Dynamic Systems and Applications **21** (2012), 645–656

A NOTE ON CONTINUATION PRINCIPLES FOR COINCIDENCES

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ABSTRACT. This paper uses the ideas and results in [5] to establish some new continuations principles which are useful from an application viewpoint.

AMS (MOS) Subject Classification. 47H10

Keywords: Continuation methods, coincidence theory.

1. INTRODUCTION

In this paper we consider homotopies $H : \overline{U} \to 2^Y$ where the maps H_t may have different domains $\overline{U_t}$. The main idea for continuation principles is to reduce the study of the family $\{H_t\}$ to that of a certain family of maps (of course depending on the old one) from the same domain $\overline{U}$ into $Y \times \mathbf{R}$. This paper extends some work in [5] and in particular we establish some continuation results motivated from initial ideas in [1, 10].

Let X and Y be Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$. We let

$$\mathbf{F}(\mathbf{X}) = \{Z : \text{ Fix } F \neq \emptyset \text{ for all } F \in \mathbf{X}(Z, Z)\}$$

where Fix$F$ denotes the set of fixed points of F.

The class $\mathbf{U}$ of maps is defined by the following properties:

(i). $\mathbf{U}$ contains the class $\mathbf{C}$ of single valued continuous functions;

(ii). each $F \in \mathbf{U}_c$ is upper semicontinuous and compact valued; and

(iii). $B^n \in \mathbf{F}(\mathbf{U}_c)$ for all $n \in \{1, 2, \dots\}$; here $B^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$.

We say $F \in \mathbf{U}_c^k(X, Y)$ if for any compact subset K of X there is a $G \in \mathbf{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Recall $\mathbf{U}_c^k$ is closed under compositions. The class $\mathbf{U}_c^k$ contains almost all the well known maps in the literature (see [8] and the references therein). It is also possible

Received August 25, 2012

to consider more general maps (see [6, 8]) and in this paper we will consider a class of maps which we will call $\mathbf{A}$.

2. CONTINUATION PRINCIPLES

We begin this section by recalling some definitions and results from [5]. Let E be a completely regular topological space and U an open subset of E.

We will consider a class $\mathbf{A}$ of maps. In some results the following condition will be assumed:

$$\begin{cases} \text{for Hausdorff topological spaces } X_1, X_2 \text{ and } X_3, \\ \text{if } F \in \mathbf{A}(X_1, X_3) \text{ and } f \in \mathbf{C}(X_2, X_1), \\ \text{then } F \circ f \in \mathbf{A}(X_2, X_3). \end{cases} \tag{2.1}$$

Definition 2.1. We say $F \in A(\overline{U}, E)$ if $F \in \mathbf{A}(\overline{U}, E)$ and $F : \overline{U} \to K(E)$ is an upper semicontinuous map; here $\overline{U}$ denotes the closure of U in E and $K(E)$ denotes the family of nonempty compact subsets of E.

Definition 2.2. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.3. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi \in A(\overline{U} \times [0,1], E)$, $x \notin \Psi_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$, $\Psi_1 = F$, $\Psi_0 = G$ (here $\Psi_t(x) = \Psi(x,t)$) and $\{x \in \overline{U} : x \in \Psi(x,t)$ for some $t \in [0,1]\}$ is relatively compact.

Remark 2.4. We note if $\Phi : \overline{U} \times [0,1] \to K(E)$ is a upper semicontinuous map then $M = \big\{x \in \overline{U} : x \in \Phi(x,t) \text{ for some } t \in [0,1]\big\}$ is closed so if M is relatively compact then M is compact. If $\Phi : \overline{U} \times [0,1] \to K(E)$ is an upper semicontinuous compact map then

$$\big\{x \in \overline{U} : x \in \Phi(x,t) \text{ for some } t \in [0,1]\big\}$$

is compact.

Remark 2.5. The result below (with (2.1) removed) also holds true if we use the following definition of $\cong$. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists an upper semicontinuous map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $x \notin \Psi_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$, $\Psi_1 = F$, $\Psi_0 = G$ and $\big\{x \in \overline{U} : x \in \Psi(x,t) \text{ for some } t \in [0,1]\big\}$ is relatively compact.

The following condition will be assumed:

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, E). \tag{2.2}$$

Definition 2.6. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to K(E)$ is essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists $x \in U$ with $x \in J(x)$. Otherwise F is inessential in $A_{\partial U}(\overline{U}, E)$ i.e. there exists a fixed point free map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$.

In [5] we established the following theorem which extended and generalized results in the literature [2, 3, 4, 7, 9, 10].

Theorem 2.7. *Let E be a completely regular topological space, U an open subset of E and assume* (2.1) *and* (2.2) *hold. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Then F is essential in $A_{\partial U}(\overline{U}, E)$ if and only if G is essential in $A_{\partial U}(\overline{U}, E)$.*

Remark 2.8. The result in Theorem 2.7 (with (2.1) removed) holds if the definition of $\cong$ is as in Remark 2.5.

Remark 2.9. If E is a normal topological space then the assumption that

$$\big\{x \in \overline{U} : x \in \Psi(x,t) \text{ for some } t \in [0,1]\big\}$$

is relatively compact can be removed in Definition 2.3 (and Remark 2.5) and we still obtain Theorem 2.7.

In many applications fixed point results are needed for homotopies H for which the maps H_t may be defined on different domains. The idea is to reduce the study of this family to that of a new family (of course depending on the old one) defined on the same domain. For notational purposes let Z be a topological space and Ω a subset of $Z \times [0,1]$. We write $\Omega_\lambda = \{x \in Z : (x,\lambda) \in \Omega\}$ to denote the section of Ω at λ.

Let E be a completely regular topological space and let U be an open subset of $E \times [0,1]$. For our next result we assume (2.1) holds and in addition

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, E \times [0,1]) \tag{2.3}$$

and

$$\begin{cases} \text{for Hausdorff topological spaces } X_1 \text{ and } X_2, \text{ if } F \in A(X_1, X_2) \\ \text{and if } \Psi(y,\mu) = (F(y),\mu) \text{ for } (y,\mu) \in X_1 \times [0,1], \text{ then} \\ \Psi_\mu \in A(X_1, X_2 \times [0,1]) \text{ for each } \mu \in [0,1] \text{ and} \\ \Psi \in A(X_1 \times [0,1], X_2 \times [0,1]); \text{ here } \Psi_\mu(x) = \Psi(x,\mu). \end{cases} \tag{2.4}$$

In [5] we established the following theorem.

Theorem 2.10. *Suppose $N \in A(\overline{U}, E)$ with*

$$x \notin N(x,\lambda) \textit{ for } (x,\lambda) \in \partial U. \tag{2.5}$$

Let $H : \overline{U} \times [0,1] \to K(E \times [0,1])$ *be given by* $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ *for* $(x,\lambda) \in \overline{U}$ *and* $\mu \in [0,1]$. *In addition assume the following conditions hold:*

$$(2.6) \qquad \begin{cases} H_0 \textit{ is essential in } A_{\partial U}(\overline{U}, E \times [0,1]);\ \textit{here} \\ H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0) \textit{ for } (x,\lambda) \in \overline{U} \end{cases}$$

and

$$(2.7) \qquad \big\{(x,\lambda) \in \overline{U} : (x,\lambda) \in H(x,\lambda,\mu) \textit{ for some } \mu \in [0,1]\big\} \textit{ is relatively compact.}$$

Then H_1 *is essential in* $A_{\partial U}(\overline{U}, E \times [0,1])$ *so in particular there exists a* $x \in U_1$ *with* $x \in N(x,1)$*; here* $H_1(x,\lambda) = H(x,\lambda,1) = (N(x,\lambda),1)$ *for* $(x,\lambda) \in \overline{U}$.

Remark 2.11. In fact H_t is essential in $A_{\partial U}(\overline{U}, E \times [0,1])$ for every $t \in [0,1]$; here $H_t(x,\lambda) = H(x,\lambda,t) = (N(x,\lambda),t)$ for $(x,\lambda) \in \overline{U}$. If E is a normal topological space then the assumption (2.7) can be removed in the statement of Theorem 2.10.

Remark 2.12. The result in Theorem 2.10 holds (with (2.1) and (2.4) removed) if the definition of $\cong$ is as in Remark 2.5 and if the following condition holds: $H(\cdot,\cdot,\eta(\cdot,\cdot)) \in A(\overline{U}, E \times [0,1])$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$.

Let E be a completely regular topological vector space and let U be an open subset of $E \times [0,1]$. Our next theorem is a special case of Theorem 2.10 where it gives conditions so that (2.6) holds.

Theorem 2.13. *Let* $p \in U_0$. *Suppose* $N \in A(\overline{U}, E)$ *and assume* (2.1), (2.3), (2.4) *and* (2.5) *hold. Let* $H : \overline{U} \times [0,1] \to K(E \times [0,1])$ *be given by* $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ *for* $(x,\lambda) \in \overline{U}$ *and* $\mu \in [0,1]$ *and assume* (2.7) *holds. Let* $Q : \overline{U} \times [0,1] \to K(E \times [0,1])$ *be given by* $Q(x,\lambda,\mu) = (\mu N(x,\lambda) + (1-\mu)p, 0)$ *for* $(x,\lambda) \in \overline{U}$ *and* $\mu \in [0,1]$. *Now suppose the following conditions hold:*

$$(2.8) \qquad x \notin \mu N(x,0) + (1-\mu)p \textit{ for } (x,0) \in \partial U \textit{ and } \mu \in (0,1)$$

$$(2.9) \qquad \begin{cases} Q_0 \textit{ is essential in } A_{\partial U}(\overline{U}, E \times [0,1]);\ \textit{here} \\ Q_0(x,\lambda) = (p,0) \textit{ for } (x,\lambda) \in \overline{U} \end{cases}$$

$$(2.10) \qquad \begin{cases} \textit{if } F \in A(\overline{U}, E) \textit{ and if } \Phi(y,\mu) = (\mu F(y) + (1-\mu)p, 0) \\ \textit{for } (y,\mu) \in \overline{U} \times [0,1],\ \textit{then } \Phi_\mu \in A(\overline{U}, E \times [0,1]) \textit{ for} \\ \textit{each } \mu \in [0,1] \textit{ and } \Phi \in A(\overline{U} \times [0,1], E \times [0,1]); \\ \textit{here } \Phi_\mu(x) = \Phi(x,\mu) \end{cases}$$

and

$$(2.11) \qquad \begin{cases} \big\{(x,\lambda) \in \overline{U} : (x,\lambda) \in Q(x,\lambda,\mu) \textit{ for some } \mu \in [0,1]\big\} \\ \textit{is relatively compact.} \end{cases}$$

Then H_1 *is essential in* $A_{\partial U}(\overline{U}, E \times [0,1])$ *so in particular there exists a* $x \in U_1$ *with* $x \in N(x,1)$*; here* $H_1(x,\lambda) = H(x,\lambda,1) = (N(x,\lambda),1)$ *for* $(x,\lambda) \in \overline{U}$.

Proof. Note (2.10) guarantees that $Q \in A(\overline{U} \times [0,1], E \times [0,1])$. Also

$$(x,\lambda) \notin Q_\mu(x,\lambda) \text{ for } (x,\lambda) \in \partial U \text{ and } \mu \in [0,1]. \tag{2.12}$$

To see this suppose there exists $(x,\lambda) \in \partial U$ and $\mu \in [0,1]$ with $(x,\lambda) \in (\mu N(x,\lambda) + (1-\mu)p, 0)$. Then $\lambda = 0$ and $x \in \mu N(x,\lambda) + (1-\mu)p = \mu N(x,0) + (1-\mu)p$ which is a contradiction (note (2.8) is contradicted if $\mu \in (0,1)$, (2.5) is contradicted if $\mu = 1$ and $p \in U_0$ (i.e. $(p,0) \in U$) is contradicted if $\mu = 0$). Thus (2.12) is true and note $Q_0 \cong Q_1$ in $A_{\partial U}(\overline{U}, E \times [0,1])$ (see above and (2.11)); here $Q_1(x,\lambda) = Q(x,\lambda,1) = (N(x,\lambda),0) = H_0(x,\lambda)$. Now Theorem 2.7 (note (2.1), (2.3) and (2.9)) guarantees that

$$Q_1 (= H_0) \text{ is essential in } A_{\partial U}(\overline{U}, E \times [0,1]).$$

Finally Theorem 2.10 (note (2.1), (2.3), (2.4) and (2.7)) guarantees that H_1 is essential in $A_{\partial U}(\overline{U}, E \times [0,1])$. Thus there exists a $(x,\lambda) \in U$ with $(x,\lambda) \in (N(x,\lambda),1)$ i.e. $x \in N(x,\lambda)$ with $\lambda = 1$ i.e. $x \in U_1 = \{y \in E : (y,1) \in U\}$ and $x \in N(x,1)$. □

Remark 2.14. From the proof above note that for each $t \in [0,1]$ there exists $x_t \in U_t$ with $x_t \in N(x_t,t)$. If E is a normal topological space then (2.7) and (2.11) can be removed in the statement of Theorem 2.13.

Remark 2.15. The result in Theorem 2.13 holds (with (2.1), (2.4) and (2.10) removed) if the definition of $\cong$ is as in Remark 2.5 and if the following condition holds: $H(\cdot,\cdot,\eta(\cdot,\cdot)) \in A(\overline{U}, E \times [0,1])$, $Q(\cdot,\cdot,\eta(\cdot,\cdot)) \in A(\overline{U}, E \times [0,1])$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$.

Our next result is motivated by ideas in [1, 7, 10]. For convenience we let E be a normal topological vector space, $N : E \times [0,1] \to K(E)$ an upper semicontinuous map and we fix $p \in E$. Let

$$S(p) = \{(x,0) \in E \times [0,1] : x \in \mu N(x,0) + (1-\mu)p \text{ for some } \mu \in [0,1]\}$$

and

$$A = \{(x,\lambda) \in E \times [0,1] : x \in N(x,\lambda)\}.$$

For our next result we consider a continuous functional $\phi : E \times [0,1] \to \mathbf{R}$.

Theorem 2.16. *Suppose there exist constants a, b with $a < b$ such that if we set $V = \phi^{-1}(a,b)$ the following conditions are satisfied:*

$$\phi(A) \cap \{a,b\} = \emptyset \tag{2.13}$$

and

$$S(p) \subset V. \tag{2.14}$$

Assume (2.1) *and* (2.4) *are satisfied and in addition for any subset U of V with $p \in U_0$ (and $A \cap V \subseteq U$, $S(p) \subseteq U$) we assume $N \in A(\overline{U}, E)$ and* (2.3), (2.9) *and* (2.10) *hold. Then for each $\lambda \in [0,1]$ there exists a fixed point of N_λ in V_λ.*

Proof. Let $B = A \cap \phi^{-1}[a,b]$. Note B is closed since N is upper semicontinuous and ϕ is continuous. In addition note (2.13) guarantees that $B = A \cap V$ (if $x \in B$ then $x \in A$ and $x \in \phi^{-1}[a,b]$ so if $x \in \phi^{-1}(a,b)$ then trivially $x \in A \cap V$, whereas if $x \in \phi^{-1}(a)$ then $x \in A$ and $\phi(x) = a$ which contradicts (2.13), and finally if $x \in \phi^{-1}(b)$ then $x \in A$ and $\phi(x) = b$ which contradicts (2.13)). Also note $B \subset V$ is closed and $S(p) \subset V$ is closed. A standard result in topology (recall E is normal) guarantees that there exists open subsets W_1 and W_2 of $E \times [0,1]$ with

$$B \subseteq W_1 \subseteq \overline{W_1} \subseteq V \text{ and } S(p) \subseteq W_2 \subseteq \overline{W_2} \subseteq V. \tag{2.15}$$

We wish to apply Theorem 2.13 with $U = W_1 \cup W_2$. To do so we need to show (2.5) and (2.8) hold. First note

$$\begin{aligned}\partial U &= \overline{W_1 \cup W_2} \backslash (W_1 \cup W_2) = (\overline{W_1} \cup \overline{W_1}) \backslash (W_1 \cup W_2) \\ &\subseteq V \backslash (W_1 \cup W_2) = (V \backslash W_1) \cap (V \backslash W_2).\end{aligned}$$

Thus

$$\partial U \subseteq V \backslash W_1 \text{ and } \partial U \subseteq V \backslash W_2. \tag{2.16}$$

Now $S(p) \subseteq W_2$ from (2.15) and so $S(p) \cap \partial U = \emptyset$ from (2.16). Thus for $(y,0) \in \partial U$ we have $(y,0) \notin S(p)$ i.e. $(y,0) \in E \times [0,1]$ with $y \notin \mu N(y,0) + (1-\mu)p$ for all $\mu \in [0,1]$. Consequently (2.8) holds. Also (2.15) and (2.16) imply $B \cap \partial U = \emptyset$. Thus for $(y,\lambda) \in \partial U$ we have $(y,\lambda) \notin B = A \cap V$. This implies $(y,\lambda) \notin A$ since if $(y,\lambda) \in A$ then $(y,\lambda) \in \partial U \subseteq V$ and $(y,\lambda) \in A$ i.e. $(y,\lambda) \in A \cap V = B$, a contradiction. Thus $(y,\lambda) \in \partial U$ and $(y,\lambda) \notin A$ i.e. $y \notin N(y,\lambda)$. Consequently (2.5) holds. For each $t \in [0,1]$, Theorem 2.13 (see Remark 2.14) guarantees that there exists $x \in U_t$ with $x \in N(x,t)$ i.e. $x \in N_t(x)$ with $x \in U_t \subseteq V_t$. □

We now show that the ideas above can be applied to other natural situations. First let E be a completely regular topological vector space, Y a topological vector space, and U an open subset of E. Also let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear, continuous single valued map with $L + T : dom\, L \to Y$ an isomorphism (i.e. a linear homeomorphism); for convenience we say $T \in H_L(E,Y)$.

Definition 2.17. Let $F : \overline{U} \to 2^Y$. We say $F \in A(\overline{U}, Y; L, T)$ if $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$.

Definition 2.18. We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with $Lx \notin F(x)$ for $x \in \partial U \cap dom\, L$.

Definition 2.19. Let $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ if there exists a map $\Psi : \overline{U} \times [0,1] \to 2^Y$ with $\Psi \in A(\overline{U} \times [0,1], Y; L, T)$, $Lx \notin \Psi_t(x)$ for any $x \in \partial U \cap dom\, L$ and $t \in [0,1]$, $\Psi_1 = F$, $\Psi_0 = G$ (here $\Psi_t(x) = \Psi(x,t)$) and

$$\{x \in \overline{U} \cap dom\, L : Lx \in \Psi(x,t) \text{ for some } t \in [0,1]\}$$

is relatively compact.

For our next result we assume the following condition holds:

$$(2.17) \qquad \begin{cases} \text{if } X_2 = \overline{U} \text{ or } X_2 = \overline{U} \times [0,1] \text{ and if} \\ F \in A(\overline{U} \times [0,1], Y; L, T) \text{ and } f \in \mathbf{C}(X_2, \overline{U} \times [0,1]), \\ \text{then } F \circ f \in A(X_2, Y; L, T). \end{cases}$$

Remark 2.20. The result below (with (2.17) removed) also holds true if we use the following definition of $\cong$. Let $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ if there exists a map $\Psi : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(\Psi + T) : \overline{U} \times [0,1] \to K(E)$ upper semicontinuous and with $(L+T)^{-1}(\Psi(\cdot, \eta(\cdot)) + T) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $Lx \notin \Psi_t(x)$ for any $x \in \partial U \cap dom\, L$ and $t \in [0,1]$, $\Psi_1 = F$, $\Psi_0 = G$ and

$$\{x \in \overline{U} \cap dom\, L : Lx \in \Psi(x,t) \text{ for some } t \in [0,1]\}$$

is relatively compact.

The following condition will be assumed:

$$(2.18) \qquad \cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, Y; L, T).$$

Definition 2.21. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$. We say F is L-essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ there exists $x \in U \cap dom\, L$ with $Lx \in J(x)$. Otherwise F is L–inessential in $A_{\partial U}(\overline{U}, Y; L, T)$ i.e. there exists a map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ such that $Lx \notin J(x)$ for $x \in \overline{U} \cap dom\, L$.

In [5] we established the following result.

Theorem 2.22. *Let E be a completely regular topological vector space, Y a topological vector space, U an open subset of E, $L : dom\, L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E, Y)$, and assume* (2.17) *and* (2.18) *hold. Suppose Φ and Ψ are two maps in $A_{\partial U}(\overline{U}, Y; L, T)$ with $\Phi \cong \Psi$ in $A_{\partial U}(\overline{U}, Y; L, T)$. Then Φ is L–essential in $A_{\partial U}(\overline{U}, Y; L, T)$ if and only if $\Psi^\star$ is L–essential in $A_{\partial U}(\overline{U}, Y; L, T)$.*

Remark 2.23. The result in Theorem 2.22 (with (2.17) removed) holds if the definition of $\cong$ is as in Remark 2.20. If E is a normal topological vector space then the assumption that

$$\{x \in \overline{U} \cap dom\, L : Lx \in \Psi(x,t) \text{ for some } t \in [0,1]\}$$

is relatively compact can be removed in Definition 2.19 (and Remark 2.20) and we still obtain Theorem 2.22.

Let E be a completely regular topological vector space, Y a topological vector space, and U an open subset of $E \times [0,1]$. Also let $L : dom\, L \subseteq E \to Y$ be a linear (not necessarily continuous) single valued map; here $dom\, L$ is a vector subspace of E. Now let $\mathbf{L} : dom\, \mathbf{L} = dom\, L \times [0,1] \to Y \times [0,1]$ be given by $\mathbf{L}(y,\lambda) = (Ly,\lambda)$. Let $T : E \to Y$ be a linear, continuous single valued map with $L+T : dom\, L \to Y$ an isomorphism (i.e. a linear homeomorphism) and let $\mathbf{T} : E \times [0,1] \to Y \times [0,1]$ be given by $\mathbf{T}(y,\lambda) = (Ty, 0)$. Notice $(\mathbf{L}+\mathbf{T})^{-1}(y,\lambda) = ((L+T)^{-1}y, \lambda)$ for $(y,\lambda) \in Y \times [0,1]$.

For our next result we assume (2.17) (with Y replaced by $Y \times [0,1]$, L replaced by $\mathbf{L}$ and T replaced by $\mathbf{T}$) holds and in addition

$$\cong \text{ is an equivalence relation in } A_{\partial U}(\overline{U}, Y \times [0,1]; \mathbf{L}, \mathbf{T}) \tag{2.19}$$

and

$$\begin{cases} \text{if } F \in A(\overline{U}, Y; L, T) \text{ and if } \Psi(y,\mu) = (F(y),\mu), (y,\mu) \in \overline{U} \times [0,1], \\ \text{then } \Psi_\mu \in A(\overline{U}, Y \times [0,1]; \mathbf{L}, \mathbf{T}) \text{ for each } \mu \in [0,1] \text{ and} \\ \Psi \in A(\overline{U} \times [0,1], Y \times [0,1]; \mathbf{L}, \mathbf{T});\ \text{here } \Psi_\mu(x) = \Psi(x,\mu). \end{cases} \tag{2.20}$$

In [5] we established the following result.

Theorem 2.24. *Suppose $N \in A(\overline{U}, Y; L, T)$ with*

$$Lx \notin N(x,\lambda) \text{ for } (x,\lambda) \in \partial U \cap dom\, \mathbf{L}. \tag{2.21}$$

Let $H : \overline{U} \times [0,1] \to 2^{Y \times [0,1]}$ be given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$. In addition assume the following conditions hold:

$$\begin{cases} H_0 \text{ is essential in } A_{\partial U}(\overline{U}, Y \times [0,1]; \mathbf{L}, \mathbf{T});\ \text{here} \\ H_0(x,\lambda) = H(x,\lambda,0) = (N(x,\lambda),0) \text{ for } (x,\lambda) \in \overline{U} \end{cases} \tag{2.22}$$

and

$$\begin{cases} \{(x,\lambda) \in \overline{U} \cap dom\, \mathbf{L} : \mathbf{L}(x,\lambda) \in H(x,\lambda,\mu) \text{ for some } \mu \in [0,1]\} \\ \text{is relatively compact.} \end{cases} \tag{2.23}$$

Then H_1 is essential in $A_{\partial U}(\overline{U}, Y \times [0,1]; \mathbf{L}, \mathbf{T})$ so in particular there exists a $x \in U_1 \cap dom\, L$ with $Lx \in N(x,1)$; here $H_1(x,\lambda) = H(x,\lambda,1) = (N(x,\lambda),1)$ for $(x,\lambda) \in \overline{U}$.

Remark 2.25. In fact H_t is essential in $A_{\partial U}(\overline{U}, Y \times [0,1]; \mathbf{L}, \mathbf{T})$ for every $t \in [0,1]$; here $H_t(x,\lambda) = H(x,\lambda,t) = (N(x,\lambda),t)$ for $(x,\lambda) \in \overline{U}$. If E is a normal topological vector space then the assumption (2.23) can be removed in the statement of Theorem 2.24.

Remark 2.26. The result in Theorem 2.24 holds (with (2.17) and (2.20) removed) if the definition of $\cong$ is as in Remark 2.20 and if the following condition holds: $(\mathbf{L}+\mathbf{T})^{-1}(H(\cdot,\cdot,\eta(\cdot,\cdot))+\mathbf{T}) \in A(\overline{U}, E\times[0,1])$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$.

Our next applicable result is a special case of Theorem 2.24 and generalizes Theorem 2.13. Let E be a completely regular topological vector space, Y a topological vector space, and U an open subset of $E\times[0,1]$. Also let L, $\mathbf{L}$, T and $\mathbf{T}$ be as described before Theorem 2.24.

Theorem 2.27. *Suppose $N \in A(\overline{U}, Y; L, T)$ and assume* (2.17) *(with Y replaced by $Y\times[0,1]$, L replaced by $\mathbf{L}$ and T replaced by $\mathbf{T}$),* (2.19), (2.20) *and* (2.21) *hold. Let $H : \overline{U}\times[0,1] \to 2^{Y\times[0,1]}$ be given by $H(x,\lambda,\mu) = (N(x,\lambda),\mu)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$ and assume* (2.23) *holds. Let $G : E \to 2^Y$ and let $Q : \overline{U}\times[0,1] \to 2^{Y\times[0,1]}$ be given by $Q(x,\lambda,\mu) = (\mu N(x,\lambda) + (1-\mu)G(x), 0)$ for $(x,\lambda) \in \overline{U}$ and $\mu \in [0,1]$. Now suppose the following conditions hold:*

$$Lx \notin \mu N(x,0) + (1-\mu)G(x) \text{ for } (x,0) \in \partial U \cap dom\,\mathbf{L} \text{ and } \mu \in [0,1) \tag{2.24}$$

$$\begin{cases} Q_0 \text{ is essential in } A_{\partial U}(\overline{U}, Y\times[0,1]; \mathbf{L},\mathbf{T}); \text{ here} \\ Q_0(x,\lambda) = (G(x),0) \text{ for } (x,\lambda) \in \overline{U} \end{cases} \tag{2.25}$$

$$\begin{cases} Q_\mu \in A_{\partial U}(\overline{U}, Y\times[0,1]; \mathbf{L},\mathbf{T}) \text{ for each } \mu \in [0,1] \text{ and} \\ Q \in A_{\partial U}(\overline{U}\times[0,1], Y\times[0,1]; \mathbf{L},\mathbf{T}); \text{ here } Q_\mu(x,\lambda) = Q(x,\lambda,\mu) \end{cases} \tag{2.26}$$

and

$$\begin{cases} \{(x,\lambda) \in \overline{U} \cap dom\,\mathbf{L} : \mathbf{L}(x,\lambda) \in Q(x,\lambda,\mu) \text{ for some } \mu \in [0,1]\} \\ \text{is relatively compact.} \end{cases} \tag{2.27}$$

Then H_1 is essential in $A_{\partial U}(\overline{U}, Y\times[0,1]; \mathbf{L},\mathbf{T})$ so in particular there exists a $x \in U_1 \cap dom\,L$ with $Lx \in N(x,1)$; here $H_1(x,\lambda) = H(x,\lambda,1) = (N(x,\lambda),1)$ for $(x,\lambda) \in \overline{U}$.

Proof. Note (2.26) guarantees that $Q \in A(\overline{U}\times[0,1], E\times[0,1]; \mathbf{L},\mathbf{T})$. Also

$$\mathbf{L}(x,\lambda) \notin Q_\mu(x,\lambda) \text{ for } (x,\lambda) \in \partial U \cap dom\,\mathbf{L} \text{ and } \mu \in [0,1]. \tag{2.28}$$

To see this suppose there exists $(x,\lambda) \in \partial U \cap dom\,\mathbf{L}$ and $\mu \in [0,1]$ with $\mathbf{L}(x,\lambda) \in (\mu N(x,\lambda) + (1-\mu)G(x), 0)$. Then $\lambda = 0$ and $Lx \in \mu N(x,\lambda) + (1-\mu)G(x) = \mu N(x,0) + (1-\mu)G(x)$ which is a contradiction (note (2.24) is contradicted if $\mu \in [0,1)$ and (2.21) is contradicted if $\mu = 1$). Thus (2.28) is true and note $Q_0 \cong Q_1$ in $A_{\partial U}(\overline{U}, E\times[0,1]; \mathbf{L},\mathbf{T})$ (see above and (2.27)); here $Q_1(x,\lambda) = Q(x,\lambda,1) = (N(x,\lambda),0) = H_0(x,\lambda)$. Now Theorem 2.22 (note (2.17) (with Y replaced by $Y\times[0,1]$, L replaced by $\mathbf{L}$ and T replaced by $\mathbf{T}$), (2.19) and (2.25)) guarantees that

$$Q_1 (= H_0) \text{ is essential in } A_{\partial U}(\overline{U}, E\times[0,1]; \mathbf{L},\mathbf{T}).$$

Finally Theorem 2.24 (note (2.17) (with Y replaced by $Y \times [0,1]$, L replaced by $\mathbf{L}$ and T replaced by $\mathbf{T}$), (2.19), (2.20) and (2.23)) guarantees that H_1 is essential in $A_{\partial U}(\overline{U}, E \times [0,1]; \mathbf{L}, \mathbf{T})$. Thus there exists a $(x,\lambda) \in U \cap dom\,\mathbf{L}$ with $\mathbf{L}(x,\lambda) \in (N(x,\lambda), 1)$ i.e. $Lx \in N(x,\lambda)$ with $\lambda = 1$ i.e. $x \in dom\,L$ and $x \in U_1 = \{y \in E : (y,1) \in U\}$ and $Lx \in N(x,1)$. □

Remark 2.28. From the proof above note that for each $t \in [0,1]$ there exists $x_t \in U_t \cap dom\,L$ with $Lx_t \in N(x_t, t)$. If E is a normal topological space then (2.23) and (2.27) can be removed in the statement of Theorem 2.27.

Remark 2.29. The result in Theorem 2.27 holds (with (2.17) (with Y replaced by $Y \times [0,1]$, L replaced by $\mathbf{L}$ and T replaced by $\mathbf{T}$), (2.20) and (2.26) removed) if the definition of $\cong$ is as in Remark 2.20 and if the following condition holds: $(\mathbf{L}+\mathbf{T})^{-1}(H(\cdot,\cdot,\eta(\cdot,\cdot)) + \mathbf{T}), (\mathbf{L}+\mathbf{T})^{-1}(Q(\cdot,\cdot,\eta(\cdot,\cdot)) + \mathbf{T}) \in A(\overline{U}, E \times [0,1])$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$.

For our final result for convenience let E be a normal topological vector space, Y a topological vector space, U an open subset of $E \times [0,1]$, $G : E \to 2^Y$ and $N : E \times [0,1] \to 2^Y$. Also let L, $\mathbf{L}$, T and $\mathbf{T}$ be as described before Theorem 2.24. We will also assume $(\mathbf{L}+\mathbf{T})^{-1}(N+\mathbf{T})$ and $(\mathbf{L}+\mathbf{T})^{-1}(\mathbf{G}+\mathbf{T})$ are upper semicontinuous maps; here $\mathbf{G}(x,\lambda) = (G(x), 0)$ for $(x,\lambda) \in E \times [0,1]$. Let

$$\mathbf{S} = \{(x,0) \in E \times [0,1] \cap dom\,\mathbf{L} : Lx \in \mu N(x,0) + (1-\mu)G(x) \text{ for some } \mu \in [0,1]\}$$

and

$$A = \{(x,\lambda) \in E \times [0,1] \cap dom\,\mathbf{L} : Lx \in N(x,\lambda)\}.$$

For our next result we consider a continuous functional $\phi : E \times [0,1] \to \mathbf{R}$.

Theorem 2.30. *Suppose there exist constants a, b with $a < b$ such that if we set $V = \phi^{-1}(a,b)$ the following conditions are satisfied:*

$$\phi(A) \cap \{a,b\} = \emptyset \tag{2.29}$$

and

$$\mathbf{S} \subset V. \tag{2.30}$$

In addition for any subset U of V with $A \cap V \subseteq U$ and $\mathbf{S} \subseteq U$ we assume $N \in A(\overline{U}, Y; L, T)$ and (2.17) *(with Y replaced by $Y \times [0,1]$, L replaced by $\mathbf{L}$ and T replaced by $\mathbf{T}$),* (2.19), (2.20), (2.25) *and* (2.26) *hold. Then for each $\lambda \in [0,1]$ there exists a $x_\lambda \in V_\lambda \cap dom\,L$ with $Lx_\lambda \in N_\lambda(x_\lambda)$.*

Proof. Let $B = A \cap \phi^{-1}[a,b]$ and as in Theorem 2.16 we note $B = A \cap V$. Also $B \subset V$ and $\mathbf{S} \subset V$ are closed so there exists open subsets W_1 and W_2 of $E \times [0,1]$ with

$$B \subseteq W_1 \subseteq \overline{W_1} \subseteq V \text{ and } \mathbf{S} \subseteq W_2 \subseteq \overline{W_2} \subseteq V. \tag{2.31}$$

Let $U = W_1 \cup W_2$. As in Theorem 2.16 we have

$$\partial U \subseteq V \backslash W_1 \text{ and } \partial U \subseteq V \backslash W_2. \tag{2.32}$$

Now (2.31) and (2.32) imply $\mathbf{S} \cap \partial U = \emptyset$. Thus for $(y,0) \in \partial U$ we have $(y,0) \notin \mathbf{S}$ i.e. $(y,0) \in E \times [0,1] \cap dom\,\mathbf{L}$ with $Ly \notin \mu N(y,0) + (1-\mu)G(y)$ for all $\mu \in [0,1]$. Consequently (2.24) holds. Also (2.31) and (2.32) imply $B \cap \partial U = \emptyset$. Thus for $(y,\lambda) \in \partial U$ we have $(y,\lambda) \notin B = A \cap V$. This implies $(y,\lambda) \notin A$ since if $(y,\lambda) \in A$ then $(y,\lambda) \in \partial U \subseteq V$ and $(y,\lambda) \in A$ i.e. $(y,\lambda) \in A \cap V = B$, a contradiction. Thus $(y,\lambda) \in \partial U$ and $(y,\lambda) \notin A$ i.e. $(y,\lambda) \in \partial U$ and $(y,\lambda) \in dom\,\mathbf{L}$ and $Ly \notin N(y,\lambda)$. Consequently (2.21) holds. For each $t \in [0,1]$, Theorem 2.27 guarantees that there exists $x \in U_t \cap dom\,L$ with $Lx \in N(x,t)$ i.e. $Lx \in N_t(x)$ with $x \in V_t \cap dom\,L$. □

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