Dynamic Systems and Applications 21 (2012) 607-630

A CLASS OF HIGHER ORDER STOCHASTIC DIFFERENTIAL EQUATIONS

JEAN-CLAUDE PEDJEU AND G. S. LADDE

Department of Mathematics & Statistics, University of South Florida
Tampa, Florida, USA

ABSTRACT. This work is devoted to the study of higher order stochastic differential equations (HOSDE). The variation of constant parameter technique is utilized to develop a method for finding closed form solution processes of classes of HOSDE. The probability distribution of the solution processes in the context of the second order equations is discussed.

AMS (MOS) Subject Classification. Primary 60H35, Secondary 60H10.

1. INTRODUCTION

In real life, mathematical models of several dynamic random processes are influenced by not only their state but also rates of change of states leading to higher order linear homogeneous stochastic differential equations. In this paper we are interested in finding exact or closed form solution processes to such equations. In Section 2, we formulate the problem, and present a few basic preliminary results. In Section 3, we develop a method of finding exact solutions of higher order (order $n \geq 2$) stochastic linear differential equations with constant coefficients. These solutions are classified in Section 4 based on nature of the roots of the characteristic polynomial of associated with the higher order deterministic differential equation corresponding to the stochastic differential. We also illustrate the method developed in Sections 3 and 4 with $n = 2$ and provide some useful examples. Finally, Section 5 provides ideas about finding the probability distribution of the solution processes in the context of second order $(n = 2)$ equation.

2. PROBLEM FORMULATION

Since the introduction of Itô-Doob calculus, the modeling and study of random dynamic phenomena have been very impressive, leading to the development of fundamental results for linear and nonlinear stochastic differential equations and their applications including science, Engineering, and finance [1, 3, 11, 12, 16]. Although

Received June 22, 2012

most of the studies are about the linear, nonlinear, and systems of stochastic differential equations, very limited work on higher order stochastic differential equations with multiplicative noise is available. A treatment of higher order ordinary deterministic and stochastic differential equations can be found in Ladde et al. in [10, 11]. This paper is devoted to the study of the following higher order stochastic differential equations with constant coefficients of the form

$$dy^{(n-1)} + \sum_{i=0}^{n-1} a_i y^{(i)} dt + \sum_{i=0}^{n-1} \sigma_j y^{(j)} dw = 0, \tag{2.1}$$

where $n \in \mathbb{N}$, $n \geq 2$, a_i, and σ_j are constants, $i, j = 0, \ldots n-1$, and w is a normalized Wiener process.

Our goal is to develop a method of finding closed form solutions of (2.1). In doing so, we will be interested in investigating conditions under which the exact solutions of such equations are feasible. For this purpose, we set $x_1(t) = y(t)$, $x_{i+1}(t) = \dot{x}_i(t)$, for $i = 1, 2, 3, \ldots, n-1$, and write $x(t) = [x_1(t), \ldots, x_n(t)]^T$. Under these considerations, equation (2.1) can be rewritten as a stochastic system of differential equations (SSDE).

$$dx = Ax\, dt + Bx\, dw(t) \tag{2.2}$$

where matrices A (the companion matrix associated with $dy^{(n-1)} + \sum_{i=0}^{n-1} a_i y^{(i)} dt = 0$) and B (stochastic perturbations) are defined by:

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & \ldots & \ldots & 0 & 1 \\ -a_0 & -a_1 & \ldots & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix} \tag{2.3}$$

and

$$B = \begin{pmatrix} 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & 0 & 0 \\ -\sigma_0 & -\sigma_1 & \ldots & \ldots & -\sigma_{n-2} & -\sigma_{n-1} \end{pmatrix}, \tag{2.4}$$

respectively.

Definition 2.1. Let $T > 0$ and $[t_0, t_0 + T] = J \subseteq \mathbb{R}$. A solution of the n-th ($n \geq 2$) order linear stochastic differential equation of type (2.1) is a stochastic process $y = y(t, w)$ defined on J, whose sample paths are is $(n-1)$ times continuously differentiable, and it satisfies (2.1) in the sense of Itô-Doob calculus.

In the following, we present a well-known result [10] that is useful for finding a solution process of the deterministic system of differential equations (2.2):

$$dx = Ax\,dt \tag{2.5}$$

where A is $n \times n$ companion matrix defined in (2.3)

Proposition 2.2. *Let $\Lambda(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda^1 + a_0$. Then for any number λ,*

$$A\begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} - \Lambda(\lambda) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.6}$$

Proof. Detailed proof is provided in [10, 11].

The following result establishes the fact that the solution processes of (2.1) and (2.2) are equivalent. In fact, a stochastic process is solution of (2.1) if and only if, it is a solution of (2.2). □

Theorem 2.3. *Let $y(t)$ and $x(t) = [x_1(t), \dots, x_n(t)]^T$ be any solutions of* (2.1) *and* (2.2)*, respectively. Then,*

(a) $[x_y(t)]^T = [y(t), y^{(1)}(t), \dots, y^{(n-1)}(t)]$ *is solution process of* (2.2)*,*

(b) $dx_1^{(n-1)} + \sum_{i=0}^{n-1} a_i x_1^{(i)} dt + \sum_{j=0}^{n-1} \sigma_j x_1^{(j)} dw = 0,$

where $x_1(t)$ is the first component of the solution process $x(t)$ of (2.2).

The proof is straightforward from (2.1) and (2.2). A method finding closed form solution processes of (2.1) is presented in the following section.

3. METHOD OF SOLVING HIGHER ORDER ITÔ-DOOD TYPE HOMOGENEOUS STOCHASTIC DIFFERENTIAL EQUATIONS

In this section, we utilize the eigenvalue type method [10, 11] to find solutions of the higher order linear homogeneous stochastic differential equations (2.1) with constant coefficients. The procedure is a modification of the method of solving deterministic systems of differential equations (one time scale t) with two time-scales: t

and $w(t)$. By employing the procedure developed in [11], we decompose (2.2) into two time-scale components: deterministic component as defined in (2.5) and stochastic component as follows:

$$dx = Bxdw, \tag{3.1}$$

where B is $n \times n$ constant matrix defined in (2.4).

The next step consists of finding fundamental matrix solution processes $\Phi_d(t)$ and $\Phi_s(t)$ of (2.5) and (3.1), respectively. Then, create a candidate for the fundamental matrix solution process

$$\Phi(t) = \Phi_d(t)\Phi_s(t) \tag{3.2}$$

of (2.2), and test the correctness of the fundamental matrix $\Phi(t)$.

It is known [10, 11] that under the following assumptions:

(i) at least one of the matrices $\Phi_d(t)$ and $\Phi_s(t)$ is normalized fundamental matrix, and

(ii) $AB = BA$,

the matrix $\Phi(t)$ defined in (3.2) is the fundamental matrix solution process of (2.2). However, for the matrices A and B in (2.2), we have $AB \neq BA$, unless $B \equiv 0$, where 0 is the zero matrix. But, $B = 0$ if and only if, systems (2.2) and (2.5) are identical, i.e. the problem is reduced to deterministic system of differential equations. Therefore, in order to find the solution process of the non-trivial stochastic differential equations (2.1), we need to modify the above described procedure. For this purpose, in the sequel, we assume that $B \neq 0$. The procedure of finding solutions of (2.2) is based on the method of variation of constant parameters. For easy reference, we state and prove the following result which is a special case of the result in [11].

Theorem 3.1 (Method of Variation of Constant Parameter)**.** *Let us assume that*

(H_1) $\Phi_d(t)$ *is the fundamental matrix solution of* (2.5) *, and*

(H_2) let $x(t) = \Phi_d(t)c(t)$*, where* $c(t)$ *is an* n*-dimensional unknown vector function.*

Then $x(t)$ *is a solution process of* (2.2) *if, and only if,* $c(t)$ *is a solution process of the stochastic system of linear differential equations with time-varying coefficients*

$$dc = \Phi_d^{-1}(t)B\Phi_d(t)c\,dw(t) \tag{3.3}$$

Proof. From hypotheses (H_1) and (H_2), we have

$$\begin{aligned} dx(t) &= d(\Phi_d(t)c(t)) \\ &= d\Phi_d(t)c(t) + \Phi_d(t)dc(t) \\ &= A\Phi_d(t)dtc(t) + \Phi_d(t)dc(t) \quad \text{(by using the assumption on } \Phi_d(t)) \\ &= A\Phi_d(t)c(t)dt + \Phi_d(t)dc(t) \qquad (3.4) \end{aligned}$$

Now, assume that $\boldsymbol{x}(t) = \Phi_d(t)c(t)$ is a solution of (2.2). Then, it satisfies

$$
\begin{aligned}
dx &= Ax\,dt + Bx\,dw(t) \quad \text{(from (2.2))} \\
&= A\Phi_d(t)c(t)dt + B\Phi_d(t)c(t)dw(t) \quad \text{(from the hypothesis } (H_2)\text{)}
\end{aligned}
\tag{3.5}
$$

Equating the right hand sides of (3.4) to the right hand side of (3.5) leads to

$$A\Phi_d(t)c(t)dt + \Phi_d(t)dc(t) = A\Phi_d(t)c(t)dt + B\Phi_d(t)c(t)dw(t) \tag{3.6}$$

which yields

$$\Phi_d(t)dc(t) = B\Phi_d(t)c(t)dw(t). \tag{3.7}$$

Hence applying $\Phi_d^{-1}(t)$ to both sides of (3.7), we obtain the equation (3.3). Conversely, let us suppose that $c(t)$ is solution of (3.3). Then, replacing $dc(t)$ with $\Phi_d^{-1}(t)B\Phi_d(t)c\,dw(t)$ in (3.4) shows that $x(t) = \Phi_d(t)c(t)$ is solution process of (2.2). □

Remark 3.2. (i) From Theorems 2.3 and 3.1, we note that finding a general solution process of stochastic system (3.3) is key to developing general solution processes of (2.2).

(ii) To find closed form or exact solution processes of linear system of time varying coefficient matrix (3.3), we need to examine the algebraic structure of matrix $\Phi_d^{-1}(t)B\Phi_d(t)$.

For the sake of examining the algebraic structure of matrix $\Phi_d^{-1}(t)B\Phi_d(t)$, let's denote by B_j^r the j-th row of matrix B in (2.4) for each $j = 1, 2, \ldots, n$. We observe that B_j^r is the zero vector for $j = 1, 2, \ldots, n-1$. The matrix $B\Phi_d(t)$ can be rewritten as

$$
B\Phi_d(t) = \begin{bmatrix} B_1^r \\ \vdots \\ B_j^r \\ \vdots \\ B_n^r \end{bmatrix} [\Phi_{d1}^c(t) \ldots \Phi_{dk}^c(t) \ldots \Phi_{dn}^c(t)]
$$

i.e

$$
B\Phi_d(t) = \begin{bmatrix}
B_1^r\Phi_{d1}^c(t) & \ldots & B_1^r\Phi_{dk}^c(t) & \ldots & B_1^r\Phi_{dn}^c(t) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
B_j^r\Phi_{d1}^c(t) & \ldots & B_j^r\Phi_{dk}^c(t) & \ldots & B_j^r\Phi_{dn}^c(t) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
B_n^r\Phi_{d1}^c(t) & \ldots & B_n^r\Phi_{dk}^c(t) & \ldots & B_n^r\Phi_{dn}^c(t)
\end{bmatrix}
$$

$$
(3.8) \qquad = \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 0 \\ B_n^r \Phi_{d1}^c(t) & \dots & B_n^r \Phi_{dk}^c(t) & \dots & B_n^r \Phi_{dn}^c(t) \end{bmatrix},
$$

where $\Phi_{dk}^c(t)$ denotes the k-th column vector of matrix $\Phi_d(t)$.

Now, by denoting $\Phi_d^{-1}(t) = (\phi_{ij})_{n\times n}$ and using (3.8), the matrix $M = \Phi_d^{-1}(t)B\Phi_d(t) = \left([\Phi_d^{-1}(t)B\Phi_d(t)]_{ij}\right)_{n\times n}$ is written as:

$$
(3.9) \quad M = \begin{bmatrix} \phi_{1n}(t)B_n^r\Phi_{d1}^c(t) & \dots & \phi_{1n}(t)B_n^r\Phi_{dk}^c(t) & \dots & \phi_{1n}(t)B_n^r\Phi_{dn}^c(t) \\ \vdots & \dots & \vdots & \vdots & \vdots \\ \phi_{in}(t)B_n^r\Phi_{d1}^c(t) & \dots & \phi_{in}(t)B_n^r\Phi_{dk}^c(t) & \dots & \phi_{in}(t)B_n^r\Phi_{dn}^c(t) \\ \vdots & \dots & \vdots & \vdots & \vdots \\ \phi_{nn}(t)B_n^r\Phi_{d1}^c(t) & \dots & \phi_{nn}(t)B_n^r\Phi_{dk}^c(t) & \dots & \phi_{nn}(t)B_n^r\Phi_{dn}^c(t) \end{bmatrix}.
$$

From (3.9), we are ready to state and prove the following result. The result provides a tool for the classifications of nth order solvable linear Itô-Doob type stochastic differential equations with constant coefficients.

Lemma 3.3. *Let the hypotheses of Theorem* 3.1 *be satisfied. Then, all but one column vectors of matrix* $\Phi_d^{-1}(t)B\Phi_d(t)$ *in* (3.3) *are zeroes if and only if for any given* $1 \leq k \leq n$,

$$
(3.10) \quad B_n^r\Phi_{dk}^c(t) \neq 0 \quad \text{and} \quad B_n^r\Phi_{dj}^c(t) = 0 \qquad \text{for all } j \neq k, j = 1, 2, \dots, n
$$

Proof. The validity of the necessary condition follows from

$$
(3.11) \qquad \Phi_d^{-1}(t)B\Phi_d(t) = \begin{bmatrix} 0 & \dots & 0 & \phi_{1n}(t)B_n^r\Phi_{dk}^c(t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \phi_{in}(t)B_n^r\Phi_{dk}^c(t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \phi_{nn}(t)B_n^r\Phi_{dk}^c(t) & 0 & \dots & 0 \end{bmatrix},
$$

for any given $k = 1, 2, \dots, n$.

For the sufficient condition, we note that if (3.10) is true for any $j = 1, 2, \dots, n$ and $j \neq k$ for $1 \leq k \leq n$, the entire j-th column is zero because $B_n^r\Phi_{dj}^c(t)$ is a factor of each entry of the j-th column. This complete the proof. □

The following lemma provides some important information on condition (3.10).

Lemma 3.4. *Condition* (3.10) *is equivalent to the following:*

(a) $\sum_{i=1}^{n}\sigma_{i-1}\lambda_j^{i-1} = 0$ *for all* $j = 1, 2, \dots, n$, $j \neq k$, *and* $\sum_{i=1}^{n}\sigma_{i-1}\lambda_k^{i-1} \neq 0$

(b) $\Sigma^T\Lambda_j = 0$ *for all* $j = 1, 2, \dots, n$, $j \neq k$ *and* $\Sigma^T\Lambda_k \neq 0$ *where* $\Sigma^T = [\sigma_0, \sigma_1, \dots, \sigma_{n-1}]$ *and* $\Lambda_j^T = [1, \lambda_j, \lambda_j^2, \dots, \lambda_j^{n-1}]$

(c) *For all $j \neq k$, eigenvectors Λ_j corresponding to λ_j are orthogonal to random environmental parameter vector $\sum$ and it belongs to the span of the eigenvector Λ_k.*

The proof is straightforward from (3.10).

Remark 3.5. The matrix $\Phi_d^{-1}(t)B\Phi_d(t)$ is a function of matrices B and $\Phi_d(t)$, and the latter depends of the eigenvalues of A. Therefore, the algebraic condition (3.10) depends on the coefficients of matrices A and B. Moreover, closed form solution processes of a solvable n-th order linear Itô-Doob type stochastic differential equations are classified into n classes.

In the following, we present the main result. It deals with a general procedure of finding closed form solution process of (2.2).

Theorem 3.6. *Let the hypotheses of Lemma* 3.3 *be satisfied. Then,*

(a) *the higher order stochastic differential equation* (2.1) *is solvable;*

(b) *n solutions of* (2.1) *are represented by:*

$$\begin{cases} y_j(t) = \psi_{1j}(t), \quad \textit{for} \;\; j \neq k, j, k = 1, 2, \ldots, n, \\ y_k(t) = \sum\limits_{j \neq k}^{n} \psi_{1j}(t) \int_0^t \phi_{jn}(s) B_n^r \Phi_{dk}^c(s) \exp[\nu_k(s, w(s)) dw(s)] \\ \qquad\qquad + \psi_{1k}(t) \exp[\nu_k(t, w(t))], \quad \textit{for} \;\; j = k, \end{cases} \tag{3.12}$$

where $\psi_{1j}(t)$ is an entry of the matrix $\Phi_d(t) = (\psi_{ij}(t))_{n \times n}$ and

$$\nu_k(t, w(t)) = -\frac{1}{2} \int_0^t \left[\phi_{kn}(s) B_n^r \Phi_{dk}^c(s)\right]^2 ds + \int_0^t \phi_{kn}(s) B_n^r \Phi_{dk}^c(s) dw(s)$$

(c) *a closed form general solution of* (2.1) *is*

$$y(t) = \sum_{j=1}^{n} c_j y_j(t) \tag{3.13}$$

where c_j's are arbitrary constants with $c_k \neq 0$.

Proof. From (3.3) and (3.12), we have

$$dc_j = \phi_{jn}(t) B_n^r \Phi_{dk}^c(t) c_k dw(t), \quad \text{for} \;\; j \neq k, j, k = 1, 2, \ldots, n, \tag{3.14}$$

$$dc_k = \phi_{kn}(t) B_n^r \Phi_{dk}^c(t) c_k dw(t), \quad \text{for some} \;\; k = 1, 2, \ldots, n. \tag{3.15}$$

Solving (3.15) for c_k yields

$$c_k(t) = c_{k0} \exp[\nu_k(t, w(t))] = c_{k0} \rho_{kk}(t) \quad \text{(by notation)} \tag{3.16}$$

where c_{k0} is an arbitrary constant.

Substituting $c_k(t)$ in (3.16) into (3.14) and solving for c_j, we have

$$c_j(t) = c_{j0} + c_{k0} \int_0^t \phi_{jn}(s) B_n^r \Phi_{dk}^c(s) \exp[\nu_k(t, w(t))] dw(s)$$

$$= c_{j0} + c_{k0}\rho_{jk}(t) \quad \text{(by notation)} \tag{3.17}$$

where c_{j0}'s are arbitrary constants $j \neq k$, $j, k = 1, 2, \ldots, n$.

From (3.14), (3.14) and (3.16), the fundamental solution process of transformed system (3.3), denoted by $\Phi_T(t) \equiv \Phi_T(t, w(t))$, in the context of (3.11) is

$$\Phi_T(t) = \begin{bmatrix} 1 & 0 & \ldots & 0 & \varphi_{1k}(t) & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & \varphi_{2k}(t) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \varphi_{(k-1)k}(t) & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \varphi_{kk}(t) & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & \varphi_{(k+1)k}(t) & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \varphi_{nk}(t) & 0 & \ldots & 1 \end{bmatrix}, \tag{3.18}$$

where

$$\rho_{jk}(t) = \begin{cases} \int_0^t \phi_{jn}(s) B_n^r \Phi_{dk}^c(s) \exp[\nu_k(t, w(t))] dw(s), & \text{for } j \neq k; \\ \exp[\nu_k(t, w(t))], & \text{for } j = k \end{cases} \tag{3.19}$$

From (2.2), Lemma 3.3, (3.16)–(3.19), Theorem 3.1, we conclude that the following matrix $\Phi(t)$ is a fundamental matrix solution process of (2.2).

$$\begin{aligned} \Phi(t) &:= \Phi_d(t)\Phi_T(t) \\ &= \begin{bmatrix} \psi_{11}(t) & \ldots & \psi_{1(k-1)}(t) & \sum_{i=1}^{n}\psi_{1i}(t)\varphi_{ik}(t) & \psi_{1(k+1)}(t) & \ldots & \psi_{1n}(t) \\ \psi_{21}(t) & \ldots & \psi_{2(k-1)}(t) & \sum_{i=1}^{n}\psi_{2i}(t)\varphi_{ik}(t) & \psi_{2(k+1)}(t) & \ldots & \psi_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{j1}(t) & \ldots & \psi_{j(k-1)}(t) & \sum_{i=1}^{n}\psi_{ji}(t)\varphi_{ik}(t) & \psi_{j(k+1)}(t) & \ldots & \psi_{jn}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_{n1}(t) & \ldots & \psi_{n(k-1)}(t) & \sum_{i=1}^{n}\psi_{ni}(t)\varphi_{ik}(t) & \psi_{n(k+1)}(t) & \ldots & \psi_{nn}(t) \end{bmatrix}, \end{aligned} \tag{3.20}$$

where $\Phi_d(t) = (\psi_{ij}(t))_{n\times n}$ is a fundamental matrix solution of (2.5). Therefore, applying Theorem 2.3, we conclude that the general solution of the higher order stochastic differential equation (2.1) is

$$\begin{aligned} y(t) &= \sum_{j\neq k}^{n} c_j\psi_{1j}(t) + c_k\sum_{i=1}^{n}\psi_{1i}\varphi_{ik}(t) \\ &= \sum_{j=1}^{n} c_j y_j(t) \end{aligned} \tag{3.21}$$

where c_j's are arbitrary constants, for $j \neq k$, $y_j(t) = \psi_{1j}(t)$ and $y_k(t) = c_k\sum_{i=1}^{n}\psi_{1i}\varphi_{ik}(t)$.

This establishes (a), (b) and (c) and completes the proof of the theorem. □

Next, we utilize the procedure developed in this section to find closed form or exact solution processes of (2.2).

4. CLOSED FORM SOLUTION PROCEDURE

In applying the procedure developed in Section 3 for finding general solution processes of classes of (2.2) in the context of eigenvalues of A, we identify the following three cases: (i) distinct eigenvalues, (ii) repeated eigenvalues, and (iii) complex eigenvalues of the companion matrix A.

Let us consider the case where the companion matrix A has n distinct real eigenvalues.

Case 1: Matrix A has n distinct real eigenvalues. Here we assume that the companion matrix A has n distinct real eigenvalues (i.e. $m = n$) λ_i, $i = 1, 2, \ldots, n$. In this case, $n_j = 1$ and $f_{jk}(t) = f_j(t) = t^k e^{\lambda_j t} = e^{\lambda_j t}$ for all $j = 1, 2, \ldots, m(= n)$ since $k = 0$. Then, the fundamental matrix solution $\Phi_d(t)$ of (2.5) is given by

$$\Phi_d(t) = \left[e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \\ \vdots \\ \lambda_1^{n-2} \\ \lambda_1^{n-1} \end{bmatrix} \quad e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \\ \vdots \\ \lambda_2^{n-2} \\ \lambda_2^{n-1} \end{bmatrix} \ldots e^{\lambda_n t} \begin{bmatrix} 1 \\ \lambda_n \\ \vdots \\ \lambda_n^{n-2} \\ \lambda_n^{n-1} \end{bmatrix} \right]. \tag{4.1}$$

The following theorem provides the condition of existence of a k-th class ($k = 1, 2, \ldots, n$) of equation (2.2) when matrix A has distinct real eigenvalues.

Theorem 4.1. *Let the hypotheses of Theorem* 3.6 *be satisfied. Furthermore, assume that the companion matrix A in* (2.3) *has n distinct real eigenvalues and that condition* (a) *of Lemma* 3.4 *holds for any $k = 1, 2, 3, \ldots, n$. Then, there exist solution processes for the stochastic system of differential equations* (3.3).

Proof. Suppose that the condition in part (a) of Lemma 3.4 holds for some column k ($k = 1, 2, \ldots, n$). We note that $\phi_{kn} B_n^r \Phi_{dk}^c(t) = -\phi_{kn}(t) e^{\lambda_k t} \sum_{i=1}^n \sigma_{i-1} \lambda_k^{i-1}$. Then, (3.15) becomes

$$dc_k = -\phi_{kn}(t) e^{\lambda_k t} \sum_{i=1}^{n} \sigma_{i-1} \lambda_k^{i-1} c_k \, dw(t). \tag{4.2}$$

Solving this equation for c_k yields

$$c_k(t) = c_{k0} e^{\left[-\frac{1}{2} \int_0^t \left(\phi_{kn}(s) e^{\lambda_k s} \sum_{i=1}^n \sigma_{i-1} \lambda_k^{i-1} \right)^2 ds - \int_0^t \phi_{kn}(s) e^{\lambda_k s} \sum_{i=1}^n \sigma_{i-1} \lambda_k^{i-1} dw(s) \right]}. \tag{4.3}$$

and for each $j = 1, 2, \dots, k-1, k+1, \dots, n-1, n$,

$$c_j(t) = c_{j0} - \int_0^t \phi_{jn}(s) e^{\lambda_k s} \sum_{i=1}^{n} \sigma_{i-1} \lambda_k^{i-1} c_k(s) dw(s) \tag{4.4}$$

where c_{j0} is constant for $j = 1, 2, \dots, k-1, k+1, \dots, n$, and $c_k(t)$ is as in (4.3).

The general solution process of equation (2.1) in this case is therefore given in the form

$$\begin{aligned} y(t) &= \Phi^r_{d1}(t) c(t) \\ &= c_k(t) e^{\lambda_k t} + \sum_{j=1, j\neq k}^{n} \left(c_{j0} - \int_0^t \phi_{jn}(s) e^{\lambda_k s} \sum_{i=1}^{n} \sigma_{i-1} \lambda_k^{i-1} c_k(s) dw(s) \right) e^{\lambda_j t}, \end{aligned} \tag{4.5}$$

where $c_k(t)$ is as in (4.3). □

Illustration 1: Let us consider a general 2-nd order stochastic differential equations

$$d\dot{y} + a_1 \dot{y} dt + a_0 y\, dt + \sigma_1 \dot{y} dw(t) + \sigma_0 y\, dw(t) = 0, \tag{4.6}$$

where a_0, a_1, σ_1, and σ_0 are constants real numbers, and w is a Wiener process. If $a_1^2 - 4a_0 > 0$, the associated companion matrix A has distinct real eigenvalues $\lambda_1 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2}$ and $\lambda_2 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}$. Then, after computing the matrix $\Phi_d(t)$ and its inverse, the matrix $\Phi_d^{-1}(t) B \Phi_d(t)$ in (3.9) reduces to

$$\Phi_d^{-1}(t) B \Phi_d(t) = \begin{bmatrix} \frac{(\sigma_0 + \sigma_1 \lambda_1)}{\lambda_2 - \lambda_1} & \frac{(\sigma_0 + \sigma_1 \lambda_2) e^{-(\lambda_1 - \lambda_2)t}}{\lambda_2 - \lambda_1} \\ -\frac{(\sigma_0 + \sigma_1 \lambda_1) e^{(\lambda_1 - \lambda_2)t}}{\lambda_2 - \lambda_1} & -\frac{(\sigma_0 + \sigma_1 \lambda_2)}{\lambda_2 - \lambda_1} \end{bmatrix}. \tag{4.7}$$

Moreover, equation (3.3) becomes

$$dc = \begin{bmatrix} \frac{\sigma_0 + \sigma_1 \lambda_1}{(\lambda_2 - \lambda_1)} & \frac{\sigma_0 + \sigma_1 \lambda_2}{(\lambda_2 - \lambda_1)} e^{-(\lambda_1 - \lambda_2)t} \\ \frac{-(\sigma_0 + \sigma_1 \lambda_1)}{(\lambda_2 - \lambda_1)} e^{(\lambda_1 - \lambda_2)t} & \frac{-(\sigma_0 + \sigma_1 \lambda_2)}{(\lambda_2 - \lambda_1)} \end{bmatrix} c dw(t) \tag{4.8}$$

Our procedure the yields two conditions: $\sigma_0 + \sigma_1 \lambda_2 = 0$ and $\sigma_0 + \sigma_1 \lambda_1 = 0$. From the application of Theorem 3.6, the general solution of (4.6) corresponding to condition $\sigma_0 + \sigma_1 \lambda_1 = 0$ is:

$$\begin{aligned} y(t) &= \Phi^r_{d1} c(t) = c_1(t) e^{\lambda_1 t} + c_2(t) e^{\lambda_2 t} \\ &= c_{10} \exp\left[-\frac{\sigma_0 t}{\sigma_1} \right] + c_{20} \exp\left[\left(-a_1 + \frac{\sigma_0}{\sigma_1} - \frac{\sigma_1^2}{2} \right) t - \sigma_1 w(t) \right] \\ &\quad + c_{20} \sigma_1 \exp\left[-\frac{\sigma_0 t}{\sigma_1} \right] \int_0^t \exp\left[\left(-a_1 + \frac{2\sigma_0}{\sigma_1} - \frac{\sigma_1^2}{2} \right) s - \sigma_1 w(s) \right] dw(s) \\ &= c_{10} e^{-\sigma_0 t/\sigma_1} + c_{20} (\xi + \sigma_1^2/2) e^{-\sigma_0 t/\sigma_1} \int_0^t \exp\left[\xi s - \sigma_1 w(s) \right] ds, \end{aligned} \tag{4.9}$$

where $\xi = -a_1 + 2\sigma_0/\sigma_1 - \sigma_1^2/2$, and c_{10} and $c_{20} \neq 0$ are arbitrary constants.

Similarly, the general solution of (4.6) corresponding to condition $\sigma_0 + \sigma_1 \lambda_2 = 0$ is:

$$y(t) = c_1(t) e^{\lambda_1 t} + c_2(t) e^{\lambda_2 t}$$

$$= c_{20}e^{-\sigma_0 t/\sigma_1} + c_{10}(\xi + \sigma_1^2/2)e^{-\sigma_0 t/\sigma_1}\int_0^t \exp\left[\xi s - \sigma_1 w(s)\right] ds, \tag{4.10}$$

where $\xi = -a_1 + 2\sigma_0/\sigma_1 - \sigma_1^2/2$, and c_{20} and $c_{10} \neq 0$ are arbitrary constants.

Example 1: Condition $\sigma_0 + \sigma_1\lambda_1 = 0$.

Let us assume that $\sigma_0 = -\sigma_1\lambda_1$ with $\sigma_1 \neq 0$. Then equation (4.6) becomes

$$d\dot{y} + a_1\dot{y}\,dt + a_0 y\,dt + \sigma_1\dot{y}dw(t) - \sigma_1\lambda_1 y\,dw(t) = 0, \tag{4.11}$$

where $\lambda_1 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} = -\sigma_0/\sigma_1$. For instance if $a_0 = 2, a_1 = 3, \sigma_1 = -2$. The equation (4.6) becomes

$$d\dot{y} + 3\dot{y}\,dt + 2y\,dt - 2\dot{y}dw(t) - 4y\,dw(t) = 0, \tag{4.12}$$

$\lambda_1 = -2$, $\lambda_2 = -a_1 - \lambda_1 = -1$. Under these conditions, the general solution process of the equation (4.12) is given by

$$\begin{aligned} y(t) &= c_1 e^{-2t} + c_2 e^{(-3t+2w(t))} - 2c_2 e^{-2t}\int_0^t e^{(-s+2w(s))}dw(s) \\ &= c_1 e^{-2t} + c_2 e^{-2t}\int_0^t e^{(-s+2w(s))}ds \end{aligned} \tag{4.13}$$

where c_1 and $c_2 \neq 0$ are arbitrary constants. One sample path is displayed in Figure 1.

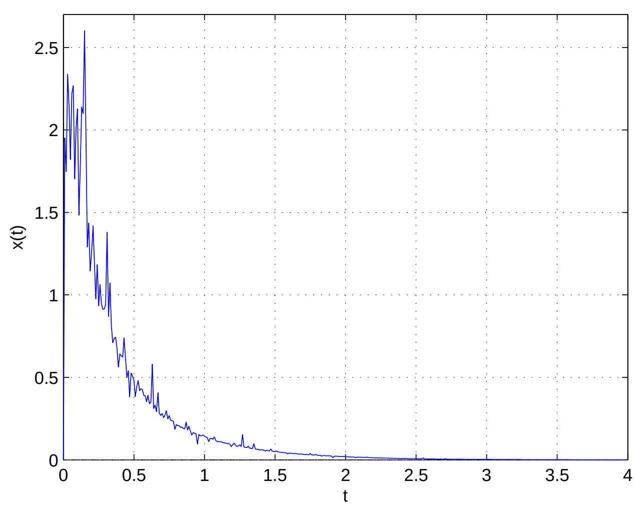

FIGURE 1. Plot of a sample path of the solution process in example 1

The mean and variance of the $y(t)$ are

$$E[y(t)] = c_1 e^{-2t} + c_2 e^{-t}$$

and

$$\mathrm{Var}(y(t)) = 5c_2^2 e^{2t} - 2c_2^2 e^{-4t} - c_2^2 e^{-2t},$$

respectively.

Example 2: Condition $\sigma_0 + \sigma_1\lambda_2 = 0$.

Let us consider the case where $\sigma_0 + \sigma_1\lambda_2 = 0$. Then equation (4.6) becomes

$$d\dot{y} + a_1\dot{y}\,dt + a_0 y\,dt + \sigma_1\dot{y}\,dw(t) - \sigma_1\lambda_2 y\,dw(t) = 0, \tag{4.14}$$

For example, taking $a_0 = 6$, $a_1 = -5, \sigma_1 = -2$, we have $\lambda_2 = \frac{-a_1+\sqrt{a_1^2-4a_0}}{2} = 3$, $\lambda_1 = -a_1 - \lambda_2 = 2$. Then equation (4.14) reduces to

$$d\dot{y} - 5\dot{y}\,dt + 6y\,dt - 2\dot{y}\,dw + 4y\,dw(t)(t) = 0, \tag{4.15}$$

Therefore, by applying the result obtained in (4.10), the general solution process of (4.15) is

$$y(t) = c_1 e^{-3t+2w(t)} + c_2 e^{3t} - 2c_1 e^{3t}\int_0^t e^{(-3s+2w(s))}dw(s) \tag{4.16}$$

$$= c_1 e^{3t} + c_2 e^{3t}\int_0^t e^{(-3s+2w(s))}ds \tag{4.17}$$

where c_1 and $c_2 \neq 0$ are arbitrary constants. One sample path is displayed in Figure 2.

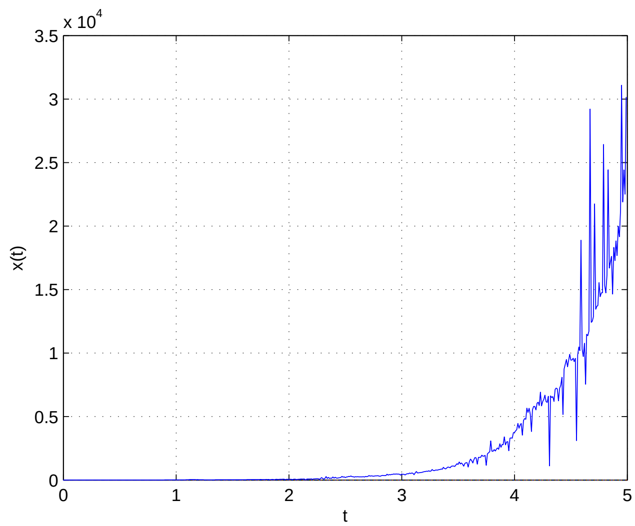

FIGURE 2. Plot of sample a path of the solution process in example 2

The mean and variance of the $y(t)$ are

$$E[y(t)] = c_1 e^{2t} + c_2 e^{3t}$$

and

$$\mathrm{Var}(y(t)) = \frac{1}{3}c_1^2[5e^{10t} + c_1^2 e^{4t} - 3e^{6t}],$$

respectively.

Next, we investigate the solution procedure when the companion matrix A has some eigenvalue with multiplicity greater than one.

Case 2: Matrix A has Repeated real eigenvalues without loss of generality, we assume that the companion matrix A has an eigenvalue λ of multiplicity m, $1 < m \leq n$ (for simplicity, we take $\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_m$). Furthermore, assume that the remaining $n-m$ eigenvalues are distinct and real, and we denote them by $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n$.

We recall here some useful results in the theory of deterministic higher order differential equations.

Proposition 4.2 ([2]). *For any nth-order linear homogeneous differential equation of the form*

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y \;=\; 0 \tag{4.18}$$

whose characteristic equation has a repeated root r of multiplicity k, the linearly independent solutions of the equation corresponding to r are

$$e^{rt}, te^{rt}, t^2e^{rt}, \ldots, t^{k-1}e^{rt}$$

Now, suppose the characteristic equation of (4.18) has m distinct roots r_1 (multiplicity n_1), r_2 (multiplicity n_2), $\ldots, r_m$ (multiplicity n_m), where $n_1+n_2+\cdots+n_m = n$ ($mj \geq 1$ for all $j = 1, 2, \ldots, m$. Then the general solution of (4.18) is

$$y(t) = \left[f_{10}(t)f_{11}(t)\ldots f_{1(n_1-1)}(t)\ldots f_{m0}(t)\ldots f_{m(n_m-1)}(t)\right]\left[c_1\ldots c_n\right]^T, \tag{4.19}$$

where $[c_1c_2\ldots c_n]^T$ is an n-dimensional constant vector, $f_{jk}(t) = t^k e^{\lambda_j t}$, $1 \leq j \leq m$ and $0 \leq k \leq n_j - 1$. Furthermore, from (2.1), (4.18) and (4.19), the fundamental matrix solution of the deterministic system of differential equations (2.5) is then given by

$$\Phi_d(t) = \left[\begin{bmatrix} f_{10}(t) \\ f'_{10}(t) \\ \vdots \\ f_{10}^{(n-2)}(t) \\ f_{10}^{(n-1)}(t) \end{bmatrix} \cdots \begin{bmatrix} f_{1(n_1-1)}(t) \\ f'_{1(n_1-1)}(t) \\ \vdots \\ f_{1(n_1-1)}^{(n-2)}(t) \\ f_{1(n_1-1)}^{(n-1)}(t) \end{bmatrix} \cdots \begin{bmatrix} f_{m0}(t) \\ f'_{m0}(t) \\ \vdots \\ f_{m0}^{(n-2)}(t) \\ f_{m0}^{(n-1)}(t) \end{bmatrix} \cdots \begin{bmatrix} f_{m(n_m-1)}(t) \\ f'_{m(n_m-1)}(t) \\ \vdots \\ f_{m(n_m-1)}^{(n-2)}(t) \\ f_{m(n_m-1)}^{(n-1)}(t) \end{bmatrix}\right].$$

We begin by discussing the case when $m = 2$. In this case the fundamental matrix solution $\Phi_d(t)$ of (2.5) is as follows:

$$\Phi_d(t) = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & e^{\lambda_3 t} & \ldots & e^{\lambda_(n-1)t} \\ \lambda_1 e^{\lambda_1 t} & e^{\lambda_1 t} + \lambda_1 te^{\lambda_1 t} & \lambda_3 e^{\lambda_3 t} & \ldots & \lambda_{n-1}e^{\lambda_{n-1}t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-2}e^{\lambda_1 t} & (n-2)\lambda_1^{n-3}e^{\lambda_1 t} + \lambda_1^{n-2}te^{\lambda_1 t} & \lambda_3^{n-2}e^{\lambda_3 t} & \ldots & \lambda_{n-1}^{n-2}e^{\lambda_{n-1}t} \\ \lambda_1^{n-1}e^{\lambda_1 t} & (n-1)\lambda_1^{n-2}e^{\lambda_1 t} + \lambda_1^{n-1}te^{\lambda_1 t} & \lambda_3^{n-1}e^{\lambda_3 t} & \ldots & \lambda_{n-1}^{n-1}e^{\lambda_{n-1}t} \end{bmatrix}. \tag{4.20}$$

And, by using the results in equations (3.9), we obtain

$$B_n^r\Phi_{dj}^c(t) = -\sum_{i=1}^{n}\sigma_{i-1}\lambda_j^{i-1}e^{\lambda_1 t}, \quad \text{for } j = 1, 3, 4, \ldots, n, \tag{4.21}$$

and

$$B_n^r\Phi_{d2}^c(t) = -\sum_{i=2}^{n}(i-1)\sigma_{i-1}\lambda_1^{i-2}e^{\lambda_1 t} - \sum_{i=1}^{n}\sigma_{i-1}\lambda_1^{i-1}te^{\lambda_1 t} \tag{4.22}$$

$$= -\sum_{i=2}^{n}(i-1)\sigma_{i-1}\lambda_1^{i-2}e^{\lambda_1 t} + tB_n^r\Phi_{d1}^c(t) \tag{4.23}$$

We note that (3.10) of Lemma 3.3 holds only for $k = 2$. This means that, in the case of exactly one repeated real eigenvalue of multiplicity 2 of companion matrix A, the technique utilized to find solution processes of classes of equation (2.1) yields solutions only for the 2nd class equation.

Theorem 4.3. *Let the hypotheses of Theorem* 3.6 *be satisfied. Furthermore, assume that the companion matrix A in* (2.3) *has $n-1$ distinct real eigenvalues, one of them, say without loss of generality λ_1, with multiplicity* 2*. Then, the stochastic system of differential equations* (3.3) *has at least one solution process, provided that*

$$\sum_{i=1}^{n}\sigma_{i-1}\lambda_j^{i-1} = 0 \qquad \textit{for all } j = 1,\, 3, 4, \ldots, n. \tag{4.24}$$

Proof. Assume that the hypotheses of Theorem 4.3 hold and that condition (4.24) is satisfied. Then equation (3.3) reduces to

$$\begin{aligned} dc_j &= \phi_{jn}(t)B_n^r\Phi_{d2}^c(t)c_2\,dw(t) \qquad \text{for } j = 1,\, 2\,, \ldots,\, n,\\ &= -\phi_{jn}(t)e^{\lambda_1 t}\left(\sum_{i=2}^{n}(i-1)\sigma_{i-1}\lambda_1^{i-2} + \sum_{i=1}^{n}\sigma_{i-1}\lambda_1^{i-1}t\right)c_2\,dw(t), \end{aligned} \tag{4.25}$$

and for $j = 2$ we solve the following stochastic differential equation:

$$dc_2 = -\phi_{2n}(t)e^{\lambda_1 t}\left(\sum_{i=2}^{n}(i-1)\sigma_{i-1}\lambda_1^{i-2} + \sum_{i=1}^{n}\sigma_{i-1}\lambda_1^{i-1}t\right)c_2\,dw(t). \tag{4.26}$$

Let's define

$$G_j(t) = -\phi_{jn}(t)e^{\lambda_1 t}\left(\sum_{i=2}^{n}(i-1)\sigma_{i-1}\lambda_1^{i-2} + \sum_{i=1}^{n}\sigma_{i-1}\lambda_1^{i-1}t\right) \qquad \text{for all } j = 1, 2, \ldots, n.$$

Then the solution of equation (4.26) is given by

$$c_2(t) = c_{20}\exp\left(-\frac{1}{2}\int_0^t G_2^2(s)ds + \int_0^t G_2(s)dw(s)\right). \tag{4.27}$$

By using this solution, we obtain the solution process of (4.25) as

$$c_j(t) = c_{j0} + \int_0^t G_j(s)c_2(s)dw(s) \text{ for all } j = 1, 3, 4, \ldots, n. \tag{4.28}$$

Then, the general solution process of (2.1) is given by

$$
\begin{aligned}
y(t) &= \Phi_{d1}^r(t)c(t) \\
&= [c_1(t) + tc_2(t)]e^{\lambda_1 t} + \sum_{j=3}^{n} e^{\lambda_j t} c_j(t).
\end{aligned}
\tag{4.29}
$$

where $c_j(t)$ are provided in (4.28).

Now, let's consider the situation where $m > 2$. In this case, a fundamental matrix solution $\Phi_d(t)$ of (2.5) is as follows:

$$
\Phi_d(t) = \left[\begin{bmatrix} f_1(t) \\ f_1^{(1)}(t) \\ \vdots \\ f_1^{(n-2)}(t) \\ f_1^{(n-1)}(t) \end{bmatrix} \cdots \begin{bmatrix} f_m(t) \\ f_m^{(1)}(t) \\ \vdots \\ f_m^{(n-2)}(t) \\ f_m^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} f_{m+1}(t) \\ f_{m+1}^{(1)}(t) \\ \vdots \\ f_{m+1}^{(n-2)}(t) \\ f_{m+1}^{(n-1)}(t) \end{bmatrix} \cdots \begin{bmatrix} f_n(t) \\ f_n^{(1)}(t) \\ \vdots \\ f_n^{(n-2)}(t) \\ f_n^{(n-1)}(t) \end{bmatrix} \right],
\tag{4.30}
$$

where $f_p(t) = f_p(\lambda_1, t) := t^{p-1}e^{\lambda_1 t}$ and

$$
f_p^{(k)}(t) = \frac{d^k}{dt^k} f_p(t) = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \frac{d^j}{dt^j}[t^{p-1}] \frac{d^{k-j}}{dt^{k-j}}[e^{\lambda_1 t}],
\tag{4.31}
$$

for $p = 1, 2, \ldots, m$ and $k = 0, 1, 2, \ldots, n-1$. In this case, we have

$$
B_n^r \Phi_{dj}^c(t) = -\sum_{i=1}^{n} \sigma_{i-1} f_j^{(i-1)}(t)
\tag{4.32}
$$

for $j = 2, \ldots, m$, and

$$
B_n^r \Phi_{dj}^c(t) = -\sum_{i=1}^{n} \sigma_{i-1} \lambda_j^{i-1} e^{\lambda_j t} = -e^{\lambda_j t} \sum_{i=1}^{n} \sigma_{i-1} \lambda_j^{i-1},
\tag{4.33}
$$

for $j = 1, m+1, m+2, \ldots, n$. □

Remark 4.4. (i) From (4.32), we observe that $B_n^r \Phi_{dj}^c(t) \neq 0$ for any $j = 2, 3, \ldots, m$. Therefore condition (3.10) does not hold here.
(ii) Theorem 3.6 is not applicable since Lemma 3.3 failed to hold.

Illustration 2: Consider the second degree equation (4.6) introduced in the previous case. When $a_1^2 - 4a_0 = 0$, matrix A has one repeated eigenvalue $\lambda_1 = \lambda_2 = -a_1/2$. Under these conditions, we have

$$
\Phi_d^{-1}(t) B \Phi_d(t) = \begin{bmatrix} -\left(-\sigma_0 + \frac{a_1\sigma_1}{2}\right) t & \sigma_0 t^2 + \sigma_1 \left(1 - \frac{a_1 t}{2}\right) t \\ -\sigma_0 + \frac{a_1\sigma_1}{2} & -\sigma_0 t - \sigma_1 (1 - \frac{a_1 t}{2}) \end{bmatrix}.
\tag{4.34}
$$

Therefore,

$$
\begin{bmatrix} dc_1 \\ dc_2 \end{bmatrix} = \begin{bmatrix} -\left(-\sigma_0 + \frac{a_1\sigma_1}{2}\right) t & \sigma_1 t \\ -\sigma_0 + \frac{a_1\sigma_1}{2} & -\sigma_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} dw(t).
\tag{4.35}
$$

Our procedure the yields two conditions: $-\sigma_0+\frac{a_1\sigma_1}{2}=0$ and $\sigma_1=0$. If $-\sigma_0+\frac{a_1\sigma_1}{2}=0$ i.e. $\sigma_0=\frac{a_1\sigma_1}{2}$, (4.35) becomes

$$\begin{bmatrix} dc_1 \\ dc_2 \end{bmatrix} = \begin{bmatrix} 0 & \sigma_1 t \\ 0 & -\sigma_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} dw(t) = \begin{bmatrix} \sigma_1 t c_2\, dw(t) \\ -\sigma_1 c_2\, dw(t) \end{bmatrix}. \tag{4.36}$$

Solving $dc_2=-\sigma_1 c_2\, dw(t)$ yields

$$c_2(t) = c_{20}\exp\left(-\frac{1}{2}\sigma_1^2 t - \sigma_1 w(t)\right) \tag{4.37}$$

and substituting the quantity on the right hand side of (4.37) for $c_2(t)$ in $dc_1(t) = \sigma_1 t c_2\, dw$, and solving for $c_1(t)$ we obtain

$$c_1(t) = c_{10} + c_{20}\sigma_1 \int_0^t s\exp\left(-\frac{1}{2}\sigma_1^2 s - \sigma_1 w(s)\right) dw(s). \tag{4.38}$$

And the general solution process of (4.6) is given by

$$\begin{aligned} y(t) &= c_{20}\sigma_1 e^{-\frac{a_1}{2}t}\int_0^t s\exp\left(-\frac{1}{2}\sigma_1^2 s - \sigma_1 w(s)\right) dw(s) \\ &\quad + c_{20} t\exp\left((-a_1/2-\sigma_1^2/2)t - \sigma_1 w(t)\right) + c_{10}e^{-\frac{a_1}{2}t} \\ &= c_{10}e^{-\frac{a_1}{2}t} + c_{20}e^{-\frac{a_1}{2}t}\int_0^t \exp\left(-\frac{1}{2}\sigma_1^2 s - \sigma_1 w(s)\right) ds \end{aligned} \tag{4.39}$$

where c_{10} and $c_{20}\neq 0$ are arbitrary constants.

Next, we present an example with a simulated sample path that illustrates the case of repeated eigenvalue of matrix A.

Example 3: Let $a_0=1, a_1=-2, \sigma_1=2$, and $\sigma_0=-2$. The equation (4.6) becomes

$$d\dot{y} - 2\dot{y}dt + y\,dt + 2\dot{y}\,dw(t) - 2y\,dw(t) = 0, \tag{4.40}$$

$\lambda_1=\lambda_2=1$, and $-\sigma_0+\frac{a_1\sigma_1}{2} = 2+\frac{(-2)(2)}{2}=0$. Therefore, from (4.39) the general solution solution of (4.40) is

$$y(t) = c_1 e^t + c_2 e^t \int_0^t \exp[-2s - 2w(s)]ds \tag{4.41}$$

where c_1 and c_2 are arbitrary constants ($c_2\neq 0$). A sample path of this solution process is shown in Figure 3.

For obvious reasons, we emphasize that Theorem 3.6 is not applicable to condition $\sigma_1=0$. Moreover, the coefficient rate matrix $\Phi_d^{-1}(t)B\Phi_d(t)$ under this condition is a time-varying matrix, and its structure exhibits non-trivial coupled interactions with the components of state vector c. Because of this, one is not able to find the close form solution of this types of system. This, it is not feasible to find close form solution of (4.6) corresponding to this condition.

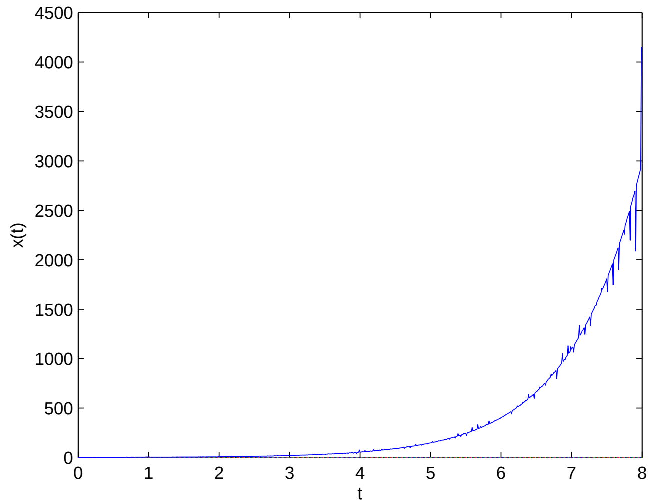

FIGURE 3. Plot of a sample path of the solution process of example 3

Remark 4.5. From (4.9), (4.10) and (4.39), we observe that the solution processes (when the companion matrix A has a repeated eigenvalue or when all its eigenvalues real and distinct) can be expressed in the following form:

$$y(t) \;=\; P(t) + Q(t)\int_0^t \exp\left[\xi s - \sigma_1 w(s)\right] ds, \tag{4.42}$$

where $P(t) = c_1 e^{-\sigma_0 t/\sigma_1}, Q(t) = c_2(\xi + \sigma_1^2/2)e^{-\sigma_0 t/\sigma_1}, \xi = -a_1 + 2\sigma_0/\sigma_1 - \sigma_1^2/2$, for distinct eigenvalues of A, and $P(t) = c_1 e^{-\frac{a_1}{2}t}, Q(t) = c_2 e^{-\frac{a_1}{2}t}, \xi = -\frac{1}{2}\sigma_1^2$ when A has a repeated eigenvalue; c_1 and $c_2 \neq 0$ are arbitrary constants,

In the next section, we examine the situation where the companion matrix A has complex eigenvalues.

Case 3: Matrix A has distinct complex eigenvalues. Without loss of generality, let us assume that A has at least one complex eigenvalue, say for simplicity, $\lambda = \lambda_1 = \alpha + i\beta$. Then, its conjugate $\overline{\lambda} = \lambda_2 = \alpha - i\beta$ is also an eigenvalue of A. Writing λ_1 in the polar form, we have $\lambda_1 = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{\alpha^2+\beta^2}$ and $\theta \in [0, 2\pi)$ is the angle of λ_1 in the polar coordinate system. Furthermore, for $k = 0, 1, 2, \dots, n-1$, $e^{\lambda_1 t}\lambda_1^k = r^k e^{\alpha t} e^{i(\beta t + k\theta)}$, $\lambda_2 = r(\cos\theta - i\sin\theta)$, and $e^{\lambda_2 t}\lambda_2^k = r^k e^{\alpha t} e^{i(-\beta t - k\theta)}$. Note that the real and imaginary parts of the complex solution process corresponding the eigenvalue $\lambda = \lambda_1$ (or $\lambda = \lambda_2$) are also solution processes of the same equation. Furthermore, those two solution processes are linearly independent. Therefore, it is convenient to replace the solution process corresponding to λ_1 with the real part of that solution and the solution process corresponding to λ_2 with the imaginary part, respectively. In addition, if we assume that the remaining $n-2$ eigenvalues of the companion matrix A, denoted λ_i, $i = 3, 4, \dots, n$, are real and distinct, then the fundamental matrix solution $\Phi_d(t)$ of (2.5) is given by

$$\Phi_d(t) = \begin{bmatrix} \mathrm{Re}\left(e^{\lambda_1 t}\right) & \mathrm{Im}\left(e^{\lambda_1 t}\right) & e^{\lambda_3 t} & \dots & e^{\lambda_n t} \\ \mathrm{Re}\left(\lambda_1 e^{\lambda_1 t}\right) & \mathrm{Im}\left(\lambda_1 e^{\lambda_1 t}\right) & \lambda_3 e^{\lambda_3 t} & \dots & \lambda_n e^{\lambda_n t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathrm{Re}\left(\lambda_1^{n-2} e^{\lambda_1 t}\right) & \mathrm{Im}\left(\lambda_1^{n-2} e^{\alpha_1 t}\right) & \lambda_3^{n-2} e^{\lambda_3 t} & \dots & \lambda_n^{n-2} e^{\lambda_n t} \\ \mathrm{Re}\left(\lambda_1^{n-1} e^{\lambda_1 t}\right) & \mathrm{Im}\left(\lambda_1^{n-1} e^{\lambda_1 t}\right) & \lambda_3^{n-1} e^{\lambda_3 t} & \dots & \lambda_n^{n-1} e^{\lambda_n t} \end{bmatrix}, \tag{4.43}$$

where for $k = 0, 1, 2, \dots, n-1$, $\mathrm{Re}\big(\lambda_1^k e^{\lambda_1 t}\big) = r^k e^{\alpha t}\cos(k\theta + \beta t)$ and $\mathrm{Im}\big(\lambda_1^k e^{\lambda_1 t}\big) = r^k e^{\alpha t}\sin(k\theta + \beta t)$ are the real and imaginary parts of the complex variable function $\lambda_1^k e^{\lambda_1 t}$. By using the results in equation (3.9), the entries of the matrix $\Phi_d^{-1} B \Phi_d$ are as follows:

for the first column,

$$\begin{aligned} \phi_{jn}(t) B_n^r \Phi_{d1}^c(t) &= -\phi_{jn}(t) \sum_{k=1}^{n} \sigma_{k-1} Re\left(\lambda_1^{k-1} e^{\lambda_1 t}\right) \\ &= -\phi_{jn}(t) e^{\alpha t} \sum_{k=1}^{n} \sigma_{k-1} r^{k-1} \cos[(k-1)\theta + \beta t], \quad j = 1, 2, ..., n \end{aligned} \tag{4.44}$$

for the second column,

$$\begin{aligned} \phi_{jn}(t) B_n^r \Phi_{d2}^c(t) &= -\phi_{jn}(t) \sum_{k=1}^{n} \sigma_{k-1} Im\left(\lambda_1^{k-1} e^{\lambda_1 t}\right) \\ &= -\phi_{jn}(t) e^{\alpha t} \sum_{k=1}^{n} \sigma_{k-1} r^{k-1} \sin[(k-1)\theta + \beta t], \quad j = 1, 2, ..., n \end{aligned} \tag{4.45}$$

and for the l-th column, $l = 3, 4, \dots, n$,

$$\phi_{jn}(t) B_n^r \Phi_{d1}^c(t) = -\phi_{jn}(t) e^{\lambda_l t} \sum_{k=1}^{n} \sigma_{k-1} \lambda_l^{k-1}, \text{ for } j = 1, 2, \dots, n, \tag{4.46}$$

Remark 4.6. (i) From (4.44) and (4.45), we note that Theorem 4.1 is not applicable in this case. The time-varying coefficient rate matrix $\Phi_d^{-1}(t) B_n^r \Phi_d(t)$ of the transformed system (3.3) in Theorem 3.1 cannot be reduced to (3.10). Thus, we cannot utilize the our method to find closed form solution for (2.1). To illustrate this, let's consider the second degree equation (4.6) given above and assume that $a_1^2 - 4a_0 < 0$ so that the companion matrix A has distinct complex solutions $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with $\beta \neq 0$, then

$$\Phi_d^{-1}(t) B \Phi_d(t) = \frac{-1}{\beta} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix},$$

where

$$\begin{aligned} \phi_{11} &= -\sin\beta t[\sigma_0 \cos\beta t + \sigma_1(\alpha\cos\beta t - \beta\sin\beta t)] \\ \phi_{12} &= -\sin\beta t[\sigma_0 \sin\beta t + \sigma_1(\beta\cos\beta t + \alpha\sin\beta t)] \end{aligned}$$

$$\phi_{21} = \cos\beta t[\sigma_0 \cos\beta t + \sigma_1(\alpha\cos\beta t - \beta\sin\beta t)]$$
$$\phi_{22} = \cos\beta t[\sigma_0 \sin\beta t + \sigma_1(\beta\cos\beta t + \alpha\sin\beta t)].$$

The matrix $\Phi_d^{-1}(t)B\Phi_d(t)$ is singular. Therefore, (3.3) does not have a unique solution.

(ii) As we shall see in the following example, there are cases where the matrix $\Phi_d^{-1}(t)B_n^r\Phi_d(t)$ has exactly one non null column vector.

Consider the 3rd order linear homogeneous Itô-Doob type stochastic differential equation

$$dy^{(2)} + [y^{(2)} + y^{(1)} + y]dt + [y^{(2)} + y]dw(t) = 0. \tag{4.47}$$

In the vector form, this equation is written as

$$dx = Ax\,dt + Bx\,dw(t) = 0, \tag{4.48}$$

where $x \in \mathbb{R}^3$; A and B are 3×3 deterministic and stochastic companion matrices in (2.5) and (3.1) relative to (4.48), and they are as:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}. \tag{4.49}$$

The fundamental matrix solution of the deterministic part of (4.48) is

$$\Phi_d(t) = \begin{bmatrix} \cos t & \sin t & e^{-t} \\ \sin t & \cos t & -e^{-t} \\ -\cos t & -\sin t & e^{-t} \end{bmatrix} \tag{4.50}$$

and its inverse is

$$\Phi_d^{-1}(t) = \frac{1}{2}\begin{bmatrix} \cos t - \sin t & -2\sin t & -(\cos t + \sin t) \\ \cos t + \sin t & 2\cos t & \cos t - \sin t \\ e^t & 0 & e^t \end{bmatrix}. \tag{4.51}$$

The stochastic system of differential equations (3.3) in the context of (4.48) is

$$dc = \Phi_d^{-1}(t)B\Phi_d(t)c\,dw(t) = \begin{bmatrix} 0 & 0 & (\cos t + \sin t)e^{-t} \\ 0 & 0 & -(\cos t - \sin t)e^{-t} \\ 0 & 0 & -1 \end{bmatrix} c\,dw(t). \tag{4.52}$$

Thus, the time varying coefficient matrix rate matrix (3.3) corresponding to the given stochastic differential equation satisfies the hypothesis of Lemma 3.3. Therefore, the application of Theorem 3.6, the condition (a) of Theorem 3.6 assures the feasibility of a closed form solution. Moreover, condition (c) of the same theorem provides the closed form representation of the general solution of the given differential equation. Thus, we have

$$y(t) = c_3\left[e^{-(3/2)t-w(t)} + \cos t\int_0^t (\cos s + \sin s)e^{-(3/2)s-w(s)}dw(s)\right. \tag{4.53}$$

$$-\sin t \int_0^t (\cos s - \sin s) e^{-(3/2)s - w(s)} dw(s) \Bigg] + c_1 \cos t + c_2 \sin t.$$

The following example is a modified version of Chandrasekhar equation. This version incorporates multiplicative noise rather than the additive noise as in the original version.

APPLICATIONS: A MODIFIED CHANDRASEKHAR EQUATION

The theory of the Brownian motion of a free particle (i.e., in the absence of an external field of force) generally starts with the Langevin's equation

$$\frac{du}{dt} = -\beta u + a(t), \tag{4.54}$$

where u denotes the velocity of the particle. According to this equation, the influence of the surrounding medium on the motion can be split up into two parts: first, a systematic part $-\beta u$ representing a dynamical friction experienced by the particle and second, a fluctuating part $a(t)$ which is characteristic of the Brownian motion. $a(t)$ is independent of u and varies extremely rapidly compared to the variations of u. Chandrasekhar [4] generalized the Langevin equation by considering the presence of an external field of force which leads to the following equation

$$\frac{du}{dt} = -\beta u + a(t) + K(r, t), \tag{4.55}$$

where $K(r, t)$ is the acceleration produced by the field. The method of solution is illustrated sufficiently by a one-dimensional harmonic oscillator describing Brownian motion.

$$\frac{du}{dt} = -\beta u + a(t) - \omega^2 t, \tag{4.56}$$

where ω is denotes the circular frequency of the oscillator. Alternatively, equation (4.55) can be written in the form

$$\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + \omega^2 y = a(t), \tag{4.57}$$

or as it is known today

$$d\dot{y} + (\beta\dot{y} + \nu y)dt = \sigma dw(t), \tag{4.58}$$

for $\sigma \neq 0$ and $\beta > 0$. If the system is also subject to (the influence of) external environmental random perturbations, the equation (4.58) becomes

$$d\dot{y} + (\beta_1\dot{y} + \beta_0 y)dt + (\sigma_1\dot{y} + \sigma_0 y)dw(t) = 0, \tag{4.59}$$

with $\beta_1, \beta_0, \sigma_1 > 0$. The companion matrices for this equation are

$$A = \begin{bmatrix} 0 & 1 \\ -\beta_0 & -\beta_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ -\sigma_0 & -\sigma_1 \end{bmatrix}. \tag{4.60}$$

The eigenvalues of matrix A are $\lambda_1 = \frac{-\beta_1 - \sqrt{\beta_1^2 - 4\beta_0}}{2}$ and $\lambda_2 = \frac{-\beta_1 + \sqrt{\beta_1^2 - 4\beta_0}}{2}$. If $\beta_1^2 > 4\beta_0$, these eigenvalues are real and distinct. In this case, the solution process of (4.59) for is of the form

$$y(t) = c_1 e^{-\sigma_0 t/\sigma_1} + c_2(\xi + \sigma_1^2/2) e^{-\sigma_0 t/\sigma_1} \int_0^t \exp\left[\xi s - \sigma_1 w(s)\right] ds, \tag{4.61}$$

where $\xi = -\beta_1 + \frac{2\sigma_0}{\sigma_1} - \frac{\sigma_1^2}{2}$, c_1 and $c_2 \neq 0$ are arbitrary constants. If $\beta_1^2 = 4\beta_0$ there is one repeated eigenvalue, namely $\lambda = -\beta_1/2$. In this case, equation (4.59) yields the following solution

$$y(t) = c_1 e^{-\beta_1 t/2} + c_2 e^{-\beta_1 t/2} \int_0^t \exp\left(-\frac{1}{2}\sigma_1^2 s - \sigma_1 w(s)\right) ds, \tag{4.62}$$

where c_1 and $c_2 \neq 0$ are arbitrary constants.

If $\beta_1^2 < 4\beta_0$ the companion matrix A has two complex eigenvalues. As discussed earlier, in this case our method for finding solution of (4.59) is not feasible.

Remark 4.7. From Remark 4.5, we note that when the companion matrix A has repeated or distinct real eigenvalues, the solution process of equation (4.6) is an isomorphic transformation of the exponential functional of Brownian motion

$$B_t^{(\mu)} = \int_0^t \exp(\sigma w(s) + \mu s) ds, \tag{4.63}$$

where $\sigma \neq 0$ and μ are arbitrary real constants. Therefore, to find probability distribution of $y(t)$, it is enough to obtain the one of $B_t^{(\mu)}, t \geq 0$.

The distribution of the process $\{B_t^{(\mu)}, t \geq 0\}$ has been subject of extensive research in the past two decades thanks to it applications in fields of mathematical finance, diffusion processes in random environments.

In the following, we examine the probability distribution of the solution processes $y(t)$ of the second order ($n = 2$) stochastic differential equation (4.6) in the context of repeated or distinct real eigenvalues of companion matrix A.

5. THE PROBABILITY DISTRIBUTION $y(t)$ WHEN $n = 2$

The probability distribution of the solution process $y(t)$ (for $n = 2$) is identical to the that of $A_t = \int_0^t \exp\left[\xi s - \sigma_1 w(s)\right] ds$. A_t is exponential functional of Brownian motion which has been of interest to researchers in Mathematical finance, diffusion processes in random environments, stochastic analysis related to Brownian motions on hyperbolic spaces. Various approaches were adopted to determine the law of A_t. One approach by Dufresne [7] focusses on the reciprocal of the integral A_t. A partial differential equation is derived for its Laplace transform and then used to derive an expression for the density. Another very popular approach derives the law of A_t thought

Bessel processes [14, 16]. Thanks to scaling properties of Brownian motion, to determine the law of A_t, it suffices to determine the law of $A_t^{(\nu)} = \int_0^t \exp\left[2(w(s)+\nu s)\right] ds$. The conversion rule between A_t and $A_t^{(\nu)}$ is given by (ref. [3], pp. 43)

$$\int_0^T \exp[\mu s + \sigma w(s)] ds \overset{\text{law}}{=} \frac{4}{\sigma^2} A_t^{(\nu)}, \quad t = \frac{\sigma^2 T}{4}, \quad \nu = \frac{2\mu}{\sigma^2}, \tag{5.1}$$

where $T > 0$ and $t \in [0, T]$.

The probability distribution of $A_t^{(\nu)}$, taken at a fixed time t (was obtained by Yor [15]) is determined by

$$P\left(A_t^{(\nu)} \in du | w(t) + \nu t = x\right) \overset{\text{def.}}{=} a_t(x,u) du. \tag{5.2}$$

In relation (5.2), $a_t(x,u)$ satisfies

$$\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) a_t(x,u) = \frac{1}{u} \exp\left(-\frac{1}{2u}(1+e^{2x})\right) \theta\left(\frac{e^x}{u}, t\right) \tag{5.3}$$

where

$$\theta(r,t) = \frac{r e^{\pi^2/2t}}{\sqrt{2\pi^3 t}} \int_0^\infty e^{-\xi^2/2t} e^{-r\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi \xi}{t}\right) d\xi \tag{5.4}$$

This means that the probability density function of $A_t^{(\nu)}$ is the integral of $a_t(\cdot, u)$ times the normal density function with mean νt and variance t. It follows [16] from (5.2) that

$$P\left(A_t^{(\nu)} \in du\right) = du \gamma_t^{(\nu)}(u), \tag{5.5}$$

where

$$\gamma_t^{(\nu)}(u) = \int_0^\infty \theta(z,t)(uz)^{\nu-1} \exp\left(\frac{-\nu^2 t}{2}\right) \exp\left[-\frac{1}{2}\left(\frac{1}{u} + uz^2\right)\right] dz \tag{5.6}$$

Given the complexity of the distribution of $A_t^{(\nu)}$, it is worth looking into its moments to gain some partial information about its distribution.

The Moments. Although the first two moments of $A_t^{(\nu)}$ can easily by obtained by integration, higher moments are not obtained that way. The higher moments were can be computed using Itô's Lemma, time reversal, and a recurrence argument were (Dufresne [5, 6]) or using Laplace transform (Geman and Yor [16], pp. 49-54). They obtained the formula

$$E\left[\left(\int_0^t \exp\left[\lambda(\nu s + w(s))\right] ds\right)^n\right] = \frac{n!}{\lambda^{2n}} \left\{\sum_{j=0}^n d_j^{(\nu/\lambda)} \exp\left[\left(\frac{\lambda^2 j^2}{2} + \lambda j \nu\right) t\right]\right\}, \tag{5.7}$$

where

$$d_j^{(\beta)} = 2^n \prod_{\substack{i \neq j \\ 0 \leq i \leq n}} \left[(\beta + j)^2 - (\beta + i)^2\right]^{-1}, \tag{5.8}$$

By applying this result to the solution process $\{y(t), t \geq 0\}$ in its formulation presented in Remark 4.5 in the case of distinct and repeated eigenvalues of matrix A, we have the following formula for the moments of $y(t)$:

$$
\begin{aligned}
E\left[(y(t))^n\right] &= \sum_{k=0}^{n} \binom{n}{k} P^{n-k}(t) Q^k(t) E\left[\left(\int_0^t \exp\left[\lambda(\nu s + w(s))\right] ds\right)^k\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} P^{n-k}(t) Q^k(t) \frac{k!}{\lambda^{2k}} \left\{\sum_{j=0}^{k} d_j^{(\nu/\lambda)} \exp\left[\left(\frac{\lambda^2 j^2}{2} + \lambda j \nu\right) t\right]\right\},
\end{aligned} \tag{5.9}
$$

where $d_j^{(\beta)}$ is defined in (5.8) and $\lambda = -\sigma_1$, $\nu\lambda = \xi$; $P(t)$, $Q(t)$ and ξ are defined in Remark 4.5.

6. CONCLUSION

In this paper, we provided a method of finding classes of solution processes of higher order stochastic differential equations (HOSDE). Although higher order deterministic and matrix differential equations are available in literature, we note that HOSDE remains an interesting problem. Our current research project includes applications of such equations and as well as the stability of their solution processes.

Acknowledgments. This research was supported by the Mathematical Science Division, US Army Office, Grant No. W911NF12-1-0090.

REFERENCES

[1] Arnold, L. *Stochastic Differential Equations: Theory and Applications*. John Wiley. 1974.

[2] Boelkins, M. R., Potter, M. C. and Goldberg, J. L. *Differential Equations with Linear Algebra*. John Wiley. 1974.

[3] Breton, M. and Ben-Ameur, H. *Numerical methods in finance*. Springer. 2005.

[4] Chandrasekhar, S. *Stochastic Problems in Physiscs and Astronomy*, Reviews of Modern Physics 15 (1943):1–89.

[5] Dufresne, D.. *Weak Convergence of Random Growth Processes with Applications to Insurance. Insurance*, Math. Econom., 8 (1989): 187–201.

[6] Dufresne, D.. *The Distribution of a Perpetuity, with Applications to Risk Theory and Pension Funding*, Scand. Actuarial J.,(1990): 39–79.

[7] Dufresne, D.. *The integral of geometric Brownian motion*, Adv.Appl. Prob., 33 (2001): 223–241.

[8] Friedman, A. *Stochastic Differential Equations and Applications*, vol.1, Academic Press. 1975.

[9] Kloeden, P. E. and Platen, E. *Numerical Solution of Stochastic Differential Equations*, Springer. 1992.

[10] Ladde, Anil G. and Ladde, G. S. *An Introduction to Differential Equations I: Diterministic Modeling, Methods and Analysis*, World Scientific. 2011.

[11] Ladde, Anil G. and Ladde, G. S. *An Introduction to Differential Equations II: Stochastic Modeling, Methods and Analysis*, World Scientific. 2011.

[12] Ladde, G. S. and Sambandham, M. *Stochastic versus deterministic systems of differential equations*, Marel Dekker Inc., New York. 2004.

[13] Ladde, G. S. and Ladde, A. G. *Dynamic Processes Under Random Environmental Perturbations*, Bulletin of Marthwada Mathematical Society 8: 96–123.

[14] Matsumoto, H. and Yor, M. *Exponential functionals of Brownian motion,I: Probability laws at fxed time*, Probability Surveys, Vol. 2 (2005): 312–347.

[15] Yor, M. *On Some Exponential Functionals of Brownian*, Springer. 1992. Advances in Applied Probability, Vol. 24, No. 3 (1992), pp. 509–531.

[16] Yor, M. *Exponential Functionals of Brownian Motion and Related Processes*, Springer. 2001.