Dynamic Systems and Applications 21 (2012) 657-668

CLASSIFICATION OF SOLUTIONS OF SECOND ORDER IMPULSIVE NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

E. THANDAPANI[a], R. SAKTHIVEL[b], AND E. CHANDRASEKARAN[c]

[a]Ramanujan Institute for Advanced Study in Mathematics, University of Madras
Chennai - 600 005, India
ethandapani@yahoo.co.in
[b]Presidency College, Chennai-600 005, India
e_chandrasekaran@yahoo.com
[c]Presidency College, Chennai-600 005, India
r_ s_ vel@yahoo.com

ABSTRACT. This paper is focused on the following nonlinear impulsive neutral delay differential equation with positive and negative coefficients

$$\begin{cases} [r(t)(x(t)+c(t)x(t-\tau))']' + p(t)f(x(t-\delta) - q(t)g(x(t-\sigma)) = 0, \\ x(t_k) = I_k(x(t_k^-)), \quad x'(t_k) = J_k(x'(t_k^-)), \quad k = 1, 2, 3, \ldots, \end{cases}$$

where $0 \le t_0 < t_1 < t_2 < \cdots < t_k \cdots$ with $\lim_{k\to\infty} t_k = \infty$. For this equation, oscillation criteria are established.

AMS Subject Classification: 34C10, 34A37

Key Words and Phrases: Oscillation, Impulsive, Neutral Delay Differential Equations, positive and negative coefficients

1. INTRODUCTION

In the last few years, there has been an increasing interest in the study of oscillatory behavior of solutions of second order neutral delay differential equations with positive and negative coefficients, see for example [21, 20, 15] and the references cited therein. Recently, there has been increasing interest on the oscillation and nonoscillation of second order neutral delay differential equations with impulses, see the paper [1, 2, 3, 4, 5, 10, 11, 12, 17, 18], and references contained therein. However, to the best of our knowledge, there is little in the way of results for the oscillation of impulsive neutral delay differential equations with positive and negative coefficients, see for example [13] and the references cited therein. For the theory of delay differential equations and impulsive differential equations, see the recent books by Györi and Ladas [6] and Lakshmikantham et al. [10], respectively.

Received August 24, 2011

In this paper, we consider the second order nonlinear neutral delay differential equations with positive and negative coefficients

$$(1)\quad \begin{cases} [r(t)(x(t)+c(t)x(t-\tau))']' + p(t)f(x(t-\delta)) - q(t)g(x(t-\sigma)) = 0, \\ x(t_k) = I_k(x(t_k^-)), \quad x'(t_k) = J_k(x'(t_k^-)), \quad k = 1,2,3,\ldots, \\ x(t) = \phi(t), \quad t_0 - \tau \le t \le t_0, \end{cases}$$

where τ, δ, σ are positive real numbers, $m = \max(\tau, \delta, \sigma)$, $0 \le t_0 < t_1 < t_2 < \cdots < t_k \cdots$ with $\lim_{k\to\infty} t_k = \infty$ and $t_{k+1} - t_k > m$ for all $k \in N$, $c \in PC'([t_0,\infty), R_+)$ and $\phi, \phi' : [t_0 - \sigma, t_0] \to R$ have at most a finite number of discontinuities of the first kind and are right continuous at these points.

Through out this paper, we always assume that:

- (H_1) $f, g : [t_0 - m, +\infty) \times R \times R$ is continuous, $uf(t,u,v) > 0$, $ug(t,u,v) > 0$ for all $uv > 0$ and there exist a function $M(t) > 0$ such that $\frac{g(t,u,v)}{f(t,u,v)} \le M(t)$ for $u \ne 0$, where $M(t)$ is continuous in $[t_0, +\infty)$;
- (H_2) $r, p, q \in (R, [t_0, +\infty))$ with r is positive, $c(t) \ge 0$ and is nondecreasing for all $t \ge t_0$, $q(t) \ge 0$ for all $t \ge t_0$ and f is nondecreasing;
- (H_3) $I_k, J_k \in C(R,R)$ and there exist a positive constant a_k, $\overline{a_k}$, b_k, such that $\overline{a_k} \le \dfrac{I_k(x)}{x} \le a_k$, $J_k(x) = b_k(x)$ for all $x \ne 0$, $k = 1,2,3,\ldots$;
- (H_4) $c(t)$ and $c'(t)$ are right continuous on (t_k, t_{k+1}) with left lateral limits
$$c(t_k^-) = \frac{1}{b_k} c(t_k) \ \text{ and } \ c'(t_k^-) = \frac{1}{b_k} c'(t_k)$$
for all $k \in N$.

Let $J \subset R$ be an interval, we define $PC(J,R) = \{x : J \to R; x(t)$ is continuous everywhere except some t_k's at which $x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k)\}$; $PC'(J,R) = \{x \in PC(J,R) : x(t)$ is continuously differentiable everywhere except some t_k's at which $x'(t_k^-)$ and $x'(t_k^+)$ exist and $x'(t_k^-) = x'(t_k)\}$.

By a solution of equation of (1), we mean a function $x \in PC([t_0,+\infty), R) \bigcap PC'([t_0,+\infty), R)$ which satisfies (1).

In [18], Mingshu Peng and R. P. Agarwal considered equation (1) when $c(t) = 1$, $p(t) = 1$, and $q(t) = 0$ and classified all solutions of equation (1) into four classes and established conditions for the existence and nonexistence of solutions in these classes.

Motivated by the above observations, in this paper we divided all the solutions of equation (1) into the following four cases: Let S denote the set of all nontrivial solutions of equation (1). Then

$$\begin{aligned} M^+ &= \{x \in S : \text{ there exists } t_x \ge t_0 \text{ such that } x(t)x'(t) \ge 0 \text{ for } t \ge t_x\}; \\ M^- &= \{x \in S : \text{ there exists } t_x \ge t_0 \text{ such that} x(t)x'(t) \le 0 \text{ for } t \ge t_x\}; \\ OS &= \{x \in S : \text{ there exists } t_n \to \infty \text{ such that } x(t_n) = 0\}; \end{aligned}$$

$$WOS = \{x \in S : x(t) \text{ is nonoscillates but } x'(t) \text{ is oscillates}\}.$$

A nontrivial solution of equation (1) is said to be nonoscillatory, if it is eventually positive or eventually negative; otherwise, the solution is said to be oscillatory.

This paper is organized as follows. In Section 2, we prove two interesting lemmas, which will be used in Section 3 to prove our main theorems. To illustrate our results, examples are provided in Section 4.

2. SOME LEMMAS

We start by presenting a lemma which is borrowed from [10] replacing the left continuity by the right continuity of $m(t)$ and $m'(t)$ at t_k for all $k \in N$.

Lemma 1. *Suppose*

(i) *the sequence* $\{t_k\}_{k\in N}$ *satisfies* $0 \le t_0 < t_1 < t_2 < \cdots < t_k \cdots$ *with* $\lim_{t\to\infty} t_k = \infty$;

(ii) $m, m' : R_+ \to R$ *are right continuous on* $R_+ \backslash \{t_k : k \in N\}$, *there exist the lateral limits* $m(t_k^-)$, $m'(t_k^-)$, $m(t_k^+)$, $m'(t_k^+)$ *and* $m(t_k^+) = m(t_k)$, $k = 1, 2, 3, \ldots$;

(iii) *for* $k = 1, 2, 3, \ldots$ *and* $t \ne t_0$, *we have*

$$m'(t) \le p(t)m(t) + q(t), \quad t \ne t_k, \tag{2}$$

$$m(t_k) \le \alpha_k m(t_k^-) + \beta_k, \tag{3}$$

where $p, q \in C(R_+, R)$, α_k *and* β_k *are real constants with* $\alpha_k \ge 0$. *Then the following inequality holds*

$$\begin{aligned} m(t) \le m(t_0) \prod_{t_0<t_k<t} \alpha_k \exp\left(\int_{t_0}^t p(s)ds\right) + \int_{t_o}^t \prod_{s<t_k<t} \exp\left(\int_s^t p(u)du\right) q(s)ds \\ + \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} \alpha_j \exp\left(\int_{t_k}^t p(s)ds\right) \beta_k, \quad t \ge t_0. \end{aligned} \tag{4}$$

Lemma 2. *Let* $x(t)$ *be a positive solution of equation* (1) *for all* $t \ge T$ *and* $b_k \ge 1$. *If*

(i) $p(t) \ge Mq(t) \ge 0$ *for all* $t \ge t_0$;

(ii) $\sigma \ge \delta$.

If

$$\lim_{t\to\infty} \int_{t_j}^t \frac{1}{r(s)} \prod_{t_j<t_k\le s} \frac{b_k}{\max\{a_k, b_k\}} ds = +\infty, \tag{5}$$

then $y'(t) \ge 0$ *for* $t \in [t_k, t_k + 1)$, $t_k \ge T$, *where* $y(t) = x(t) + c(t)x(t - \tau)$.

Proof. Since $x(t)$ is positive we may assume without loss of generality that $x(t-\tau) > 0$, for all $t \ge T \ge t_0$. Then

$$y(t) = x(t) + c(t)x(t - \tau) > 0 \quad \text{for } t \ge T \ge t_0.$$

At first we prove that $y'(t_k^-) \geq 0$ for $t_k \geq T$. If it is not true, then there exist some j such that $t_j \geq T$, $y'(t_j^-) < 0$. Let $y'(t_j^-) = -\alpha$ with $\alpha > 0$. Since $t_{k+1} - t_k > m \geq \tau$ for each $k \in N$, we have

$$t_k < t_{k+1} - \tau < t_{k+1} \tag{6}$$

for all $k \in N$. Thus from the continuity of x and x' on $[t_{k-1}, t_k)$, inequality (6), assumptions (H_3) and (H_4) and equation (1), we have $y'(t_k) = b_k y'(t_k^-)$.

On the other hand, if $t \in [t_k, t_{k+1})$, $k \in N$ and $t_k \geq T$, it follows by (H_1) and equation (1) that

$$\begin{aligned} (r(t)y'(t))' &= -p(t)f(x(t-\delta)) + q(t)g(x(t-\sigma)) \\ &\leq -[p(t) - Mq(t)]f(x(t-\delta)) \leq 0. \end{aligned}$$

Hence

$$(r(t)y'(t))' \leq 0, \quad t > t_j, \quad t \neq t_k, \quad k = j+1, j+2, \ldots.$$

and

$$y'(t_k) = b_k y'(t_k^-), \quad k = j+1, j+2, \ldots.$$

Let $m(t) = r(t)y'(t)$. Then

$$m'(t) \leq 0, \quad t > t_j, \quad t \neq t_k, \quad k = j+1, j+2, \ldots \tag{7}$$

$$m(t_k) = b_k m(t_k^-), \quad k = j+1, j+2, \ldots. \tag{8}$$

Using Lemma 1 in (7) and (8), we obtain

$$m(t) \leq m(t_j^-) \prod_{t_j < t_k < t} b_k$$

or

$$y'(t) \leq \frac{r(t_j^-)y'(t_j^-)}{r(t)} \prod_{t_j < t_k < t} b_k. \tag{9}$$

For $k = j+1, j+2, \ldots$

$$\begin{aligned} y(t_k) &= x(t_k) + c(t_k)x(\tau(t_k - \tau)) \\ &\leq a_k(x(t_k^-)) + b_k c(t_k^-)x(\tau(t_k^- - \tau)) \end{aligned}$$

$$y(t_k) \leq \max\{a_k, b_k\} y(t_k^-), \quad k = j+1, j+2, \ldots. \tag{10}$$

Again by using Lemma 1 in (9) and (10), we have

$$\begin{aligned} y(t) &\leq y(t_j^-) \prod_{t_j < t_k < t} \max\{a_k, b_k\} + \int_{t_j}^{t} \prod_{s < t_k < t} \max\{a_k, b_k\} \left[\frac{r(t_j)y'(t_j^-)}{r(s)} \prod_{t_j < t_k < s} b_k \right] ds \\ &\leq \prod_{t_j < t_k < t} \max(a_k, b_k) \left[y(t_j^-) - \alpha r(t_j) \int_{t_j}^{t} \frac{1}{r(s)} \prod_{t_j < t_k \leq s} \frac{b_k}{\max\{a_k, b_k\}} ds \right]. \end{aligned}$$

Letting $t \to \infty$ in the last inequality, we see that by condition (5), $y(t) \to -\infty$ which is a contradiction. Therefore $y'(t_k^-) \geq 0$ for all $t_k \geq T$. Since $r(t)\, y'(t)$ is nonincreasing on $[t_k, t_{k+1})$, it is clear that

$$y'(t) \geq \frac{r(t_{k+1}^-)y'(t_{k+1}^-)}{r(t)} \geq 0 \quad \text{for } t \in [t_k, t_{k+1}), \quad t \geq T.$$

The proof is now complete. □

3. MAIN RESULTS

In this section, we present some sufficient conditions for the existence and nonexistence of solutions of equation (1) in the four classes.

Theorem 1. *Assume that*

(i) $x(t_k^- - \delta) = x(t_k - \delta)$;

(ii) $\sigma \geq \delta$.

If

$$\lim_{t\to\infty} \int_{t_j}^{t} \prod_{t_j < t_k \leq s} \frac{1}{b_k}[p(s) - Mq(s)]ds = +\infty, \tag{11}$$

then $M^+ = \emptyset$.

Proof. Suppose that equation (1) has a solution $x \in M^+$. Without loss of generality we may assume that $x(t) > 0$ for $t \geq t_0$ (the proof is similar for the case $x(t) < 0$). Then $x'(t) \geq 0$ for $t \geq T_1 \geq t_0$. It follows from Lemma 2 that $y'(t) \geq 0$ for $t \in [t_k, t_{k+1})$, $k = 1, 2, \ldots$. Define

$$w(t) = \frac{r(t)y'(t)}{f(x(t-\delta))}, \quad t \neq t_k \geq T_1.$$

Then $w(t_k^-) \geq 0$ $(k = 1, 2, 3, \ldots)$ and $w(t) \geq 0$ for $t \geq T_1$. Using (H_1) and the equation (1) for $t \neq t_k$, we have

$$\begin{aligned} w'(t) &= \frac{(r(t)y'(t))'}{f(x(t-\delta))} - \frac{r(t)y'(t)f'(x(t-\delta))x'(t-\delta)}{f^2(x(t-\delta))} \\ &\leq -p(t) + q(t)\frac{g(x(t-\sigma))}{f(x(t-\sigma))} \\ &\leq -(p(t) - Mq(t)). \end{aligned}$$

Also

$$w(t_k) = \frac{r(t_k)y'(t_k)}{f(x(t_k-\delta))} = \frac{b_k r(t_k)y'(t_k^-)}{f(x(t_k^- - \delta))} = b_k w(t_k^-).$$

Therefore $w(t)$ satisfies the following differential inequalities:

$$\begin{aligned} w'(t) &\leq -(p(t) - Mq(t)), \quad t > t_j, \quad t \neq t_k, \quad k = j+1, j+2, \ldots \\ w(t_k) &= b_k w(t_k^-), \quad k = j+1, j+2, \ldots. \end{aligned}$$

Using Lemma 1, we have,

$$\begin{aligned} w(t) &\le w(t_j^-) \prod_{t_j<t_k<t} b_k - \int_{t_j}^{t} \prod_{s<t_k<t} b_k (p(s) - Mq(s))ds \\ &\le \prod_{t_j<t_k<t} b_k \left[w(t_j^-) - \int_{t_j}^{t} \prod_{t_j<t_k\le s} \frac{1}{b_k}(p(s) - Mq(s))ds \right], \end{aligned}$$

which, in view of condition (11) and $w(t) \ge 0$, leads to contradiction as $t \to \infty$. The proof of the theorem is complete. □

Theorem 2. *Assume that*

(i) $\tau \le \sigma \le \delta$;
(ii) $\int_0^\alpha \frac{du}{f(u)} < \infty$ *and* $\int_{-\alpha}^0 \frac{du}{f(u)} > -\infty$;
(iii) f *is sub multiplicative, i.e.,* $f(uv) \le f(u)f(v)$ *for* $uv > 0$.

If (7) *holds and*

$$\sum_{k=0}^{+\infty} \int_{t_k^-}^{t_{k+1}} \frac{1}{r(s)f(1 + c(s))} \left[\lim_{t\to\infty} \int_s^t \prod_{s<t_k\le v} \frac{1}{b_k}(p(v) - Mq(v))dv \right] ds = +\infty, \tag{12}$$

then $M^- = \emptyset$.

Proof. Suppose that equation (1) has a solution $x \in M^-$. Without loss of generality we assume that there exist $t \ge T_1 \ge t_0$ such that $x(t) > 0$, $x'(t) \le 0$, $x(t - m) > 0$ and $x'(t - m) \le 0$ for $t \ge T_1$. Then $y(t) = x(t) + c(t)x(t - \tau) > 0$ and $y'(t) \le 0$ for $t \in [t_k, t_{k+1})$, $t_k \ge T_1$. Define

$$w(t) = \frac{r(t)y'(t)}{f(x(t - \delta))}, \quad t \ne t_k \ge T_1.$$

Since $\sigma \le \delta$ and $x'(t) \le 0$, using the argument in Theorem 1, we obtain the inequality

$$w(t) \le w(s) \prod_{s<t_k<t} b_k - \int_{t_j}^{t} \prod_{v<t_k<t} b_k(p(v) - Mq(v))dv.$$

From the above inequality, we have

$$\begin{aligned} w(s) &\ge \frac{w(t)}{\prod_{s<t_k<t} b_k} + \frac{\int_s^t \prod_{v<t_k<t} b_k\ (p(v) - Mq(v))dv}{\prod_{s<t_k<t} b_k} \\ &\ge \int_s^t \prod_{s<t_k\le v} \frac{1}{b_k}(p(v) - Mq(v))dv \end{aligned}$$

or

$$\frac{y'(s)}{f(x(s - \delta))} \ \ge\ \frac{1}{r(s)} \int_s^t \prod_{s<t_k\le v} \frac{1}{b_k}(p(v) - Mq(v))dv. \tag{13}$$

Since $x(t)$ is nonincreasing and $\tau \le \delta$, we see that

$$y(t) \le (1 + c(t)x(t - \delta)), \quad \text{for } t \in [t_k, t_{k+1}),$$

and

$$f(y(t)) \le f(1+c(t))f(x(t-\delta)), \quad \text{for } t \in [t_k, t_{k+1}).$$

Then

$$\frac{y'(t)}{f(1+c(t))f(x(t-\delta))} \ge \frac{y'(t)}{f(y(t))}. \tag{14}$$

From (13) and (14), we obtain

$$\frac{y'(s)f(1+c(s))}{f(y(s))} \quad \ge \quad \frac{1}{r(s)}\int_s^t \prod_{s<t_k\le v}\frac{1}{b_k}(p(v)-Mq(v))dv,$$

or

$$\frac{y'(s)}{f(y(s))} \quad \ge \quad \frac{1}{r(s)f(1+c(s))}\int_s^t \prod_{s<t_k\le v}\frac{1}{b_k}(p(v)-Mq(v))dv.$$

For $s \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots$, we have by condition (ii),

$$\sum_{k=0}^{+\infty}\int_{t_k^-}^{t_{k+1}} \frac{1}{r(s)f(1+c(s))}\left[\lim_{t\to\infty}\int_s^t \prod_{s<t_k\le v}\frac{1}{b_k}(p(v)-Mq(v))dv\right]ds$$
$$\le \sum_{k=0}^{+\infty}\int_{t_k^-}^{t_{k+1}}\frac{y'(s)}{f(y(s))}ds \le \sum_{k=0}^{+\infty}\int_{y(t_k^-)}^{y(t_{k+1})}\frac{du}{f(u)} < \infty.$$

This contradicts (12) and the proof is complete. □

Theorem 3. *Assume that* (11) *holds. If equation* (1) *has a nonoscillatory solution* $x(t) \in M^-$, *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. We may assume that $x(t) > 0$ for $t \ge t_0$. Then $x'(t) \le 0$, which implies there exist a constant β such that $\lim_{t\to\infty} x(t) = \beta$.

If $\beta = 0$, then the proof is complete. If not, then $\beta > 0$ and $x(t-\sigma) \ge \beta$ and $f(x(t-\delta)) \ge f(\beta)$. From equation (1),

$$\begin{aligned}(r(t)y'(t))' \quad &\le \quad -(p(t)-Mq(t))f(x(t-\delta))\\ &\le \quad -(p(t)-Mq(t))f(\beta).\end{aligned}$$

Let $m(t) = r(t)y'(t)$. In view of Lemma 2, $m(t) > 0$ and $m(t)$ satisfying the following differential inequality

$$m'(t) \le -(p(t)-Mq(t))f(\beta), \quad t > t_j, \quad t \ne t_k, \quad k = j+1, j+2, \ldots$$

and

$$m(t_k) = b_k m(t_k^-), \quad k = j+1, j+2, \ldots.$$

Then applying Lemma 1, we obtain

$$m(t) \le m(t_j^-)\prod_{t_j<t_k<t} b_k - f(\beta)\int_{t_j}^t \prod_{s<t_k<t} b_k(p(s)-Mq(s))ds,$$

or

$$m(t) \leq \prod_{t_j<t_k<t} b_k \left[m(t_j^-) - f(\beta) \int_{t_j}^{t} \prod_{t_j<t_k\leq s} \frac{1}{b_k}(p(s) - Mq(s))ds \right].$$

In view of (11), the right side of the above inequality is eventually negative, whereas the left side is nonnegative, which is a contradiction. This completes the proof. □

Theorem 4. *Assume that*

(i) $c(t)$ *is bounded;*
(ii) $x(t_k^- - \delta) = x(t_k - \delta)$.

If

$$\lim_{t\to\infty} \int_{t_j}^{t} \prod_{t_j<t_k<s} \frac{1}{b_k} \left[(p(s) - Mq(s)) \int_T^t \frac{1}{r(\xi)} d\xi \right] ds = +\infty, \tag{15}$$

then every solution in the class M^+ *is unbounded.*

Proof. Let x be a solution of equation (1), such that $x \in M^+$. Without loss of generality, we may assume that $x(t) \geq 0$, $x'(t) \geq 0$, $x(t-m) \geq 0$ and $x'(t-m) \geq 0$ for $t \geq T_1 \geq t_0$.

Let $y(t) = x(t) + c(t)x(t-\tau)$. Then by Lemma 2, $y(t) > 0$ and $y'(t) > 0$ for $t \in [t_k, t_{k+1})$. Define

$$w(t) = -\frac{r(t)y'(t)}{f(x(t-\delta))} \int_T^t \frac{1}{r(s)} ds, \quad t \neq t_k \geq T_1,$$

Then $w(t_k^-) \leq 0$ $(k = 1, 2, \dots)$ and $w(t) \leq 0$ for $t \geq T_1$. For $t \neq t_k$, we have

$$w'(t) = \left[-\frac{(r(t)y'(t))'}{f(x(t-\delta)} + \frac{r(t)y'(t)f'(x(t-\delta))x'(t-\delta)}{f^2(x(t-\delta))} \right] \int_T^t \frac{1}{r(s)} ds - \frac{y'(t)}{f(x(t-\delta)},$$

or

$$w'(t) \geq (p(t) - Mq(t)) \int_T^t \frac{1}{r(s)} ds - \frac{y'(t)}{f(x(t-\delta)}.$$

Now for $t = t_k$, we have,

$$w(t_k) = -\frac{r(t_k)y'(t_k)}{f(x(t_k-\delta))} \int_T^{t_k} \frac{1}{r(s)} ds = b_k w(t_k^-).$$

Therefore $w(t)$ satisfies the following differential inequality

$$w'(t) \geq v(t), \quad t > t_j, \quad t \neq t_k, \quad k = j+1, j+2, \dots$$

and

$$w(t_k) = b_k w(t_k^-), \quad t = t_k, \quad k = j+1, j+2, \dots$$

where

$$v(t) = (p(t) - Mq(t)) \int_T^t \frac{1}{r(s)} ds - \frac{y'(t)}{f(x(t-\delta)}.$$

Using Lemma 1, we obtain

$$\begin{aligned} w(t) &\geq w(t_j^-) \prod_{t_j<t_k<t} b_k + \int_{t_j}^t \prod_{s<t_k<t} b_k v(s) ds, \\ &\geq \prod_{t_j<t_k<t} b_k \left[w(t_j^-) + \int_{t_j}^t \prod_{t_j<t_k\leq s} \frac{1}{b_k} \left((p(s) - Mq(s)) \int_T^t \frac{1}{r(\xi)} d\xi - \frac{y'(s)}{f'(x(s-\delta))} \right) ds \right], \end{aligned}$$

or

$$\begin{aligned} w(t) \quad \geq \quad & \prod_{t_j<t_k<t} b_k \Big[w(t_j^-) + \int_{t_j}^t \prod_{t_j<t_k\leq s} \frac{1}{b_k} \left((p(s) - Mq(s)) \int_T^t \frac{1}{r(\xi)} d\xi \right) ds \\ & - \int_{t_j}^t \prod_{t_j<t_k\leq s} \frac{1}{b_k} \frac{y'(s)}{f'(x(s-\delta))} ds \Big]. \end{aligned}$$

Since $\frac{y'(t)}{f(x(t-\delta))}$ is positive in $[t_k, t_{k+1})$, the limit as $t \to \infty$ of the last integral on the right hand side of the last inequality exists. Assume that

$$\int_{t_j}^t \prod_{t_j<t_k\leq s} \frac{1}{b_k} \frac{y'(s)}{f'(x(s-\delta))} ds = M_1 < \infty.$$

This gives $\lim\limits_{t\to\infty} w(t) = \infty$, which contradicts $w(t)$ being negative. Thus

$$\lim_{t\to\infty} \int_{t_j}^t \prod_{t_j<t_k\leq s} \frac{1}{b_k} \frac{y'(s)}{f'(x(s-\delta))} ds = \infty. \tag{16}$$

Since $f(x)$ is nondecreasing, we have

$$f(x(t-\delta)) \quad \geq \quad f(x(t_1 - \delta))$$

or

$$\frac{1}{f(x(t-\delta))} \quad \leq \quad \frac{1}{f(x(t_1-\delta))} = M_2.$$

Therefore

$$\int_{t_j}^t \prod_{t_j<t_k\leq s} \frac{1}{b_k} \frac{y'(s)}{f'(x(s-\delta))} ds \quad \leq \quad M_2 \int_{t_j}^t \prod_{t_j<t_k\leq s} \frac{1}{b_k} y'(s) ds.$$

From (16), we obtain

$$\lim_{t\to\infty} y(t) = \infty. \tag{17}$$

Since $y(t) = x(t) + c(t)x(t-\tau)$ and $x'(t) \geq 0$, we have $y(t) \leq (1 + c(t))x(t)$. In view of (15), and the fact that $c(t)$ is bounded, this implies that $\lim\limits_{t\to\infty} x(t) = \infty$. This completes the proof. $\square$

The following result is an immediate consequence of Theorem 1, Theorem 2 and Theorem 3.

Theorem 5. *If* (5) *and* (11) *hold with* $x(t_k^- - \tau) = x(t_k - \tau)$ *and* $\sigma = \delta$, *then every solution of equation* (1) *is either oscillatory or weakly oscillatory.*

4. EXAMPLES

In this section we present some examples to illustrate the main results.

Example 1. Consider the following second order impulsive type neutral delay differential equation

$$\begin{cases} [\frac{1}{t}x(t) + x(t-1)]'' + tx(t) \\ \qquad -(t-1)x(t-1) = 0, \quad t \geq 1, \quad t \neq 2^k, \quad k = 1, 2, 3, \ldots \\ x(2^k) = (\frac{k+1}{k})x((2^k)^-), \quad x'(2^k) = (\frac{k+1}{k})x'((2^k)^-), \quad k = 1, 2, 3, \ldots. \end{cases} \tag{18}$$

Here $r(t) = \frac{1}{t}$, $c(t) = 1$, $a_k = b_k = \frac{k+1}{k}$, $p(t) = t$, $q(t) = t - 1$, $t_k = 2^k$, $t_0 = 1$, $\tau = 1$, $\sigma = 1$, $\delta = 0$, $f(u) = g(u) = u$, and $M = 1$. Clearly conditions (i) and (ii) of Theorem 1 are satisfied. Further a straightforward calculation shows that

$$\begin{aligned} \lim_{t\to\infty} \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{1}{b_k}[p(s) - Mq(s)]ds &= \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{1}{b_k} ds \\ &= \int_{1}^{t_1} \prod_{1<t_k<s} \frac{1}{b_k} ds + \int_{t_1}^{t_2} \prod_{1<t_k<s} \frac{1}{b_k} ds + \cdots \\ &= \int_{1}^{t_1} \prod_{1<t_k<s} \frac{k}{k+1} ds + \int_{t_1}^{t_2} \prod_{1<t_k<s} \frac{k}{k+1} ds + \cdots \\ &= 1 + \frac{1}{2} \times 2 + \frac{1}{2} \times \frac{2}{3} \times 2^2 + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times 2^3 \\ &= +\infty. \end{aligned}$$

Therefore all the conditions of Theorem 1 are satisfied and hence $M^+ = \emptyset$.

Example 2. Consider the following second order impulsive type neutral delay differential equation

$$\begin{cases} [t^2(t-1)^2(x(t) + x(t-1))']' + 4t(t-2)^3 x^{\frac{1}{3}}(t-2) \\ \qquad -2(t-1)^3 x^{\frac{1}{3}}(t-1) = 0, \quad t \geq 1, \quad t \neq 2^k, \quad k = 1, 2, 3, \ldots \\ x(2^k) = x((2^k)^-), \quad x'(2^k) = x'((2^k)^-), \quad k = 1, 2, 3, \ldots \end{cases} \tag{19}$$

Here $r(t) = t^2(t-1)^2$, $c(t) = 1$, $a_k = b_k = 1$, $p(t) = 4t(t-2)^3$, $q(t) = 2(t-1)^3$, $t_k = 2^k$, $t_0 = 1$, $\tau = 1$, $\sigma = 1$, $\delta = 2$, $f(u) = g(u) = u^{\frac{1}{3}}$, and $M = 1$. It is easy to see that all conditions of Theorem 2 are satisfied, therefore $M^- = \emptyset$.

Example 3. Consider the following second order impulsive type neutral delay differential equation

$$\begin{cases} [\frac{1}{t}(x(t)+x(t-1))']' + \frac{1}{t^2}x(t) \\ \qquad\qquad -\frac{t-1}{t^3}x(t-1) = 0, \quad t \geq 1, \quad t \neq 2^k, \quad k = 1, 2, 3, \ldots \\ x(2^k) = (\frac{k+1}{k})x((2^k)^-), \quad x'(2^k) = (\frac{k+1}{k})x'((2^k)^-), \quad k = 1, 2, 3, \ldots \end{cases} \tag{20}$$

Here $r(t) = \frac{1}{t}$, $c(t) = 1$, $a_k = b_k = \dfrac{k+1}{k}$, $p(t) = \frac{1}{t^2}$, $q(t) = \frac{t-1}{t^3}$, $t_k = 2^k$, $t_0 = 1$, $\tau = 1$, $\sigma = 1$, $\delta = 0$, $f(u) = g(u) = u$, and $M = 1$. It can be easily see that all conditions of Theorem 4 are satisfied. Hence every solution in the class M^+ is unbounded.

We conclude this paper with the following remark.

Remark: It would be interesting to extend the results of this paper to equation (1) when the function $c(t)$ is negative.

Acknowledgements: The authors thank the referee for his/her comments and suggestions which improved the content of the paper.

REFERENCES

[1] L. Berenzansky and E. Braverman, On oscillation of a second order impulsive delay differential equation, *J. Math. Anal. Appl.*, **39**, 217–225, (2000).

[2] E. M. Bonotto, L. P. Gimenses and M. Federson, Oscillation for a second order neutral differential equation with impulses, *Cad. Mat.* , **9**, 169–190,(2008).

[3] Jin-Fa Cheng and Yu-Ming Chu, Oscillation of second order neutral impulsive differential equations, *J. Ineq. Appl.*, Volume 2010 (2010), 29 pages.

[4] L. P. Gimenses and Marica Federson, Oscillation by impulses for a seond order delay differential eqution, *Cad. Mat.*, **6**, 181–191,(2005).

[5] K. Gopalsamy and B. G. Zhang, On delay differential equations with mpulises, *J. Math. Anal. Appl.*, **139**, 110–122 ,(1989).

[6] I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equtions With Applications*, Clarendon Press, Oxford, 1991.

[7] Chengjun Guo and Zhiting Xu, On the osillation of second order linear impulsive differential equations, *Diff. Eqns. Appl.*, **2**, 319–330, (2010).

[8] Zhenlai Han, Tongxing Li, Shurong Sun and Weisong Chen, *Adv.Diff. Equ.*, Volume 2010, (2010), 8 pages.

[9] Zhimin He and Weigao Ge, Oscillation of impulsive delay differential equations, *Indian. J. Pure. Appl. Math.*, **31(9)**, 1089–1101, (2000).

[10] Lakshmikantham. Y, Bainov. D, Simenov. P. S., *Theoary of Impulsive Differential Equations*, World Scientific, Singapore , 1989.

[11] Wu Xiu-Li, Chen si-Yang and Hong Ji, Oscillation of a class of second order nonlinear ODE with impulses, *Appl. Math. Comput.* **138**, 181–188, (2003).

[12] Xiaodi Li, Oscillation properties of second order delay differential-difference equations with impulses, *Advances in Appl. Math. Anal.* **3**, 55–66,(2008).

[13] Z. Luo, X. Lin and J. Shen, Oscillation of impulsive neutral differential equations with positive and negative coefficients, *Indian. J. Pure. Appl. Math.* **31(7)**, 753–766,(2000).

[14] Qiong Meng and Jurang Yan, Bounded oscillation for second order non-linear neutral delay differential equations in critical and non-critical cases, *Nonlinear Anal.* **64**, 7543–1561, (2006).

[15] Ö. Öcalan, M. K. Yildiz and B. Karpuz, On the oscillation of nonlinear neutral differential equation with positive and negative coefficients, *Dynam. Sys. Appl.* **17** ,667–676,(2008)

[16] Mingshu Peng, Oscillation caused by impulses, *J. Math. Anal. Appl.* **255**, 163–176, (2001).

[17] Mingshu Peng and Weigao Ge, Oscillation criteria for second order nonlinear differential equations with impulses, *Comput. Math. Appl.* **39**, 217–255,(2000).

[18] Mingshu Peng and R. P. Agarwal, Oscillation theorems of second order nonlinear neutral delay differential equations under impulsive perturbations, *Indian. J. Pure. Appl. Math.* **33(7)**, 1017–1029, (2002).

[19] Yuri. V. Rogovchenko, Oscillation of a second order nonlinear delay differential equations, *Funk. Ekva.* **43**, 1–29, (2000).

[20] Y. Shoukaku, Oscillation of solutions of second order neutral differential equations with positive and negative coefficients, *J. Appl. Anal.* **15(2)**, 281–298, (2009).

[21] E. Thandapani, V. Muthulakshmi and John R. Graef, Classification of solutions of second order nonlinear neutral delay differential equations with positive and negative coefficients, *Int. J. Pure. Appl. Math.* **64(1)**, 159–170, (2010).

[22] Qigui Yang, Lijun Yang and Siming Zhu, Interval criteria for oscillation of second order nonlinear neutral differential equations, *Comput. Math. Appl.* **46**, 903–918, (2003).

[23] Jurang Yan, Oscillation properties of a second order impulsive delay differential equation, *Comput. Math. Appl.* **47**, 253–258 (2004).

[24] Y. Zhang, A. Zhao and J. Yan, Oscillation criteria for impulsive delay differential equations, *J. Math. Anal. Appl.* **205**, 461–470 ,(1997).

[25] A. Zhao and J. Yan, Existence of positive solutions for delay differential equations with impulses, *J. Math. Anal. Appl.* **210**, 667–678 (1997).