s)\Delta s > \frac{1}{l^{\gamma(\gamma+1)}}, \tag{3.31}$$

then any solution $y(t)$ of (1.1) is either oscillatory or satisfies $\lim_{t\to\infty} y(t) = 0$.

3.2. **The case $\boldsymbol{\delta(t) \leq t}$.** In this subsection, we consider the case where $\delta(t) \leq t$ and establish some oscillation results in this, the delay, case. We will use the notation in Subsection 3.1 as well as the following:

$$\pi(t) := P(t) \left(\frac{p^{\frac{1}{\gamma}}(t) P(\delta(t), T)}{p^{\frac{1}{\gamma}}(t) P(t, T) + \mu(t)}\right)^{\gamma} \eta^{\gamma}(t),$$

$$\eta(t) := \frac{h_2(\delta(t), T)}{r(\delta(t))\,(\delta(t) - T)}.$$

Theorem 3.10. *Assume that* (h_1)*–*(h_4) *and* (2.2) *hold,* $r^{\Delta}(t) \geq 0$*, and* $r^{\Delta\Delta}(t) \geq 0$*. If*

$$\int_{t_0}^{\infty} \pi(s)\Delta s = \infty, \tag{3.32}$$

then any solution $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1.1) with $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$ where T is chosen so that Lemma 2.1 holds. By Lemma 2.2, $x(t) \in C_0 \cup C_2$. If $x(t) \in C_0$, then Lemma 2.8 implies $\lim_{t\to\infty} x(t) = 0$. Now let $x(t) \in C_2$ and define $w(t)$ as in (3.2). Proceeding as in the proof of Theorem 3.1, we obtain

$$w^{\Delta} \leq -P(t)\left(\frac{x(\delta)}{\left(x^{[1]}\right)^{\sigma}}\right)^{\gamma} - \gamma\frac{1}{p^{\frac{1}{\gamma}}(t)}\left(w^{\sigma}\right)^{1+\frac{1}{\gamma}}. \tag{3.33}$$

Now consider the coefficient of $P(t)$ in (3.33). Setting $z = x^{[1]} = rx^{\Delta}$, we have

$$z > 0, \quad z^{\Delta} > 0, \quad \text{and} \quad (p\left[z^{\Delta}\right]^{\gamma})^{\Delta} < 0. \tag{3.34}$$

Then,

$$\left(\frac{x^{\delta}}{\left(x^{[1]}\right)^{\sigma}}\right)^{\gamma} = \left(\frac{x^{\delta}}{z^{\delta}}\right)^{\gamma}\left(\frac{z^{\delta}}{z^{\sigma}}\right)^{\gamma}. \tag{3.35}$$

Since $z^{[1]}(t) = p\left(z^{\Delta}\right)^{\gamma}(t)$ is decreasing for $t \geq T$, we have

$$z^{\sigma}(t) - z(\delta(t)) = \int_{\delta(t)}^{\sigma(t)} \frac{z^{[1]}(s)}{p^{\frac{1}{\gamma}}(s)}\Delta s \leq z^{[1]}(\delta(t))\int_{\delta(t)}^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s,$$

and this implies that

$$\frac{z^{\sigma}(t)}{z(\delta(t))} \leq 1 + \frac{z^{[1]}(\delta(t))}{z(\delta(t))}\int_{\delta(t)}^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s. \tag{3.36}$$

On the other hand, we have that

$$\begin{aligned} z(\delta(t)) \quad &> \quad z(\delta(t)) - z(T) = \int_{T}^{\delta(t)} \frac{z^{[1]}(s)}{p^{\frac{1}{\gamma}}(s)}\Delta s \\ &\geq \quad z^{[1]}(\delta(t))\int_{T}^{\delta(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s, \end{aligned}$$

which leads to

$$\frac{z^{[1]}(\delta(t))}{z(\delta(t))} < \left(\int_{T}^{\delta(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s\right)^{-1}.$$

Using the last inequality in (3.36), we obtain

$$\begin{aligned}\frac{z^\sigma(t)}{z(\delta(t))} &< 1+\frac{\int_{\delta(t)}^{\sigma(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}{\int_T^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}=\frac{\int_T^{\sigma(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}{\int_T^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}\\ &= \frac{\int_T^t p^{-\frac{1}{\gamma}}(s)\Delta s+\int_t^{\sigma(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}{\int_T^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}\\ &= \frac{\int_T^t p^{-\frac{1}{\gamma}}(s)\Delta s+\mu(t)p^{-\frac{1}{\gamma}}(t)}{\int_T^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}, \qquad \text{for } t\geq T.\end{aligned}$$

Hence,

$$z(\delta(t))\geq \frac{p^{\frac{1}{\gamma}}(t)P(\delta(t),T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\mu(t)}z^\sigma(t), \qquad \text{for } t\geq T,$$

which implies

$$\frac{\left(z^\delta(t)\right)^\gamma}{\left(z^\sigma(t)\right)^\gamma}\geq\left(\frac{p^{\frac{1}{\gamma}}(t)P(\delta(t),T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\mu(t)}\right)^\gamma, \text{ for } t\geq T. \tag{3.37}$$

Then, from (3.35), (3.37) and (2.10), we have

$$\left(\frac{x^\delta}{(x^{[1]})^\sigma}\right)^\gamma\geq\left(\frac{p^{\frac{1}{\gamma}}(t)P(\delta(t),T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\mu(t)}\right)^\gamma\eta^\gamma(\delta(t)), \text{ for } t\geq T. \tag{3.38}$$

Substituting (3.38) into (3.33) gives

$$w^\Delta(t)+\pi(t)+\gamma\frac{1}{p^{\frac{1}{\gamma}}(t)}\left(w^\sigma\right)^{1+\frac{1}{\gamma}}(t)\leq 0 \tag{3.39}$$

for $t\geq T$. The remainder of the proof is similar to the proof of Theorem 3.1 starting with inequality (3.12). □

Remark 4. Notice the difference between the function $Q(t)$ for the advanced case and the function $\pi(t)$ for the delay case. In the definition of $Q(t)$, there is a function $R(\delta,t)$ that is replaced by the function $\eta(t)$ in $\pi(t)$ and the function $h_2(\delta,T)$ is also part of $\eta(t)$.

Next, we formulate some sufficient conditions for the oscillation of equation (1.1). Since the proofs are similar to the proofs of Theorems 3.2–3.6 in Subsection 3.1 by using the inequality (3.39) in place of (3.12), we omit the details. Similar to what we had in the previous subsection, condition (3.32) is not required to hold in these results.

Theorem 3.11. *Assume that* (h_1)–(h_4) *and* (2.2) *hold,* $r^\Delta(t)\geq 0$, $r^{\Delta\Delta}(t)\geq 0$, *and there exists a positive rd-continuous* $\Delta-$*differentiable function* $\phi(t)$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^t\left[\phi(s)\pi(s)-\frac{p(s)(\phi^\Delta(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)}\right]\Delta s=\infty. \tag{3.40}$$

Then any solution $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty}y(t)=0$.

Theorem 3.12. *Assume that* (h_1)–(h_4) *and* (2.2) *hold*, $r^{\Delta}(t) \geq 0$, $r^{\Delta\Delta}(t) \geq 0$, *and there is a function* $H \in \Re$ *such that for* $t > s$, *we have*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\pi(s) - \frac{p(s)(H^{\Delta_s}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}H^{\gamma}(t,s)} \right] \Delta s = \infty. \tag{3.41}$$

Then any solution $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

In what follows, we present oscillation criteria of Hille and Nehari types for the delay case of equation (1.1). The proof is similar to the proof of Theorem 2.1 in [23] using inequality (3.39), and so we omit the details. We do need the notation:

$$m_* := \liminf_{t\to\infty} \frac{t^{\gamma}}{p(t)} \int_{\sigma(t)}^{\infty} \pi(s)\Delta s \quad \text{and} \quad n_* := \liminf_{t\to\infty} \frac{1}{t} \int_{T}^{t} \frac{s^{\gamma+1}}{p(s)} \pi(s)\Delta s.$$

Theorem 3.13. *Assume that* (h_1)–(h_4) *and* (2.2) *hold*, $r^{\Delta}(t) \geq 0$, *and* $r^{\Delta\Delta}(t) \geq 0$. *If either*

$$m_* > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

or

$$m_* + n_* > \frac{1}{l^{\gamma(\gamma+1)}},$$

then any solution $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

The following corollary is an immediate consequence of Theorem 3.13.

Corollary 3.14. *Assume that* (h_1)–(h_4) *and* (2.2) *hold*, $r^{\Delta}(t) \geq 0$, *and* $r^{\Delta\Delta}(t) \geq 0$. *If either*

$$\liminf_{t\to\infty} \frac{1}{t} \int_{T}^{t} \frac{s^{\gamma+1}}{p(s)} \pi(s)\Delta s > \frac{1}{l^{\gamma(\gamma+1)}}$$

or

$$\liminf_{t\to\infty} \frac{t^{\gamma}}{p(t)} \int_{\sigma(t)}^{\infty} \pi(s)\Delta s > \frac{1}{l^{\gamma(\gamma+1)}},$$

then any solution $y(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} y(t) = 0$.

4. EXAMPLES

In this section, we give some examples to illustrate the main results. To obtain the conditions for oscillation we will use the following facts (see [6, Theorem 5.68 and Corollary 5.71]):

$$\text{if} \quad 0 \leq \nu \leq 1, \quad \text{then} \quad \int_{t_0}^{\infty} \frac{\Delta s}{s^{\nu}} = \infty;$$

$$\text{if} \quad \nu > 1 \quad \text{and} \quad \sigma(t) = O(t^{\alpha}) \quad \text{for some} \quad \alpha \in [1,p), \quad \text{then} \quad \int_{t_0}^{\infty} \frac{\Delta s}{s^{\nu}} < \infty.$$

First, we give some examples in case $\delta(t) > t$.

Example 1. Consider the dynamic equation

$$[y(t)+\frac{1}{2}y(\tau(t))]^{\Delta\Delta\Delta}+\frac{\alpha}{t^2}y(2t)=0,\ \text{ for } t\in[1,\infty)_{\mathbb{T}}, \tag{4.1}$$

on a time scale $\mathbb{T}$ such that $\int_{t_0}^{\infty}\frac{1}{\sigma(s)}\Delta s=\infty$ and $\alpha>0$. We have $\gamma=1$, $r(t)=p(t)=1$, $a(t)=1/2$, $q(t)=\alpha/t^2$ and $\delta(t)=2t>t$, so $P(t)=q(t)(1-a(\delta(t))^{\gamma}=\alpha/2t^2$,

$$P(t,1)=\int_1^t\frac{\Delta s}{p(s)}=t-1,\quad R(2t,t)=\int_t^{2t}\frac{1}{r(s)}\Delta s=t,$$

and

$$Q(t)=P(t)\left(\frac{R(2t,t)p^{\frac{1}{\gamma}}(t)P(t,1)}{p^{\frac{1}{\gamma}}(t)P(t,1)+\sigma(t)-t}\right)=\frac{\alpha}{2t^2}\left(\frac{t(t-1)}{\sigma(t)-1}\right).$$

It is clear that conditions (h_1)–(h_4) and (2.2) hold. Also,

$$\int_{t_0}^{\infty}Q(s)\Delta s=\int_1^{\infty}\frac{\alpha}{2s^2}\left(\frac{s(s-1)}{\sigma(s)-1}\right)\Delta s\ge\int_1^{\infty}\frac{\alpha}{2s}\left(\frac{s-1}{\sigma(s)}\right)\Delta s=\infty,$$

so (3.1) is satisfied. By Theorem 3.1, any solution $y(t)$ of (4.1) either oscillates or satisfies $\lim_{t\to\infty}y(t)=0$.

Example 2. Let $\mathbb{T}=\mathbb{R}$ and consider the differential equation

$$\left(t\left[\left(\frac{1}{t}\left(y(t)+\frac{1}{2}y(\tau(t))\right)'\right)'\right]^3\right)'+\frac{\beta}{t^3}y^3(2t)=0,\ \text{ for } t\ge1. \tag{4.2}$$

Here $\gamma=3$, $a(t)=1/2$, $r(t)=1/t$, $p(t)=t$, $q(t)=\beta/t^3$, $\tau(t)\le t$, and $\delta(t)=2t$. It is easy to see that conditions (h_1)–(h_3) and (2.2) hold. We also have

$$P(t)=\frac{\beta}{8t^3},\quad R(\delta(t),t)=\frac{3t^2}{2},\quad P(t,T)=\ln t,\quad\text{and}\quad Q(t)=\frac{27\beta t^3}{64},$$

so condition (3.1) holds. Now

$$\int_{t_0}^{\infty}r^{-1}(t)\int_t^{\infty}\left(p^{-1}(u)\int_u^{\infty}q(s)\Delta s\right)^{\frac{1}{\gamma}}\Delta u\Delta t=\int_1^{\infty}t\int_t^{\infty}\left(\frac{1}{u}\int_u^{\infty}\frac{\beta}{s^3}ds\right)^{\frac{1}{3}}dudt=\infty,$$

so (h_4) is satisfied. By Theorem 3.1, any solution of (4.2) is either oscillatory or converges to zero.

Example 3. Let $\mathbb{T}=\mathbb{R}$ and consider the differential equation

$$\left(t\left[\left(\frac{1}{t}\left(y(t)+\frac{1}{2}y(\tau(t))\right)'\right)'\right]^5\right)'+\frac{\beta}{t^4}y^5(2t)=0,\ \text{ for } t\ge1. \tag{4.3}$$

Here $\gamma=5$, $a(t)=1/2$, $r(t)=1/t$, $p(t)=t$, $q(t)=\beta/t^4$, $\tau(t)\le t$, and $\delta(t)=2t$. Clearly, (h_1)–(h_4) and (2.2) hold. We have

$$P(t)=\frac{\beta}{32t^4},\quad R(\delta(t),t)=\frac{3t^2}{2},\quad\text{and}\quad P(t,T)=\frac{5}{4}\left(t^{4/5}-1\right).$$

Since, in this example, $\sigma(t)=t$, $Q(t)$ reduces to

$$Q(t)=P(t)R^{\gamma}(\delta(t),t)=\frac{3\beta}{64t^2},$$

so condition (3.1) does not hold. Now if we take $\phi(t)=t^5$ in Theorem 3.3, condition (3.21) becomes

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[s^5\frac{\beta}{32s^4}-\frac{s^{-5}(5s^4)^6}{6^6(s^5)^5[\frac{5}{4}\,(s^{4/5}-1)]^5}\right]\Delta s=\infty,$$

so any solution of (4.3) is either oscillatory or converges to zero.

Next, we give some examples to illustrate the results for $\delta(t)\le t$.

Example 4. Consider the third order dynamic equation

$$x^{\Delta\Delta\Delta}(t)+\frac{\beta(\sigma(t)-1)}{\delta^2(t)h_2(\delta(t),1)}x(\delta(t))=0,\quad \text{for } t\in[1,\infty)_{\mathbb{T}}, \tag{4.4}$$

on a time scale $\mathbb{T}$ with $\delta(t)\le t$, and $\lim_{t\to\infty}\delta(t)=\infty$, and $\beta>1/4$. Here $p(t)=1$, $r(t)=1$, $\gamma=1$, $a(t)=0$, and $q(t)=\beta(\sigma(t)-1)/(\delta^2(t)h_2(\delta(t),1))$. We have

$$\begin{aligned}\pi(t)&:=P(t)\left(\frac{p^{\frac{1}{\gamma}}(t)P(\delta(t),T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\mu(t)}\right)^{\gamma}\eta^{\gamma}(\delta(t)\\&=\frac{\beta(\sigma(t)-1)}{\delta^2(t)h_2(\delta,T)}\left(\frac{\delta(t)-T}{t-1+\sigma(t)-t}\right)\frac{h_2(\delta,T)}{\delta(t)-T}\\&=\frac{\beta(\sigma(t)-1)}{\delta^2(t)}\frac{1}{\sigma(t)-1}=\frac{\beta}{\delta^2(t)}.\end{aligned}$$

Now conditions (h_1)–(h_4) and (2.2) hold, but condition (3.32) may or may not hold. However, choosing $\phi(t)=t$, we have

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^{t}\left[\phi(s)\pi(s)-\frac{p(s)(\phi^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right]\Delta s&=\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\beta s}{\delta^2(s)}-\frac{1}{4s}\right]\Delta s\\&\ge\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\beta}{s}-\frac{1}{4s}\right]\Delta s=\infty\end{aligned}$$

since $\beta>1/4$, so condition (3.40) is satisfied. By Theorem 3.11, any solution $y(t)$ of (4.4) is either oscillatory or converges to zero.

Example 5. Let $\mathbb{T}=\mathbb{R}$ and consider the differential equation

$$\left(\left[\left(t\left(y(t)+\frac{1}{2}y(\tau(t))\right)^{\Delta}\right)^{\Delta}\right]^{3}\right)^{\Delta}+\frac{\beta}{t^{\epsilon}}y^3(t/2)=0,\quad t\ge 1 \tag{4.5}$$

Here, $p(t) = 1$, $a(t) = 1/2$, $r(t) = t$, $\gamma = 3$, $q(t) = \beta/t^{\epsilon}$, $\tau(t) \leq t$, and $\delta(t) = t/2$. Clearly, (h_1)–(h_4) and (2.2) hold. We have

$$\begin{aligned}\int_1^{\infty} \frac{1}{t} \int_t^{\infty} \left(\int_u^{\infty} \frac{\beta}{s^{\epsilon}} ds \right)^{\frac{1}{3}} dudt &= \left(\frac{\beta}{\epsilon - 1} \right)^{\frac{1}{3}} \int_1^{\infty} \frac{1}{t} \int_t^{\infty} u^{\frac{1-\epsilon}{3}} dudt \\ &\geq \left(\frac{\beta}{\epsilon - 1} \right)^{\frac{1}{3}} \int_1^{\infty} \frac{1}{t} \int_t^{2t} u^{\frac{1-\epsilon}{3}} dudt \\ &= \infty \quad \text{if} \quad 1 < \epsilon < 4.\end{aligned}$$

That is, (h_4) is satisfied if $1 < \epsilon < 4$. We also have

$$P(t) = \frac{\beta}{8t^{\epsilon}}, \quad P(\delta(t), t) = \frac{t-2}{2}, \quad P(t, 1) = t - 1, \quad h_2(\delta(t), 1) = \frac{(t-2)^2}{8},$$

$$\eta(t) = \frac{t-2}{2t}, \quad \text{and} \quad \pi(t) = \frac{\beta(t-2)^4}{128t^{\epsilon+3}(t-1)}.$$

Now

$$\int_1^{\infty} \pi(s) ds < \infty,$$

if $\epsilon > 1$, so condition (3.32) does not hold and as a result Theorem 3.10 does not apply. However, if we take $\phi(t) = t^{\alpha}$, then

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[t^{\alpha} \frac{\beta(s-2)^4}{128s^{\epsilon+3}(s-1)} - \left(\frac{\alpha}{4} \right)^4 t^{\alpha - 4} \right] ds = \infty$$

if $\alpha > \epsilon$. Thus, if $1 < \epsilon < 4$ and $\alpha > \epsilon$, then condition (3.40) holds, and by Theorem 3.11, any solution $y(t)$ of (4.5) is either oscillatory or converges to zero.

Example 6. As a final example, consider the third order ordinary differential equation

$$x'''(t) + \frac{1}{\sqrt{3}\, t^3} x(t) = 0, \quad t \geq 1. \tag{4.6}$$

Here

$$q(t) = \frac{1}{\sqrt{3}\, t^3} = P(t), \quad P(t, 1) = t - 1, \quad h_2(t, 1) = \frac{(t-1)^2}{2},$$

$$\eta(t) = \frac{t-2}{2}, \quad \text{and} \quad \pi(t) = \frac{(t-1)}{2\sqrt{3}\, t^3}.$$

Condition (3.32) does not hold so again Theorem 3.10 does not apply. Taking $\phi(t) = \frac{8}{3\sqrt{3}} t$ in Theorem 3.11, condition (3.40) becomes

$$\limsup_{t \to \infty} \int_1^t \left[\left(\frac{4}{9} - \frac{2}{3\sqrt{3}} \right) \frac{1}{s} - \frac{4}{9} \frac{1}{s^2} \right] \Delta s = \infty,$$

so any solution of (4.6) is either oscillatory or converges to zero. Notice that $\frac{4}{9} - \frac{2}{3\sqrt{3}} \approx .0595442$. If we apply Škerlik's result [24] to (4.6), we have

$$\int_{t_0}^{\infty} \left(s^2 q(s) - \frac{2}{3\sqrt{3}s} \right) \Delta s = \int_1^{\infty} \left(s^2 \frac{3}{3\sqrt{3}s^3} - \frac{2}{3\sqrt{3}s} \right) \Delta s \approx \int_1^{\infty} \frac{0.19245}{s} \Delta s = \infty.$$

Finally, note that the roots of the characteristic equation for (4.6) are $m = 1.6073 \pm 0.32634i$ and $m = -0.21463$.

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