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# EXISTENCE, UNIQUENESS AND BLOWUP FOR A PARABOLIC PROBLEM WITH A MOVING NONLINEAR SOURCE ON A SEMI-INFINITE INTERVAL

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**ABSTRACT.** Let $v$ and $T$ be positive numbers, $D = (0, \infty)$, $\Omega = D \times (0, T]$, and $\bar{D}$ be the closure of $D$. This article studies the first initial-boundary value problem,

$$u_t - u_{xx} = \delta(x - vt) f\left(u(x, t)\right) \text{ in } \Omega,$$
$$u(x, 0) = \psi(x) \text{ on } \bar{D},$$
$$u(0, t) = 0, u(x, t) \to 0 \text{ as } x \to \infty \text{ for } 0 < t \leq T,$$

where $\delta(x)$ is the Dirac delta function, and $f$ and $\psi$ are given functions. It is shown that the problem has a unique continuous solution $u$, if $u$ exists for $t \in [0, t_b)$ with $t_b < \infty$, then $u$ blows up at $t_b$.

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## 1. INTRODUCTION

Let $v$ and $T$ be positive numbers, $D = (0, \infty)$, $\Omega = D \times (0, T]$, and $\bar{D}$ be the closure of $D$. We consider the following semilinear parabolic first initial-boundary value problem,

$$\text{(1.1)} \qquad \begin{cases} Hu = \delta(x - vt) f\left(u(x, t)\right) \text{ in } \Omega, \\ u(x, 0) = \psi(x) \text{ on } \bar{D}, \\ u(0, t) = 0, u(x, t) \to 0 \text{ as } x \to \infty \text{ for } 0 < t \leq T, \end{cases}$$

where $Hu = u_t - u_{xx}$, $\delta(x)$ is the Dirac delta function, and $f$ and $\psi$ are given functions. We assume that $f(0) \geq 0$, $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $u \geq 0$, and $\psi(x)$ is nontrivial, nonnegative and continuous such that $\psi(0) = 0$, and $\psi(x) \to 0$ as $x \to \infty$.

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A solution $u$ of the problem (1.1) is a continuous function satisfying (1.1). A solution $u$ of the problem (1.1) is said to blow up at the point $(\hat{x}, t_b)$ if there exists a sequence $\{(x_n, t_n)\}$ such that $u(x_n, t_n) \to \infty$ as $(x_n, t_n) \to (\hat{x}, t_b)$. Here, $t_b$ is called the blow-up time. If $t_b$ is finite, then $u$ is said to blow up in a finite time. On the other hand, if $t_b = \infty$, then $u$ is said to blow up in infinite time. Related problems were studied by Kirk and Olmstead [3], and Olmstead [4]. They showed that if the magnitude of the (constant) velocity $v$ exceeds a certain value, then blowup does not occur; they also showed that if $v$ is below another value, then blowup occurs. In Section 2, we convert the problem (1.1) into a nonlinear integral equation and prove that the integral equation has a unique nonnegative (continuous) solution. We then show that this solution is the unique solution $u$ of the problem (1.1). We prove that if $u$ does not exist globally, then $u$ blows up in a finite time.

## 2. EXISTENCE, UNIQUENESS AND BLOWUP

Green's function $G(x, t; \xi, \tau)$ corresponding to the problem (1.1) is determined by the following system: for $x$ and $\xi$ in $D$, and $t$ and $\tau$ in $(-\infty, \infty)$,

$$\begin{aligned} HG(x, t; \xi, \tau) &= \delta(x - \xi)\delta(t - \tau), \\ G(x, t; \xi, \tau) &= 0, \ t < \tau, \\ G(0, t; \xi, \tau) &= 0, G(x, t; \xi, \tau) \to 0 \text{ as } x \to \infty. \end{aligned}$$

For $t > \tau$, it is given by

$$G(x, t; \xi, \tau) = \frac{e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4(t-\tau)}}}{\sqrt{4\pi (t - \tau)}}$$

(cf. Duffy [2, p. 183]). To derive the integral equation from the problem (1.1), let us consider the adjoint operator $H^*$, which is given by $H^* u = -u_t - u_{xx}$. Using Green's second identity, we obtain

$$u(x, t) = \int_0^\infty G(x, t; \xi, 0)\psi(\xi) d\xi + \int_0^t G(x, t; v\tau, \tau) f (u(v\tau, \tau)) \, d\tau. \tag{2.1}$$

Let $\bar{\Omega}$ denote the closure of $\Omega$.

**Lemma 2.1.** *If $r \in C([0, T])$, then $\int_0^t G(x, t; v\tau, \tau) r(\tau) \, d\tau$ is continuous on $\bar{\Omega}$.*

*Proof.* Let $R = \max_{t \in [0,T]} |r(t)|$. Since

$$\begin{aligned} \int_0^t G(x, t; v\tau, \tau) r(\tau) \, d\tau &= \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \frac{e^{-\frac{(x-v\tau)^2}{4(t-\tau)}} - e^{-\frac{(x+v\tau)^2}{4(t-\tau)}}}{\sqrt{4\pi (t - \tau)}} r(\tau) \, d\tau \\ &\leq R \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \frac{1}{\sqrt{4\pi (t - \tau)}} d\tau = R\sqrt{\frac{t}{\pi}}, \end{aligned} \tag{2.2}$$

the integral exists for each $t \in [0, T]$. For $t > \tau$, $G(x, t; v\tau, \tau)$ is continuous in $\bar{D} \times (0, T]$. Hence for $t > \tau$, $G(x, t; v\tau, \tau) r(\tau)$ is a continuous function in $\bar{D} \times (0, T]$. Let $(x_0, t_0)$ be any arbitrarily fixed point $\in \bar{D} \times (0, T]$. Then for any given positive number $\varepsilon$, there exists some positive number $\delta_1$ such that $\sqrt{(x - x_0)^2 + (t - t_0)^2} < \delta_1$ implies

$$|G(x, t; v\tau, \tau) r(\tau) - G(x_0, t_0; v\tau, \tau) r(\tau)| < \frac{\varepsilon}{2t_0}.$$

Without loss of generality, let $t \geq t_0$. Also, let $\delta = \min\{\delta_1, \pi\varepsilon^2/(4R^2)\}$. Then for $\sqrt{(x - x_0)^2 + (t - t_0)^2} < \delta$,

$$\begin{aligned}
&\left| \int_0^t G(x, t; v\tau, \tau) r(\tau)\, d\tau - \int_0^{t_0} G(x_0, t_0; v\tau, \tau) r(\tau)\, d\tau \right| \\
&\leq \int_0^{t_0} |(G(x, t; v\tau, \tau) - G(x_0, t_0; v\tau, \tau))\, r(\tau)|\, d\tau + \int_{t_0}^t |G(x, t; v\tau, \tau) r(\tau)|\, d\tau \\
&< \frac{\varepsilon}{2t_0} \int_0^{t_0} d\tau + R \int_{t_0}^t \frac{1}{\sqrt{4\pi(t - \tau)}} d\tau \\
&\leq \frac{\varepsilon}{2} + \frac{R}{\sqrt{\pi}} \sqrt{t - t_0} \\
&\leq \frac{\varepsilon}{2} + \frac{R}{\sqrt{\pi}} \sqrt{\frac{\pi\varepsilon^2}{4R^2}} = \varepsilon.
\end{aligned}$$

Therefore, the lemma is proved. □

We modified the techniques in proving Theorems 2.4, 2.5 and 2.6 of Chan and Tian [1] for a bounded domain to obtain the following two theorems for our unbounded domain.

**Theorem 2.2.** *There exists some $t_b$ such that for $0 \leq t < t_b$, the integral equation* (2.1) *has a unique nonnegative continuous solution $u$. If $t_b$ is finite, then $u$ is unbounded in $[0, t_b)$.*

*Proof.* For $(x, t) \in \bar{\Omega}$, let us construct a sequence $\{u_n\}_{n=0}^{\infty}$ by

$$u_0(x, t) = \int_0^{\infty} G(x, t; \xi, 0) \psi(\xi) d\xi,$$

and for $n = 0, 1, 2, \ldots$,

$$\begin{gathered}
H u_{n+1}(x, t) = \delta(x - vt) f(u_n(x, t)) \text{ for } (x, t) \in \Omega, \\
u_{n+1}(x, 0) = \psi(x) \text{ for } x \in \bar{D}, \\
u_{n+1}(0, t) = 0 \text{ and } u_{n+1}(x, t) \to 0 \text{ as } x \to \infty \text{ for } t \in (0, T].
\end{gathered}$$

From (2.1),

$$u_{n+1}(x, t) = \int_0^{\infty} G(x, t; \xi, 0) \psi(\xi) d\xi + \int_0^t G(x, t; v\tau, \tau) f(u_n(v\tau, \tau)) d\tau.$$

Let us show that for any $n = 0, 1, 2, \ldots$,

$$u_0 < u_1 < u_2 < \cdots < u_n \text{ on } \bar{\Omega}. \tag{2.3}$$

Since

$$u_1(x,t) - u_0(x,t) = \int_0^t G(x,t;\upsilon\tau,\tau) f(u_0(\upsilon\tau,\tau)) d\tau,$$

it follows from the positivity of $G$, $u_0(x,t) > 0$ in $\Omega$, and $f(u) > 0$ for $u > 0$ that the right-hand side is positive. We have $u_1(x,t) > u_0(x,t)$. Let us assume that for some positive integer $j$, $u_0 < u_1 < u_2 < \cdots < u_j$ on $\bar{\Omega}$. Since $u_j > u_{j-1}$, and $f' > 0$, we have

$$u_{j+1}(x,t) - u_j(x,t) = \int_0^t G(x,t;\upsilon\tau,\tau) \left( f(u_j(\upsilon\tau,\tau)) - f(u_{j-1}(\upsilon\tau,\tau)) \right) d\tau > 0.$$

By the Principle of Mathematical Induction, we have (2.3).

Let $u$ denote $\lim_{n\to\infty} u_n$, and $M$ be any positive number such that $M > \sup_{x\in\bar{D}} \psi(x)$. We would like to show that there exists a positive constant $t_1 (\leq T)$ such that the sequence $\{u_n\}_{n=0}^{\infty}$ converges uniformly to $u$ for $t \in [0, t_1]$. Since each $u_n$ is continuous, we note that

$$u_{n+1}(x,t) - u_n(x,t) = \int_0^t G(x,t;\upsilon\tau,\tau) \left[ f(u_n(\upsilon\tau,\tau)) - f(u_{n-1}(\upsilon\tau,\tau)) \right] d\tau.$$

Let $S_n = \sup_{(x,t)\in\bar{D}\times[0,t_1]} |u_n(x,t) - u_{n-1}(x,t)|$. By using the Mean Value Theorem and (2.2),

$$S_{n+1} \leq f'(M) S_n \int_0^t G(x,t;\upsilon\tau,\tau) d\tau \leq f'(M) \sqrt{\frac{t}{\pi}} S_n.$$

Since $\lim_{t\to 0} f'(M)\sqrt{t/\pi} = 0$, there exists some positive number $\sigma_1 (\leq t_1)$ such that

$$f'(M)\sqrt{\frac{t}{\pi}} < 1 \text{ for } t \in [0, \sigma_1]. \tag{2.4}$$

Then, the sequence $\{u_n\}_{n=0}^{\infty}$ converges uniformly to $u$ for any $(x,t) \in \bar{D}) \times [0, \sigma_1]$. Thus, the integral equation (2.1) has a nonnegative continuous solution $u$ for $(x,t) \in \bar{D} \times [0, \sigma_1]$. If $\sigma_1 < t_1$, then we replace the initial condition $u(x,0) = \psi(x)$ in (2.1) by $u(\xi, \sigma_1)$, which is known. We obtain for $(x,t) \in \bar{D} \times [\sigma_1, t_1]$,

$$u_{n+1}(x,t) = \int_0^{\infty} G(x,t;\xi,\sigma_1) u(\xi,\sigma_1) d\xi + \int_{\sigma_1}^t G(x,t;\upsilon\tau,\tau) f(u_n(\upsilon\tau,\tau)) d\tau.$$

From

$$u_{n+1}(x,t) - u_n(x,t) = \int_{\sigma_1}^t G(x,t;\upsilon\tau,\tau) \left[ f(u_n(\upsilon\tau,\tau)) - f(u_{n-1}(\upsilon\tau,\tau)) \right] d\tau,$$

we have

$$S_{n+1} \leq f'(M) S_n \int_{\sigma_1}^t G(x,t;\upsilon\tau,\tau) d\tau \leq f'(M) \sqrt{\frac{t-\sigma_1}{\pi}} S_n.$$

Thus, there exists $\sigma_2 = \min\{\sigma_1, t_1 - \sigma_1\} > 0$ such that

$$f'(M)\sqrt{\frac{t-\sigma_1}{\pi}} < 1 \text{ for } t \in [\sigma_1, \min\{2\sigma_1, t_1\}].$$

Hence, $\{u_n\}_{n=0}^{\infty}$ converges uniformly to $u$ for any $(x,t) \in \bar{D} \times [\sigma_1, \min\{2\sigma_1, t_1\}]$. By proceeding in this way, $\{u_n\}_{n=0}^{\infty}$ converges uniformly to $u$ for any $(x,t) \in \bar{D} \times [0, t_1]$. Therefore, the integral equation (2.1) has a nonnegative continuous solution $u$ on $\bar{D} \times [0, t_1]$.

To show that the solution $u$ is unique, let us suppose that the integral equation (2.1) has two distinct solutions $u$ and $\widetilde{u}$ on the interval $[0, t_1]$. Let

$$\Phi = \sup_{(x,t)\in\bar{D}\times[0,t_1]} |u(x,t) - \widetilde{u}(x,t)|.$$

From

$$|u(x,t) - \widetilde{u}(x,t)| = \left| \int_0^t G(x,t;v\tau,\tau)\left[f(u(v\tau,\tau)) - f(\widetilde{u}(v\tau,\tau))\right] d\tau \right|,$$

we obtain

$$\Phi \le f'(M)\Phi \int_0^t G(x,t;v\tau,\tau)d\tau \le f'(M)\sqrt{\frac{t}{\pi}}\Phi.$$

By (2.4), we have a contradiction. Thus, the integral equation (2.1) has a unique continuous solution $u$ for any $(x,t) \in \bar{D} \times [0, t_1]$.

For each $M$, there exists some $t_1$ such that the integral equation (2.1) has a unique nonnegative continuous solution $u$. Let $t_b$ be the supremum of all $t_1$ such that the integral equation has a unique nonnegative continuous solution $u$. We would like to show that if $t_b$ is finite, then $u$ is unbounded in $[0, t_b)$. Suppose $u(x,t)$ is bounded for any $(x,t) \in \bar{D} \times [0, t_b]$. We consider the integral equation (2.1) for any $(x,t) \in \bar{D} \times [t_b, \infty)$ with the initial condition $u(x,0) = \psi(x)$ replaced by $u(x, t_b)$, which is known:

$$u(x,t) = \int_0^{\infty} G(x,t;\xi,t_b)u(\xi,t_b)d\xi + \int_{t_b}^t G(x,t;v\tau,\tau)f(u(v\tau,\tau))d\tau.$$

For any given positive constant $M_1 > \sup_{x\in\bar{D}} u(x,t_b)$, an argument as before shows that there exists some $t_2 > 0$ such that the integral equation (2.1) has a unique continuous solution $u$ for any $(x,t) \in \bar{D} \times [t_b, t_2]$. This contradicts the definition of $t_b$. Hence, if $t_b$ is finite, then $u$ is unbounded in $[0, t_b)$. □

**Theorem 2.3.** *The problem* (1.1) *has a unique solution for* $0 \le t < t_b$.

*Proof.* By Lemma 2.1, $\int_0^t G(x,t;v\tau,\tau)f(u(v\tau,\tau))\,d\tau$ exists for $x$ in any compact subset of $\bar{D}$ and $t$ in any compact subset $[t_3, t_4]$ of $[0, t_b)$. Thus for any $x \in D$ and any

$t_5 \in (0, t)$,

$$\int_0^t G(x,t;v\tau,\tau) f(u(v\tau,\tau))\, d\tau = \lim_{n\to\infty} \int_{t_5}^t \frac{\partial}{\partial \zeta} \left( \int_0^{\zeta - 1/n} G(x,\zeta;v\tau,\tau) f(u(v\tau,\tau))\, d\tau \right) d\zeta$$

$$+ \lim_{n\to\infty} \int_0^{t_5 - 1/n} G(x,t_5;v\tau,\tau) f(u(v\tau,\tau))\, d\tau. \tag{2.5}$$

For $\zeta - \tau \geq \frac{1}{n}$,

$$G_\zeta(x,\zeta;v\tau,\tau) f(u(v\tau,\tau))$$

$$= \left[ \frac{(x - v\tau)^2 e^{-\frac{(x-v\tau)^2}{4(\zeta-\tau)}} - (x + v\tau)^2 e^{-\frac{(x+v\tau)^2}{4(\zeta-\tau)}}}{8\sqrt{\pi}\,(\zeta-\tau)^{5/2}} - \frac{e^{-\frac{(x-v\tau)^2}{4(\zeta-\tau)}} - e^{-\frac{(x+v\tau)^2}{4(\zeta-\tau)}}}{4\sqrt{\pi}\,(\zeta-\tau)^{3/2}} \right] f(u(v\tau,\tau))$$

$$\leq \frac{(x - v\tau)^2 e^{-\frac{(x-v\tau)^2}{4(\zeta-\tau)}}}{8\sqrt{\pi}\,(\zeta-\tau)^{5/2}} f(u(v\tau,\tau)) \leq \frac{x^2 + v^2\left(\zeta - \frac{1}{n}\right)^2}{8\sqrt{\pi}\left(\frac{1}{n}\right)^{5/2}} f(u(v\tau,\tau)),$$

which is integrable with respect to $\tau$ over $(0, \zeta - 1/n)$. It follows from the Leibnitz rule (cf. Stromberg [5, p. 380]) that

$$\frac{\partial}{\partial\zeta}\left( \int_0^{\zeta - 1/n} G(x,\zeta;v\tau,\tau) f(u(v\tau,\tau))\, d\tau \right)$$

$$= G\left(x, \zeta; v\left(\zeta - \frac{1}{n}\right), \zeta - \frac{1}{n}\right) f\left(u\left(v\left(\zeta - \frac{1}{n}\right), \zeta - \frac{1}{n}\right)\right)$$

$$+ \int_0^{\zeta - 1/n} G_\zeta(x,\zeta;v\tau,\tau) f(u(v\tau,\tau))\, d\tau. \tag{2.6}$$

From (2.5) and (2.6),

$$\int_0^t G(x,t;v\tau,\tau) f(u(v\tau,\tau))\, d\tau$$

$$= \lim_{n\to\infty} \int_{t_5}^t G\left(x, \zeta; v\left(\zeta - \frac{1}{n}\right), \zeta - \frac{1}{n}\right) f\left(u\left(v\left(\zeta - \frac{1}{n}\right), \zeta - \frac{1}{n}\right)\right) d\zeta$$

$$+ \lim_{n\to\infty} \int_{t_5}^t \int_0^{\zeta - 1/n} G_\zeta(x,\zeta;v\tau,\tau) f(u(v\tau,\tau))\, d\tau d\zeta$$

$$+ \lim_{n\to\infty} \int_0^{t_5 - 1/n} G(x,t_5;v\tau,\tau) f(u(v\tau,\tau))\, d\tau.$$

Let us consider the problem,

$$Hw = 0 \text{ for } x \text{ and } \xi \text{ in } D, 0 \leq \tau < t,$$

$$w(0,t;\xi,\tau) = 0, w(x,t;\xi,\tau) \to 0 \text{ as } x \to \infty \text{ for } 0 \leq \tau < t < T,$$

$$\lim_{t\to\tau^+} w(x,t;\xi,\tau) = \delta(x - \xi\tau).$$

Using Green's second identity, we obtain for $t > \tau$,

$$w(x,t;\xi,\tau) = \int_0^\infty G(x,t;y,\tau)\,\delta(y-\xi\tau)\,dy = G(x,t;\xi\tau,\tau).$$

It follows that

$$\lim_{t\to\tau^+} G(x,t;\xi\tau,\tau) = \delta(x-\xi\tau).$$

This implies

$$\lim_{n\to\infty} G\left(x,\zeta;v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right) = \delta(x-v\zeta). \tag{2.7}$$

Since for $x \neq v\zeta$,

$$G\left(x,\zeta;v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right) f\left(u\left(v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right)\right)$$

converges uniformly to zero with respect to $\zeta$ as $n$ tends to infinity, it follows that for $x \neq v\zeta$,

$$\lim_{n\to\infty}\int_{t_5}^{t} G\left(x,\zeta;v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right) f\left(u\left(v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right)\right) d\zeta$$
$$= \int_{t_5}^{t} \lim_{n\to\infty} G\left(x,\zeta;v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right) f\left(u\left(v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right)\right) d\zeta.$$

For $x = v\zeta$,

$$\frac{\partial}{\partial n} G\left(v\zeta,\zeta;v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right)$$
$$= \frac{e^{-\frac{v^2\left(1+4n^2\zeta^2\right)}{4n}}\left[e^{nv^2\zeta^2}\left(2n+v^2\right)+e^{v^2\zeta}\left(-2n-v^2+4n^2v^2\zeta^2\right)\right]}{8n^{3/2}\sqrt{\pi}}.$$

It follows that for sufficiently large $n$, $G(v\zeta,\zeta;v(\zeta-1/n),\zeta-1/n)$ is an increasing function of $n$. Thus for large $n$,

$$G\left(v\zeta,\zeta;v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right) f\left(u\left(v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right)\right)$$

is an increasing sequence of nonnegative functions with respect to $n$. By the Monotone Convergence Theorem (cf. Stromberg [5, p. 288]),

$$\lim_{n\to\infty}\int_{t_5}^{t} G\left(v\zeta,\zeta;v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right) f\left(u\left(v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right)\right) d\zeta$$
$$= \int_{t_5}^{t} \lim_{n\to\infty} G\left(v\zeta,\zeta;v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right) f\left(u\left(v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right)\right) d\zeta.$$

Therefore,

$$\int_0^t G(x,t;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau$$
$$=\int_{t_5}^t \lim_{n\to\infty}\left[G\left(x,\zeta;v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right)f\left(u\left(v\left(\zeta-\frac{1}{n}\right),\zeta-\frac{1}{n}\right)\right)\right]d\zeta$$
$$+\lim_{n\to\infty}\int_{t_5}^t\int_0^{\zeta-1/n}G_\zeta(x,\zeta;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau d\zeta$$
$$+\lim_{n\to\infty}\int_0^{t_5-1/n}G(x,t_5;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau. \tag{2.8}$$

Since $f$ and $u$ are continuous, we have from (2.7) and (2.8) that

$$\int_0^t G(x,t;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau$$
$$=\int_{t_5}^t \delta\left(x-v\zeta\right)f\left(u\left(v\zeta,\zeta\right)\right)d\zeta$$
$$+\lim_{n\to\infty}\int_{t_5}^t\int_0^{\zeta-1/n}G_\zeta(x,\zeta;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau d\zeta$$
$$+\int_0^{t_5}G(x,t_5;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau. \tag{2.9}$$

Let

$$g_n\left(x,\zeta\right)=\int_0^{\zeta-1/n}G_\zeta\left(x,\zeta;v\tau,\tau\right)f\left(u(v\tau,\tau)\right)d\tau.$$

Without loss of generality, let $n>l$. We have

$$g_n\left(x,\zeta\right)-g_l\left(x,\zeta\right)=\int_{\zeta-1/l}^{\zeta-1/n}G_\zeta\left(x,\zeta;v\tau,\tau\right)f\left(u(v\tau,\tau)\right)d\tau.$$

Since $G_\zeta\left(x,\zeta;v\tau,\tau\right)\in C\left(D\times(\tau,T]\right)$, and $f\left(u\left(v\tau,\tau\right)\right)$ is nonnegative and integrable with respect to $\tau$ over $(\zeta-1/l,\zeta-1/n)$, it follows from the First Mean Value Theorem for Integrals (cf. Stromberg [5, p. 328]) that for $x$ in any compact subset of $D$ and $\zeta$ in any compact subset of $(0,t_b)$, there exists some real number $r$ such that $\zeta-r\in(\zeta-1/l,\zeta-1/n)$ and

$$g_n\left(x,\zeta\right)-g_l\left(x,\zeta\right)=G_\zeta\left(x,\zeta;v\left(\zeta-r\right),\zeta-r\right)\int_{\zeta-1/l}^{\zeta-1/n}f\left(u(v\tau,\tau)\right)d\tau.$$

Since

$$G_\zeta\left(x,\zeta;v\left(\zeta-\varepsilon\right),\zeta-\varepsilon\right)$$
$$=G_\zeta\left(x,\varepsilon;v\left(\zeta-\varepsilon\right),0\right)$$
$$=\frac{\left(\varepsilon v^2+vx-v^2\zeta\right)e^{-\frac{[x-(v\zeta-v\varepsilon)]^2}{4\varepsilon}}+\left(-\varepsilon v^2+vx+v^2\zeta\right)e^{-\frac{[x+(v\zeta-v\varepsilon)]^2}{4\varepsilon}}}{4\sqrt{\pi}\left(\varepsilon\right)^{3/2}},$$

which converges to 0 with respect to $\zeta$ as $\varepsilon \to 0$, it follows that $\{g_n\}$ is a Cauchy sequence, and hence, $\{g_n\}$ converges uniformly with respect to $\zeta$ in any compact subset $[t_6, t_7]$ of $(0, t_b)$. Hence,

$$\begin{aligned}
&\lim_{n\to\infty}\int_{t_5}^{t}\int_0^{\zeta-1/n} G_\zeta(x,\zeta;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau d\zeta \\
&= \int_{t_5}^{t}\lim_{n\to\infty}\int_0^{\zeta-1/n} G_\zeta(x,\zeta;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau d\zeta \\
&= \int_{t_5}^{t}\int_0^{\zeta} G_\zeta(x,\zeta;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau d\zeta. \qquad (2.10)
\end{aligned}$$

From (2.9) and (2.10),

$$\begin{aligned}
&\int_0^t G(x,t;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau \\
&= \int_{t_5}^{t}\delta\left(x-v\zeta\right)f\left(u\left(v\zeta,\zeta\right)\right)d\zeta + \int_{t_5}^{t}\int_0^{\zeta} G_\zeta(x,\zeta;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau d\zeta \\
&\quad + \int_0^{t_5} G(x,t_5;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_0^t G(x,t;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau \\
&= \delta\left(x-vt\right)f\left(u(vt,t)\right) + \int_0^t G_t(x,t;v\tau,\tau)f\left(u(v\tau,\tau)\right)d\tau. \qquad (2.11)
\end{aligned}$$

For $t-\tau > \varepsilon$,

$$G_x(x,t;v\tau,\tau) = \frac{\left(v\tau - x\right)e^{-\frac{(x-v\tau)^2}{4(t-\tau)}} + \left(v\tau + x\right)e^{-\frac{(x+v\tau)^2}{4(t-\tau)}}}{4\sqrt{\pi}\left(t-\tau\right)^{3/2}},$$

$$G_{xx}(x,t;v\tau,\tau) = \frac{\frac{(x-v\tau)^2 e^{-\frac{(x-v\tau)^2}{4(t-\tau)}} - (x+v\tau)^2 e^{-\frac{(x+v\tau)^2}{4(t-\tau)}}}{4(t-\tau)^2} - \frac{e^{-\frac{(x-v\tau)^2}{4(t-\tau)}} - e^{-\frac{(x+v\tau)^2}{4(t-\tau)}}}{2(t-\tau)}}{2\pi\sqrt{t-\tau}}.$$

Thus for each fixed $(\xi,\tau) \in D\times[0,T)$, $G_x(x,t;\xi,\tau)$ and $G_{xx}(x,t;\xi,\tau)$ are in $C\left(D\times(\tau,T]\right)$. For $t-\tau > \varepsilon$,

$$\begin{aligned}
G_x(x,t;v\tau,\tau)f\left(u\left(v\tau,\tau\right)\right) &= \frac{\left(v\tau - x\right)e^{-\frac{(x-v\tau)^2}{4(t-\tau)}} + \left(v\tau + x\right)e^{-\frac{(x+v\tau)^2}{4(t-\tau)}}}{4\sqrt{\pi}\left(t-\tau\right)^{3/2}}f\left(u\left(v\tau,\tau\right)\right) \\
&\le \frac{\left(v\tau - x\right)e^{-\frac{(x-v\tau)^2}{4(t-\tau)}} + \left(v\tau + x\right)e^{-\frac{(x-v\tau)^2}{4(t-\tau)}}}{4\sqrt{\pi}\left(t-\tau\right)^{3/2}}f\left(u\left(v\tau,\tau\right)\right) \\
&= \frac{v\tau e^{-\frac{(x-v\tau)^2}{4(t-\tau)}}}{2\sqrt{\pi}\left(t-\tau\right)^{3/2}}f\left(u\left(v\tau,\tau\right)\right),
\end{aligned}$$

which is integrable with respect to $\tau$ over $(0, t-\varepsilon)$. For $t-\tau > \varepsilon$,

$$\begin{aligned}
&G_{xx}(x,t;v\tau,\tau) f\left(u\left(v\tau,\tau\right)\right) \\
&= \frac{\frac{(x-v\tau)^2 e^{-\frac{(x-v\tau)^2}{4(t-\tau)}} - (x+v\tau)^2 e^{-\frac{(x+v\tau)^2}{4(t-\tau)}}}{4(t-\tau)^2} - \frac{e^{-\frac{(x-v\tau)^2}{4(t-\tau)}} - e^{-\frac{(x+v\tau)^2}{4(t-\tau)}}}{2(t-\tau)}}{2\pi\sqrt{t-\tau}} f\left(u\left(v\tau,\tau\right)\right) \\
&\leq \frac{\frac{(x-v\tau)^2 e^{-\frac{(x-v\tau)^2}{4(t-\tau)}}}{4(t-\tau)^2}}{2\pi\sqrt{t-\tau}} f\left(u\left(v\tau,\tau\right)\right) \\
&= \frac{\left(x-v\tau\right)^2 e^{-\frac{(x-v\tau)^2}{4(t-\tau)}}}{8\pi\left(t-\tau\right)^{5/2}} f\left(u\left(v\tau,\tau\right)\right),
\end{aligned}$$

which is integrable with respect to $\tau$ over $(0, t-\varepsilon)$. Using the Leibnitz rule, we have for any $x$ in any compact subset of $D$ and $t$ in any compact subset of $(0, t_b)$,

$$\frac{\partial}{\partial x}\int_0^{t-\epsilon} G(x,t;v\tau,\tau) f\left(u(v\tau,\tau)\right) d\tau = \int_0^{t-\epsilon} G_x(x,t;v\tau,\tau) f\left(u(v\tau,\tau)\right) d\tau,$$

$$\frac{\partial}{\partial x}\int_0^{t-\epsilon} G_x(x,t;v\tau,\tau) f\left(u(v\tau,\tau)\right) d\tau = \int_0^{t-\epsilon} G_{xx}(x,t;v\tau,\tau) f\left(u(v\tau,\tau)\right) d\tau.$$

For any $x_1$ in any compact subset of $D$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} G\left(x,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau \\
&= \lim_{\varepsilon\to 0}\int_{x_1}^{x}\left(\frac{\partial}{\partial\eta}\int_0^{t-\varepsilon} G\left(\eta,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau\right) d\eta \\
&\quad + \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} G\left(x_1,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau \\
&= \lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon} G_\eta\left(\eta,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau d\eta \\
&\quad + \int_0^{t} G\left(x_1,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau. \qquad (2.12)
\end{aligned}$$

We would like to show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon} G_\eta\left(\eta,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau d\eta \\
&\quad = \int_{x_1}^{x}\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} G_\eta\left(\eta,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau d\eta. \qquad (2.13)
\end{aligned}$$

By the Fubini Theorem (cf. Stromberg [5, p. 352]),

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_{0}^{t-\varepsilon} G_\eta\left(\eta,t;\upsilon\tau,\tau\right) f\left(u\left(\upsilon\tau,\tau\right)\right) d\tau d\eta \\
&= \lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon}\left( f\left(u\left(\upsilon\tau,\tau\right)\right)\int_{x_1}^{x} G_\eta\left(\eta,t;\upsilon\tau,\tau\right) d\eta\right) d\tau \\
&= \lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon} f\left(u\left(\upsilon\tau,\tau\right)\right)\left(G\left(x,t;\upsilon\tau,\tau\right)-G\left(x_1,t;\upsilon\tau,\tau\right)\right) d\tau \\
&= \int_{0}^{t} f\left(u\left(\upsilon\tau,\tau\right)\right)\left(G\left(x,t;\upsilon\tau,\tau\right)-G\left(x_1,t;\upsilon\tau,\tau\right)\right) d\tau,
\end{aligned}$$

which exists by Lemma 2.1. Therefore,

$$\begin{aligned}
&\int_{0}^{t} f\left(u\left(\upsilon\tau,\tau\right)\right)\left(G\left(x,t;\upsilon\tau,\tau\right)-G\left(x_1,t;\upsilon\tau,\tau\right)\right) d\tau \\
&\quad = \int_{x_1}^{x}\int_{0}^{t} G_\eta\left(\eta,t;\upsilon\tau,\tau\right) f\left(u\left(\upsilon\tau,\tau\right)\right) d\tau d\eta,
\end{aligned}$$

and we have (2.13). From (2.12),

$$\frac{\partial}{\partial x}\int_{0}^{t} G(x,t;\upsilon\tau,\tau) f\left(u(\upsilon\tau,\tau)\right) d\tau = \int_{0}^{t} G_x(x,t;\upsilon\tau,\tau) f\left(u(\upsilon\tau,\tau)\right) d\tau. \tag{2.14}$$

For any $x_2$ in any compact subset of $D$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon} G_x\left(x,t;\upsilon\tau,\tau\right) f\left(u\left(\upsilon\tau,\tau\right)\right) d\tau \\
&= \lim_{\varepsilon\to 0}\int_{x_2}^{x}\frac{\partial}{\partial\eta}\left(\int_{0}^{t-\varepsilon} G_\eta\left(\eta,t;\upsilon\tau,\tau\right) f\left(u\left(\upsilon\tau,\tau\right)\right) d\tau\right) d\eta \\
&\quad + \lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon} G_\eta\left(x_2,t;\upsilon\tau,\tau\right) f\left(u\left(\upsilon\tau,\tau\right)\right) d\tau \\
&= \lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_{0}^{t-\varepsilon} G_{\eta\eta}\left(\eta,t;\upsilon\tau,\tau\right) f\left(u\left(\upsilon\tau,\tau\right)\right) d\tau d\eta \\
&\quad + \int_{0}^{t} G_\eta\left(x_2,t;\upsilon\tau,\tau\right) f\left(u\left(\upsilon\tau,\tau\right)\right) d\tau.
\end{aligned} \tag{2.15}$$

We would like to show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_{0}^{t-\varepsilon} G_{\eta\eta}\left(\eta,t;\upsilon\tau,\tau\right) f\left(u\left(\upsilon\tau,\tau\right)\right) d\tau d\eta \\
&\quad = \int_{x_2}^{x}\lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon} G_{\eta\eta}\left(\eta,t;\upsilon\tau,\tau\right) f\left(u\left(\upsilon\tau,\tau\right)\right) d\tau d\eta.
\end{aligned} \tag{2.16}$$

By the Fubini Theorem,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon} G_{\eta\eta}\left(\eta,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau d\eta \\
&= \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\left(f\left(u\left(v\tau,\tau\right)\right)\int_{x_2}^{x} G_{\eta\eta}\left(\eta,t;v\tau,\tau\right) d\eta\right) d\tau \\
&= \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} f\left(u\left(v\tau,\tau\right)\right)\left(G_\eta\left(x,t;v\tau,\tau\right) - G_\eta\left(x_2,t;v\tau,\tau\right)\right) d\tau \\
&= \int_0^{t} f\left(u\left(v\tau,\tau\right)\right)\left(G_\eta\left(x,t;v\tau,\tau\right) - G_\eta\left(x_2,t;v\tau,\tau\right)\right) d\tau,
\end{aligned}$$

which exists by (2.14). Therefore,

$$\begin{aligned}
&\int_0^{t} f\left(u\left(v\tau,\tau\right)\right)\left(G_\eta\left(x,t;v\tau,\tau\right) - G_\eta\left(x_1,t;v\tau,\tau\right)\right) d\tau \\
&= \int_{x_2}^{x}\int_0^{t} G_{\eta\eta}\left(\eta,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau d\eta \\
&= \int_{x_2}^{x}\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} G_{\eta\eta}\left(\eta,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau d\eta,
\end{aligned}$$

where we have (2.16). From (2.15),

$$\begin{aligned}
&\int_0^{t} G_x\left(x,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau \\
&= \int_{x_2}^{x}\int_0^{t} G_{\eta\eta}\left(\eta,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau d\eta + \int_0^{t} G_\eta\left(x_2,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau.
\end{aligned}$$

Thus,

$$\frac{\partial}{\partial x}\int_0^{t} G_x\left(x,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau = \int_0^{t} G_{xx}(x,t;v\tau,\tau) f\left(u(v\tau,\tau)\right) d\tau.$$

Therefore,

$$\frac{\partial^2}{\partial x^2}\int_0^{t} G(x,t;v\tau,\tau) f\left(u(v\tau,\tau)\right) d\tau = \int_0^{t} G_{xx}(x,t;v\tau,\tau) f\left(u(v\tau,\tau)\right) d\tau. \tag{2.17}$$

It follows from the integral equation (2.1), (2.11) and (2.17) that for $x \in D$ and $0 < t < t_b$,

$$\begin{aligned}
Hu &= \frac{\partial}{\partial t}\left(\int_0^{t} G\left(x,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau\right) - \frac{\partial^2}{\partial x^2}\left(\int_0^{t} G\left(x,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau\right) \\
&= \delta\left(x-vt\right) f\left(u\left(vt,t\right)\right) + \int_0^{t} HG\left(x,t;v\tau,\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau \\
&= \delta\left(x-vt\right) f\left(u\left(vt,t\right)\right) + \lim_{\epsilon\to 0}\int_0^{t-\epsilon} \delta\left(x-v\tau\right)\delta\left(t-\tau\right) f\left(u\left(v\tau,\tau\right)\right) d\tau \\
&= \delta\left(x-vt\right) f\left(u\left(vt,t\right)\right) \\
&= \delta\left(x-vt\right) f\left(u\left(x,t\right)\right).
\end{aligned}$$

From (2.1), $\lim_{t\to 0} u(x,t) = \psi(x)$ for $x \in \bar{D}$. Since $G(0,t;\xi,\tau) = 0$, we have $u(0,t) = 0$. By Lemma 2.1,

$$\lim_{x\to\infty} \int_0^t G(x,t;v\tau,\tau) f(u(v\tau,\tau)) \, d\tau = \int_0^t \lim_{x\to\infty} G(x,t;v\tau,\tau) f(u(v\tau,\tau)) \, d\tau = 0.$$

Thus, the nonnegative continuous solution $u$ of the integral equation (2.1) is a solution of the problem (1.1). Since a solution of the latter is a solution of the former, the theorem is proved. □

We remark that from the above two theorems, if $t_b$ is finite, then $u$ blows up at $t_b$.

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