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ASYMPTOTIC STABILITY OF SWITCHING DIFFUSIONS HAVING SUB-EXPONENTIAL RATES OF DECAY

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ABSTRACT. This work focuses on stability of switching diffusions, in which both continuous dynamics and discrete events coexist. Our attention is devoted to the case that one has asymptotic stability but the decay rates are slower than exponential. The main effort is on obtaining asymptotic results in the almost sure sense. Sufficient conditions are provided for switching diffusion systems whose switching component depends on the diffusion process. Then Markovian regime-switching diffusions are treated, in which weaker conditions are needed. In addition, systems with delays are also treated. Examples are provided to demonstrate our results.

Key Words. Switching diffusion, stability, sub-exponential decay rate.

1. INTRODUCTION

This work is concerned with stability of switching diffusion processes (also known as hybrid switching diffusions). Our focus is on systems whose asymptotic rates of decay are much slower than exponential (termed as sub-exponential henceforth). The switching diffusion is a two component process, whose dynamics may be described by a stochastic differential equation of the continuous component together with transition rule of the discrete component. One of the distinct features of the switching diffusions is the coexistence of continuous dynamics and discrete events. In our setup, the discrete events are modeled as a random switching process $\alpha(t)$ living in a finite state space $\mathcal{M}$, When the discrete event process takes a particular value $i \in \mathcal{M}$, the continuous component evolves with the drift and diffusion coefficient depend on i.

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When the discrete component jumps from i to $j \neq i$, the continuous component is a solution of the stochastic differential equation with drift and diffusion depending on j. In contrast with the Markovian switching processes [16], we allow the switching process to be diffusion dependent. As a result, the switching process alone is not a Markov chain [20].

Recently, switching diffusions have received much attention. This is because of the needs of modeling and analysis of complex systems, in which the switching process is used to model the random environment change. Emerging applications have arisen in control and optimization, multi-agent systems, networked systems, wireless communications, financial engineering, and risk management. In applications, the underlying systems have often been in operation for a long time. Thus stability is of crucial concern. While various modes of stability for switching diffusions have been invested in the literature from different angles [9, 14, 15], less is known for asymptotic stability of systems whose decay rates are slower than exponential. This brought us to the current work.

For diffusion systems, rates of decays different from that of exponential were considered by many authors; see [3, 4, 10, 11, 13], and the references therein. In [11], Liu and Mao obtained Lyapunov component-like results with appropriate bounding constant leading to the desired decay rates. Inspired by their work, this paper concentrates on that of switching diffusions. Because of the switching process involved, the analysis becomes more involved. In addition, we need to take care of the diffusion dependent switching resulting in further complications. Nevertheless, we show that similar decay rates as that of diffusions can also be obtained.

The rest of the paper is arranged as follows. The precise problem formulation is given next. Section 3 presents the main results of sub-exponential decay rates. To further our understanding, Section 4 provide weaker conditions for Markovian switching diffusions in which the switching process is a Markov chain independent of the Brownian motion. Section 5 is devoted to several examples. Section 6 extends the results to systems with delays. Finally, concluding remarks are made in Section 7. Before proceeding further, a word of notation is in order. Throughout the paper, we use z' to denote the transpose of $z \in \mathbb{R}^{l_1 \times l_2}$ with $l_1 \geq 1$, $l_2 \geq 1$, whereas $\mathbb{R}^{l\times 1}$ is simply written as $\mathbb{R}^l$; the Euclidean norm of a vector x is denoted by $|x|$. For a matrix A, its trace norm is denoted by $|A| = \sqrt{\mathrm{tr}(AA')}$. For a set B, its indicator function is denoted by $\mathbf{I}_B(\cdot)$.

2. FORMULATION

This section presents the formulation of the problem. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $(X(t), \alpha(t))$ be a two-component Markov process such that $X(\cdot)$

takes values in $\mathbb{R}^r$, and $\alpha(\cdot)$ takes values in a finite set $\mathcal{M} := \{1, 2, \ldots, m\}$. The processes that we are interested in belong to a class of diffusion processes with regime switching, which consists of the usual diffusion processes together with a switching process that depends on the diffusion process and in which random switches take place within the finite set $\mathcal{M}$, representing the possible regimes of the environment. Let $f : \mathbb{R}^r \times [0, \infty) \times \mathcal{M} \mapsto \mathbb{R}^r$, $g : \mathbb{R}^r \times [0, \infty) \times \mathcal{M} \mapsto \mathbb{R}^{r\times d}$ be suitable functions. Consider the dynamic system given by

$$\begin{cases} dX(t) = f(X(t), t, \alpha(t))dt + g(X(t), t, \alpha(t))dW(t), & t \geq 0; \\ X(0) = x_0 \in \mathbb{R}^r, & \alpha(0) = i_0 \in \mathcal{M}. \end{cases} \tag{2.1}$$

where $W(t)$ is a d-dimensional standard Brownian motion, and $\alpha(\cdot)$ obeys the following transition rule:

$$\mathbb{P}\left(\alpha(t + \Delta t) = j | \alpha(t) = i, (X(t), \alpha(t)), s \leq t\right) = q_{ij}(X(t))\Delta t + o(\Delta t), \quad i \neq j.$$

Throughout the paper, we assume that the drift and diffusion coefficients grow at most linearly in x, satisfy the Lipschitz condition in x, and $Q(x)$ is a bounded and continuous function. Thus, (2.1) has unique global solution denoted by $X(t, x_0)$; see [20, Chapter 2] for more details).

Remark 2.1. Note that the evolution of the discrete event component $\alpha(\cdot)$ can be represented by a stochastic integral with respect to a Poisson random measure (see, for example, [2] and [17], see also [16]). For each $x \in \mathbb{R}^r$ and $i, j \in \mathcal{M}$ with $j \neq i$, let $\Delta_{ij}(x)$ be the consecutive(with respect to the lexicographic ordering on $\mathcal{M} \times \mathcal{M}$) left closed and right open intervals of the real line, each having length $q_{ij}(x)$. Define a function $h : \mathbb{R}^r \times \mathcal{M} \times \mathbb{R} \mapsto \mathbb{R}$ by

$$h(x, i, z) = \sum_{j\in\mathcal{M}} (j - i)\mathbf{I}_{\{z\in\Delta_{ij}(x)\}}$$

That is, with the partition $\{\Delta_{ij}(x) : i, j \in \mathcal{M}\}$ used, for each $i \in \mathcal{M}$, if $z \in \Delta_{ij}(x)$,$h(x, i, z) = j - i$; otherwise $h(x, i, z) = 0$. Then

$$d\alpha(t) = \int_{\mathbb{R}} h(X(t^-), \alpha(t^-), z)N_1(dt, dz)$$

where $N_1(dt, dz)$ is a Poisson measure with intensity $dt \times m_1(dz)$, and $m_1(\cdot)$ is the Lebesgue measure on $\mathbb{R}$. The Poisson measure $N_1(dt, dz)$ is independent of the Brownian motion $W(\cdot)$. In what follows, denote the associated compensated Poisson measure by $\widetilde{N}_1(ds, dz) = N_1(ds, dz) - ds \times m_1(dz)$.

The process $(X(t), \alpha(t))$ has a generator $\mathcal{L}$ as follows. For each $i \in \mathcal{M}$, $V(\cdot, \cdot, i) \in C^{2,1}(\mathbb{R}^r \times [0, \infty); \mathbb{R})$,

$$\mathcal{L}V(x, t, i) := \widetilde{\mathcal{L}}V(x, t, i) + \mathcal{Q}V(x, t, i) \tag{2.2}$$

where

$$\begin{aligned}\widetilde{\mathcal{L}}V(x,t,i) &:= \frac{\partial}{\partial t}V(x,t,i) + \sum_{k=1}^{r} f_k(x,t,i)\frac{\partial}{\partial x_k}V(x,t,i)\\ &\quad+\frac{1}{2}\sum_{k,l=1}^{r}\sum_{j=1}^{m} g^{kj}(x,t,i)g^{lj}(x,t,i)\frac{\partial^2}{\partial x_k\partial x_l}V(x,t,i)\\ \mathcal{Q}V(x,t,i) &:= \sum_{j\in\mathcal{M}} q_{ij}(x)V(x,t,j)\\ &= \sum_{j\neq i, j\in\mathcal{M}} q_{ij}(x)(V(x,t,j)-V(x,t,i))\end{aligned}$$

In what follows, when we work with Lyapunov functions, $V(\cdot,\cdot,\cdot):\mathbb{R}^r\times[0,\infty)\times\mathcal{M}\mapsto\mathbb{R}_+$. For subsequent use, we define two new operators:

$$\begin{aligned}\mathcal{Q}_{\mathcal{B}}V(x,t,i) &:= \sum_{i,j=1}^{r}\sum_{k=1}^{m} g^{ik}(x,t,i)g^{jk}(x,t,i)\frac{\partial}{\partial x_i}V(x,t,i)\frac{\partial}{\partial x_j}V(x,t,i)\\ \mathcal{Q}_{\mathcal{M}}V(x,t,i) &:= \sum_{j\neq i, j\in\mathcal{M}} q_{ij}(x)|V(x,t,j)-V(x,t,i)|^2.\end{aligned}$$

We have the following generalized Itô formula, which is deemed to be well known. For more details can be found in [17], [16] and [20].

Lemma 2.1. *Using the operator $\mathcal{L}$ defined in* (2.2)*, the generalized Itô formula is given by*

$$\begin{aligned}V(X(t),t,\alpha(t)) =& V(x_0,t,i_0) + \int_0^t \mathcal{L}V(X(s),s,\alpha(s))ds\\ &+\int_0^t\sum_{i=1}^{r}\sum_{k=1}^{d} g^{ik}(X(s),s,\alpha(s))\frac{\partial}{\partial x_i}V(X(s),s,\alpha(s))dW^k(s)\\ &+\int_0^t\int_{\mathbb{R}}\big[V(X(s^-),s^-,\alpha(s^-)+h(X(s^-),\alpha(s^-),z))\\ &-V(X(s^-),s^-,\alpha(s^-))\big]\widetilde{N}_1(ds,dz).\end{aligned}$$

3. MAIN RESULTS

Let us now give the precise definition of almost sure stability with decay rate function $\lambda(t)$. Again, we recall that our main concern is for switching diffusions. More detailed discussion for diffusion systems can be found in [11] and [3].

Definition 3.1. Suppose that $\lambda(t)$ is a positive function satisfying $\lambda(t)\uparrow+\infty$ as $t\to+\infty$. Assume there exists a sufficiently large $T>0$ such that

(1) $\log\lambda(t)$ is uniformly continuous over $t\geq T$;
(2) there exists a nonnegative constant $\tau\geq 0$ such that

$$\limsup_{t\to\infty}\frac{\log\log t}{\log\lambda(t)}\leq\tau.$$

The solution $X(t)$ of equation (2.1) is said to be *almost surely stable with rate function* $\lambda(t)$ *of order* $\gamma > 0$ if and only if

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log \lambda(t)} \le -\gamma \quad \text{a.s.}$$

for all initial conditions $X(0) = x_0 \in \mathbb{R}^r, \alpha(0) = i_0 \in \mathcal{M}$. If in addition 0 is solution to (2.1), then we call the zero solution is almost surely stable with rate function $\lambda(t)$ of order at least γ.

Theorem 3.1. *Suppose that for each $i \in \mathcal{M}$, $V(\cdot,\cdot,i) \in C^{2,1}(\mathbb{R}^r \times [0,\infty); \mathbb{R}_+)$, that $c(t) > 0$ is a continuous positive function, and that $\psi_1(t)$, $\psi_2(t)$ are two continuous nonnegative functions. Assume that for all $(x,t,i) \in \mathbb{R}^r \times [0,\infty) \times \mathcal{M}$, there exist two continuous non-increasing positive functions $\xi(t) > 0$, $\zeta(t) > 0$, positive constant $p > 0$, and real numbers ϖ, θ, ρ, μ, η such that*

(a) $c(t)|x|^p \le V(x,t,i)$;
(b) $\mathcal{L}V(x,t,i) + \xi(t)\mathcal{Q}_{\mathcal{B}}V(x,t,i) + \zeta(t)\mathcal{Q}_{\mathcal{M}}V(x,t,i) \le \psi_1(t) + \psi_2(t)V(x,t,i)$;
(c) *the following conditions hold:*

$$\liminf_{t\to\infty} \frac{\log c(t)}{\log \lambda(t)} \ge \varpi, \quad \limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_1(s)ds\right)}{\log \lambda(t)} \le \theta, \quad \limsup_{t\to\infty} \frac{\int_0^t \psi_2(s)ds}{\log \lambda(t)} \le \rho,$$
$$\liminf_{t\to\infty} \frac{\log \xi(t)}{\log \lambda(t)} \ge -\mu, \quad \liminf_{t\to\infty} \frac{\log \zeta(t)}{\log \lambda(t)} \ge -\eta.$$

Then the solution $X(t)$ of equation (2.1) *satisfies*

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log \lambda(t)} \le -\frac{\varpi - [\mu \vee \eta \vee \theta + \tau + \rho]}{p} \quad a.s.$$

Proof. The proof is inspired by the work of Liu and Mao [11]. However, we must treat the switching process with care.

Step 1. By the generalized Itô formula in Lemma 2.1 and the definition of $\mathcal{L}$, we have

$$V(X(t),t,\alpha(t)) = V(x_0,0,i_0) + \int_0^t \mathcal{L}V(X(s),s,\alpha(s))ds + M_1(t) + M_2(t) \tag{3.1}$$

where

$$M_1(t) = \int_0^t \sum_{i=1}^r \sum_{k=1}^d g^{ik}(X(s),s,\alpha(s)) \frac{\partial}{\partial x_i} V(X(s),s,\alpha(s))dW^k(s)$$
$$M_2(t) = \int_0^t \int_{\mathbb{R}} \big[V(X(s^-),s^-,\alpha(s^-) + h(X(s^-),\alpha(s^-),z)) \\ -V(X(s^-),s^-,\alpha(s^-))\big] \widetilde{N}_1(ds,dz).$$

are two martingale terms.

Step 2. By condition (1), the uniform continuity of $\log\lambda(t)$, in Definition 3.1, for any $\varepsilon > 0$ there exist two positive integers $N = N(\varepsilon)$ and $k_0 = k_0(\varepsilon)$ such that,

$$\left|\log\lambda\left(\frac{k}{2^N}\right) - \log\lambda(t)\right| \le \varepsilon \tag{3.2}$$

if $\frac{k-1}{2^N} \le t \le \frac{k}{2^N}, k \ge k_0(\varepsilon)$.

Owing to the exponential martingale inequality, by the definition of $\mathcal{Q}_{\mathcal{B}}$ and $\mathcal{Q}_{\mathcal{M}}$, for any u_i, v_i, and $w_i > 0$ with $i = 1, 2$,

$$P\left\{\omega : \sup_{0\le t\le w_1}\left[M_1(t) - \int_0^t \frac{u_1}{2}\mathcal{Q}_{\mathcal{B}}V(X(s), s, \alpha(s))ds\right] > v_1\right\} \le e^{-u_1v_1},$$

and

$$P\left\{\omega : \sup_{0\le t\le w_2}\left[M_2(t) - \int_0^t \frac{u_2}{2}\mathcal{Q}_{\mathcal{M}}V(X(s), s, \alpha(s))ds\right] > v_2\right\} \le e^{-u_2v_2}.$$

In particular, for $k = 2, 3, \ldots$, taking

$$u_1 = 2\xi\left(\frac{k}{2^N}\right), \quad v_1 = \xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}, \quad w_1 = \frac{k}{2^N};$$
$$u_2 = 2\zeta\left(\frac{k}{2^N}\right), \quad v_2 = \zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}, \quad w_2 = \frac{k}{2^N};$$

By virtue of the Borel-Cantelli lemma, there exist two integers $k_1(\varepsilon, \omega) > 0$ and $k_2(\varepsilon, \omega) > 0$ for almost all $\omega \in \Omega$ such that

$$M_1(t) \le \xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N} + \xi\left(\frac{k}{2^N}\right)\int_0^t \mathcal{Q}_{\mathcal{B}}V(X(s), s, \alpha(s))ds$$
$$M_2(t) \le \zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N} + \zeta\left(\frac{k}{2^N}\right)\int_0^t \mathcal{Q}_{\mathcal{M}}V(X(s), s, \alpha(s))ds$$

for all $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon) \vee k_1(\varepsilon, \omega) \vee k_2(\varepsilon, \omega)$.

Step 3. Substituting these equations into (3.1) and using condition (b), we have

$$\begin{aligned}
&V(X(t), t, \alpha(t))\\
&\le V(x_0, 0, i_0) + \int_0^t \mathcal{L}V(X(s), s, \alpha(s))ds\\
&\quad + \xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N} + \zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}\\
&\quad + \xi\left(\frac{k}{2^N}\right)\int_0^t \mathcal{Q}_{\mathcal{B}}V(X(s), s, \alpha(s))ds\\
&\quad + \zeta\left(\frac{k}{2^N}\right)\int_0^t \mathcal{Q}_{\mathcal{M}}V(X(s), s, \alpha(s))ds\\
&\le V(x_0, 0, i_0) + \int_0^t \mathcal{L}V(X(s), s, \alpha(s))ds
\end{aligned}$$

$$
\begin{aligned}
&+\xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}+\zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}\\
&+\int_0^t \xi(s)\mathcal{Q}_{\mathcal{B}}V(X(s),s,\alpha(s))ds+\int_0^t \zeta(s)\mathcal{Q}_{\mathcal{M}}V(X(s),s,\alpha(s))ds\\
\le{}& V(x_0,0,i_0)+\xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}+\zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}\\
&+\int_0^t\left[\psi_1(s)+\psi_2(s)V(X(s),s,\alpha(s))\right]ds \quad \text{a.s.}
\end{aligned}
$$

for all $0\le t\le \frac{k}{2^N}$, $k\ge k_0(\varepsilon)\vee k_1(\varepsilon,\omega)\vee k_2(\varepsilon,\omega)$. Hence, by Gronwall's inequality,

$$
\begin{aligned}
V(X(t),t,\alpha(t))\le{}&\Bigg[V(x_0,0,i_0)+\xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}+\zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}\\
&+\int_0^t\psi_1(s)ds\Bigg]\exp\left(\int_0^t\psi_2(s)ds\right) \quad \text{a.s.}
\end{aligned}
$$

for all $0\le t\le \frac{k}{2^N}$, $k\ge k_1(\varepsilon,\omega)\vee k_2(\varepsilon,\omega)$.

Step 4. Now using condition (c) and (3.2) yield that there exists a positive integer $k_3(\varepsilon,\omega)$ such that

$$
\begin{aligned}
&\log V(X(t),t,\alpha(t))\\
&\le\log\left[V(x_0,0,i_0)+\lambda\left(\frac{k}{2^N}\right)^{\mu+\varepsilon}+\lambda\left(\frac{k}{2^N}\right)^{\eta+\varepsilon}+\lambda(t)^{\theta+\varepsilon}\right]\\
&\quad+\log\log\frac{k-1}{2^N}+(\rho+\varepsilon)\log\lambda(t)\\
&\le\log\left[V(x_0,0,i_0)+e^{\varepsilon(\mu+\varepsilon)}\lambda\left(t\right)^{\mu+\varepsilon}+e^{\varepsilon(\eta+\varepsilon)}\lambda\left(t\right)^{\eta+\varepsilon}+\lambda(t)^{\theta+\varepsilon}\right]\\
&\quad+\log\log\frac{k-1}{2^N}+(\rho+\varepsilon)\log\lambda(t) \quad \text{a.s.}
\end{aligned}
$$

for all $\frac{k-1}{2^N}\le t\le\frac{k}{2^N}$, $k\ge k_0(\varepsilon)\vee k_1(\varepsilon,\omega)\vee k_2(\varepsilon,\omega)\vee k_3(\varepsilon,\omega)$. Immediately we have

$$
\limsup_{t\to\infty}\frac{\log V(X(t),t,\alpha(t))}{\log\lambda(t)}\le(\mu+\varepsilon)\vee(\eta+\varepsilon)\vee(\theta+\varepsilon)+(\tau+\varepsilon)+(\rho+\varepsilon) \quad \text{a.s.}
$$

Finally, by virtue of conditions (a) and (c), and letting $\varepsilon\to 0$ yield

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{\log|X(t)|}{\log\lambda(t)}&\le\limsup_{t\to\infty}\frac{1}{p}\frac{\log\left[c(t)^{-1}V(X(t),t,\alpha(t))\right]}{\log\lambda(t)}\\
&\le-\frac{\varpi-[\mu\vee\eta\vee\theta+\tau+\rho]}{p} \quad \text{a.s.}
\end{aligned}
$$

The proof is concluded. □

In fact, we can use the following lemma to get a more general result.

Lemma 3.1. *Let $h(t), u(t) \in C([0,T], \mathbb{R}_+)$, the collection of nonnegative continuous functions, let $w(t)$ be a continuous, nonnegative, and nondecreasing function defined on $[0,T]$. Suppose $0 \le \beta < 1$ and*

$$h(t) \le w(t) + \int_0^t u(s) h^\beta(s) ds, \quad 0 \le t \le T.$$

Then

$$h(t) \le \left(w(t)^{1-\beta} + (1-\beta) \int_0^t u(s) ds \right)^{\frac{1}{1-\beta}}, \quad 0 \le t \le T.$$

Proof. A proof can be found in [12]. □

Theorem 3.2. *Suppose that for each $i \in \mathcal{M}$, $V(\cdot,\cdot,i) \in C^{2,1}(\mathbb{R}^r \times [0,\infty); \mathbb{R}_+)$, that $c(t) > 0$ is a continuous positive function, and that $\psi_1(t)$, $\psi_2(t)$, $\psi_3(t)$ are three continuous and nonnegative functions. Assume that for all $(x,t,i) \in \mathbb{R}^r \times [0,\infty) \times \mathcal{M}$, there exist two continuous non-increasing positive functions $\xi(t) > 0$, $\zeta(t) > 0$, positive constants $p > 0$, $0 \le \beta < 1$, and real numbers ϖ, θ, ρ, ϑ, μ, η such that*

(a) $c(t)|x|^p \le V(x,t,i)$;
(b) $\mathcal{L}V(x,t,i) + \xi(t)\mathcal{Q}_{\mathcal{B}}V(x,t,i) + \zeta(t)\mathcal{Q}_{\mathcal{M}}V(x,t,i)$
$\le \psi_1(t) + \psi_2(t)V(x,t,i) + \psi_3(t)V(x,t,i)^\beta$;
(c) *the following conditions hold:*

$$\liminf_{t\to\infty} \frac{\log c(t)}{\log \lambda(t)} \ge \varpi, \quad \limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_1(s)ds\right)}{\log \lambda(t)} \le \theta, \quad \limsup_{t\to\infty} \frac{\int_0^t \psi_2(s)ds}{\log \lambda(t)} \le \rho(1-\beta),$$
$$\limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_3(s)ds\right)}{\log \lambda(t)} \le \vartheta(1-\beta), \quad \liminf_{t\to\infty} \frac{\log \xi(t)}{\log \lambda(t)} \ge -\mu, \quad \liminf_{t\to\infty} \frac{\log \zeta(t)}{\log \lambda(t)} \ge -\eta.$$

Then the solution $X(t)$ of equation (2.1) satisfies

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log \lambda(t)} \le -\frac{\varpi - [\mu \vee \eta \vee \theta \vee \vartheta + \tau + \rho]}{p} \quad a.s.$$

Proof. The proof is motivated by the approach of [11]. Using similar arguments as in the proof of Theorem 3.1, we can show that

$$\begin{aligned} &V(X(t), t, \alpha(t)) \\ &\quad \le V(x_0, 0, i_0) + \xi\left(\frac{k}{2^N}\right)^{-1} \log \frac{k-1}{2^N} + \zeta\left(\frac{k}{2^N}\right)^{-1} \log \frac{k-1}{2^N} \\ &\qquad + \int_0^t \left[\psi_1(s) + \psi_2(s)V(X(s),s,\alpha(s)) + \psi_3(s)V(X(s),s,\alpha(s))^\beta\right] ds \quad \text{a.s.} \end{aligned}$$

for all $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon) \vee k_1(\varepsilon,\omega) \vee k_2(\varepsilon,\omega)$. By Gronwall's inequality and Lemma 3.1, we arrive at

$$V(X(t), t, \alpha(t))$$

$$\leq \left[V(x_0,0,i_0)+\xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}+\zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}\right.$$
$$\left.+\int_0^t\psi_1(s)ds+\int_0^t\psi_3(s)V(X(s),s,\alpha(s))^\beta ds\right]\exp\left(\int_0^t\psi_2(s)ds\right)$$
$$\leq \left\{\left[V(x_0,0,i_0)+\xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}+\zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}\right.\right.$$
$$\left.+\int_0^t\psi_1(s)ds\right]^{1-\beta}\exp\left(\int_0^t\psi_2(s)ds\right)$$
$$\left.+(1-\beta)\exp\left(\int_0^t\psi_2(s)ds\right)\int_0^t\psi_3(s)ds\right\}^{\frac{1}{1-\beta}}\quad \text{a.s.}$$

for all $0\leq t\leq \frac{k}{2^N}$, $k\geq k_0(\varepsilon)\vee k_1(\varepsilon,\omega)\vee k_2(\varepsilon,\omega)$. Hence, we can find a positive random integer $k_3(\varepsilon,\omega)$ such that

$$\log V(X(t),t,\alpha(t))$$
$$\leq \frac{1}{1-\beta}\log\left\{\left[V(x_0,0,i_0)+e^{\varepsilon(\mu+\varepsilon)}\lambda(t)^{\mu+\varepsilon}+e^{\varepsilon(\eta+\varepsilon)}\lambda(t)^{\eta+\varepsilon}\right.\right.$$
$$\left.\left.+\lambda(t)^{\theta+\varepsilon}\right]^{1-\beta}+\lambda(t)^{(1-\beta)(\vartheta+\varepsilon)}\right\}+\log\log\frac{k-1}{2^N}+(\rho+\varepsilon)\log\lambda(t)\quad\text{a.s.}$$

for all $\frac{k-1}{2^N}\leq t\leq\frac{k}{2^N}$, $k\geq k_0(\varepsilon)\vee k_1(\varepsilon,\omega)\vee k_2(\varepsilon,\omega)\vee k_3(\varepsilon,\omega)$.

Using equation (3.2) again, we have

$$\limsup_{t\to\infty}\frac{\log V(X(t),t,\alpha(t))}{\log\lambda(t)}\leq(\mu+\varepsilon)\vee(\eta+\varepsilon)\vee(\theta+\varepsilon)\vee(\vartheta+\varepsilon)$$
$$+(\tau+\varepsilon)+(\rho+\varepsilon)\quad\text{a.s.}$$

Finally, using conditions (a) and (c) and letting $\varepsilon\to0$ yield

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{\log\lambda(t)}\leq\limsup_{t\to\infty}\frac{1}{p}\frac{\log\left[c(t)^{-1}V(X(t),t,\alpha(t))\right]}{\log\lambda(t)}$$
$$\leq-\frac{\varpi-[\mu\vee\eta\vee\theta\vee\vartheta+\tau+\rho]}{p}\quad\text{a.s.,}$$

which yields the desired result. □

By applying the generalized Itô formula to the function $\log V(\cdot)$, we can obtain the following useful conclusion.

Theorem 3.3. *Suppose that the solution of equation* (2.1) *satisfies $X(t)\neq0$ for all $t>0$ provided $x_0\neq0$ a.s., and that for each $i\in\mathcal{M}$, $V(\cdot,\cdot,i)\in C^{2,1}[(\mathbb{R}^r-\{0\})\times[0,\infty);\mathbb{R}_+]$, $c(t)$ and $\psi_2(t)$ are two continuous positive functions, $\psi_1(t)$ is a real-valued continuous function, and $\psi_3(t)$ is a continuous nonnegative function. Assume that for all $x\neq0$ and $t\geq0$, there exist constants $p>0$, $\rho\geq0$, $\mu\geq0$, $0<\beta<1$, and real numbers ϖ, θ such that*

(a) $c(t)|x|^p \le V(x,t,i)$;
(b) $\mathcal{L}V(x,t,i) \le \psi_1(t)V(x,t,i)$;
(c) $\mathcal{Q}_{\mathcal{B}}V(x,t,i) \ge \psi_2(t)V^2(x,t,i)$;
(d) $\mathcal{Q}_{\mathcal{M}}V(x,t,i) \le \psi_3(t)\min_{j\in\mathcal{M}} V^2(x,t,j)$;
(e) *the following conditions hold:*

$$\liminf_{t\to\infty}\frac{\log c(t)}{\log\lambda(t)} \ge \varpi, \quad \limsup_{t\to\infty}\frac{\int_0^t \psi_1(s)ds}{\log\lambda(t)} \le \theta, \quad \liminf_{t\to\infty}\frac{\int_0^t \psi_2(s)ds}{\log\lambda(t)} \ge \frac{2\rho}{1-\beta},$$
$$\limsup_{t\to\infty}\frac{\int_0^t \psi_3(s)ds}{\log\lambda(t)} \le \vartheta, \quad \limsup_{t\to\infty}\frac{\log t}{\log\lambda(t)} \le \frac{\mu\beta}{2(1+\beta)}.$$

Then the solution $X(t)$ of equation (2.1) *satisfies*

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{\log\lambda(t)} \le -\frac{\varpi+\rho-\theta-\vartheta-\mu}{p} \quad a.s.$$

Proof. Step 1. Applying the generalized Itô formula to the function $\log V(X(t),t,\alpha(t))$, we have

$$\begin{aligned}
&\log V(X(t),t,\alpha(t))\\
&\quad = \log V(x_0,0,i_0) + \int_0^t \left(\frac{\widetilde{\mathcal{L}}V(X(s),s,\alpha(s))}{V(X(s),s,\alpha(s))} - \frac{1}{2}\frac{\mathcal{Q}_{\mathcal{B}}V(X(s),s,\alpha(s))}{V^2(X(s),s,\alpha(s))}\right) ds\\
&\qquad + \int_0^t \mathcal{Q}\log V(X(s),s,\alpha(s))ds + M_1(t) + M_2(t)
\end{aligned} \tag{3.3}$$

where

$$M_1(t) = \int_0^t \frac{1}{V(X(s),s,\alpha(s))}\sum_{i=1}^r\sum_{k=1}^d g^{ik}(X(s),s,\alpha(s))\frac{\partial}{\partial x_i}V(X(s),s,\alpha(s))dW^k(s)$$
$$\begin{aligned}
M_2(t) = \int_0^t\int_{\mathbb{R}} &\big[\log V(X(s^-),s^-,\alpha(s^-) + h(X(s^-),\alpha(s^-),z))\\
&- \log V(X(s^-),s^-,\alpha(s^-))\big]\,\widetilde{N}_1(ds,dz).
\end{aligned}$$

are two martingale terms.

Step 2. Using the exponential martingale inequality, for any positive u_i, v_i, and w_i with $i=1,2$,

$$P\left\{\omega : \sup_{0\le t\le w_1}\left[M_1(t) - \int_0^t \frac{u_1}{2}\frac{\mathcal{Q}_{\mathcal{B}}V(X(s),s,\alpha(s))}{V^2(X(s),s,\alpha(s))}ds\right] > v_1\right\} \le e^{-u_1v_1},$$

and

$$P\left\{\omega : \sup_{0\le t\le w_2}\left[M_2(t) - \int_0^t \frac{u_2}{2}\mathcal{Q}_{\mathcal{M}}\log V(X(s),s,\alpha(s))ds\right] > v_2\right\} \le e^{-u_2v_2}.$$

As in *Step 2* in Theorem 3.1, we use the uniform continuity of $\log\lambda(t)$, for any $\varepsilon>0$ there exist two positive integers $N=N(\varepsilon)$ and $k_0=k_0(\varepsilon)$ such that

$$\left|\log\lambda\left(\frac{k}{2^N}\right) - \log\lambda(t)\right| \le \varepsilon \tag{3.4}$$

if $\frac{k-1}{2^N} \le t \le \frac{k}{2^N}, k \ge k_0(\varepsilon)$.

Now, for $k = 2, 3, \ldots$, taking

$$u_1 = \beta, \quad v_1 = 2\beta^{-1} \log \frac{k-1}{2^N}, \quad w_1 = \frac{k}{2^N};$$
$$u_2 = 1, \quad v_2 = 2 \log \frac{k-1}{2^N}, \quad w_2 = \frac{k}{2^N},$$

by the Borel-Cantelli lemma, there exist two integers $k_1(\varepsilon,\omega) > 0$ and $k_2(\varepsilon,\omega) > 0$ for almost all $\omega \in \Omega$ such that for all $k \ge k_0(\varepsilon) \vee k_1(\varepsilon,\omega) \vee k_2(\varepsilon,\omega)$ and $0 \le t \le \frac{k}{2^N}$,

$$M_1(t) \le 2\beta^{-1} \log \frac{k-1}{2^N} + \frac{\beta}{2} \int_0^t \frac{\mathcal{Q}_{\mathcal{B}} V(X(s), s, \alpha(s))}{V^2(X(s), s, \alpha(s))} ds,$$
$$M_2(t) \le 2 \log \frac{k-1}{2^N} + \int_0^t \mathcal{Q}_{\mathcal{M}} \log V(X(s), s, \alpha(s)) ds.$$

Step 3. Using the elementary inequality $\log(1+x) \le x$ for $x \ge -1$, we have

$$\begin{aligned}
&\mathcal{Q} \log V(x, s, i) \\
&= \sum_{j \ne i, j \in \mathcal{M}} q_{ij}(x) \left(\log V(x, s, j) - \log V(x, s, i) \right) \\
&= \sum_{j \ne i, j \in \mathcal{M}} q_{ij}(x) \log \left(1 + \frac{V(x, s, j) - V(x, s, i)}{V(x, s, i)} \right) \\
&\le \sum_{j \ne i, j \in \mathcal{M}} q_{ij}(x) \left(\frac{V(x, s, j) - V(x, s, i)}{V(x, s, i)} \right) \\
&= \frac{\mathcal{Q} V(x, s, i)}{V(x, s, i)}
\end{aligned}$$

By the definition of $\mathcal{Q}_{\mathcal{M}}$, the mean value theorem and condition (d), we have

$$\begin{aligned}
&\mathcal{Q}_{\mathcal{M}} \log V(x, t, i) \\
&= \sum_{j \ne i, j \in \mathcal{M}} q_{ij}(x) \left| \log V(x, t, j) - \log V(x, t, i) \right|^2 \\
&\le \sum_{j \ne i, j \in \mathcal{M}} q_{ij}(x) \frac{\left| V(x, t, j) - V(x, t, i) \right|^2}{V^2(x, t, j) \wedge V^2(x, t, i)} \\
&\le \frac{\mathcal{Q}_{\mathcal{M}} V(x, t, i)}{\min_{j \in \mathcal{M}} V^2(x, t, j)} \\
&\le \psi_3(t).
\end{aligned}$$

Substituting these equations into (3.3), using the definition of $\mathcal{L}$ and conditions (b) and (c), we obtain that there exists a positive integer $k_3(\varepsilon,\omega)$ such that

$$\begin{aligned}
&\log V(X(t), t, \alpha(t)) \\
&\le \log V(x_0, 0, i_0) + 2\beta^{-1} \log \frac{k-1}{2^N} + 2 \log \frac{k-1}{2^N}
\end{aligned}$$

$$+\int_0^t \psi_1(s)ds - \frac{1}{2}(1-\beta)\int_0^t \psi_2(s)ds + \int_0^t \psi_3(s)ds \quad \text{a.s.} \tag{3.5}$$

for all $\frac{k-1}{2^N} \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon) \vee k_1(\varepsilon,\omega) \vee k_2(\varepsilon,\omega) \vee k_3(\varepsilon,\omega)$. This is the equivalent to

$$\begin{aligned}
&\log V(X(t),t,\alpha(t)) \\
&\le \log V(x_0,0,i_0) + (2\beta^{-1}+2)\log\frac{k-1}{2^N} + (\theta+\varepsilon)\log\lambda(t) \\
&\quad - \frac{1}{2}(1-\beta)\frac{2\rho+\varepsilon}{1-\beta}\log\lambda(t) + (\vartheta+\varepsilon)\log\lambda(t) \quad \text{a.s.}
\end{aligned}$$

Finally, by virtue of conditions (a) and (e), and letting $\varepsilon \to 0$ yields

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\log|X(t)|}{\log\lambda(t)} &\le \limsup_{t\to\infty}\frac{1}{p}\frac{\log[c(t)^{-1}V(X(t),t,\alpha(t))]}{\log\lambda(t)} \\
&\le -\frac{\varpi+\rho-\theta-\vartheta-\mu}{p} \quad \text{a.s.}
\end{aligned}$$

The desired result thus follows. □

4. **MARKOVIAN SWITCHING DIFFUSIONS**

This section is devoted to stability in the almost sure sense of the systems when the switching process is an ergodic Markov chain.

We work on the Markovian switching system, whose $\alpha(t)$ is a continuous-time Markov chain with constant generator(i.e., $Q(x) = \widehat{Q}$ independent of x). The switching process is a Markov chain independent of the Brownian motion and the variable x. We call such processes Markovian switching diffusion processes. We work with such processes and add a remark after the theorem is obtained as in Yin and Xi [18]. Then, we use the similar approaches to give a corollary which is a more general case of Theorem 3.3. Another remark is also given to illustrate how we can generalize the previous results.

In this part, we assume the decay rate $\lambda(t)$ satisfies following condition:

$$\limsup_{t\to\infty}\frac{t}{\log\lambda(t)} \le \overline{\lambda}, \tag{4.1}$$

where $\overline{\lambda} > 0$ is a positive constant. For each $i \in \mathcal{M}$, there are positive continuous functions $K^f(t,i)$, $K^g(t,i)$, $K^d(t,i)$ and constants $\widehat{K}^f(i)$, positive constants $\widehat{K}^g(i)$, $\widehat{K}^d(i)$ such that

$$\left\{\begin{array}{c} x'f(x,t,i) \le K^f(t,i)|x|^2, \\ |g(x,t,i)|^2 \le K^g(t,i)|x|^2, \\ |x'g(x,t,i)| \ge \sqrt{K^d(t,i)}|x|^2, \end{array}\right. \quad \textit{and} \quad \left\{\begin{array}{c} \limsup_{t\to\infty} K^f(t,i) \le \widehat{K}^f(i), \\ \limsup_{t\to\infty} K^g(t,i) \le \widehat{K}^g(i), \\ \liminf_{t\to\infty} K^d(t,i) \ge \widehat{K}^d(i). \end{array}\right. \tag{4.2}$$

Theorem 4.1. *Assume conditions* (4.1) *and* (4.2) *hold. Let the switching process* $\alpha(t)$ *be independent of the Brownnian motion and* $Q(x) = \widehat{Q}$ *be irreducible. Then the solution* $X(t)$ *of equation* (2.1) *satisfies*

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log \lambda(t)} \leq \overline{\lambda} \sum_{i=1}^{m} \nu_i \left[\widehat{K}^f(i) + \frac{1}{2}\widehat{K}^g(i) - \widehat{K}^d(i) \right] \quad a.s.$$

where $\nu = (\nu_1, \ldots, \nu_m)$ *is the stationary distribution associated with* $\widehat{Q}$.

Remark 4.1. By the irreducibility of $\widehat{Q}$, we mean that the system of equations

$$\nu\widehat{Q} = 0, \ \sum_{i\in\mathcal{M}} \nu_i = 1$$

has a unique positive solution; see [19, Definition 2.7] for further details. As a consequence, $\nu = (\nu_1, \ldots, \nu_m)$ exists and moreover for each $i \in \mathcal{M}$, $P(\alpha(t) = i) \to \nu_i > 0$ as $t \to \infty$ exponentially fast. That is, $\alpha(t)$ is a ϕ-mixing process with exponential mixing rate. The implication of this is that the associated Markov chain is strongly ergodic.

Proof. The proof is divided into several steps.

Step 1. Define $V(x,t,i) = \log |x|$ for each $i \in \mathcal{M}$. Noting $V(x,t,i)$ is independent of $i \in \mathcal{M}$, then we have $\mathcal{Q}V(x,t,i) = 0$, $\mathcal{L}V(x,t,i) = \widetilde{\mathcal{L}}V(x,t,i)$. An application of the generalized Itô formula yields that for any $t > 0$,

$$\begin{aligned}
&V(X(t), t, \alpha(t)) - V(x_0, 0, i_0) \\
&\quad = \log |X(t)| - \log |x_0| \\
&\quad = \int_0^t \left[\frac{x'(s) f(X(s), s, \alpha(s))}{|X(s)|^2} - \frac{|x'(s) g(X(s), s, \alpha(s))|^2}{|X(s)|^4} \right. \\
&\qquad \left. + \frac{|g(X(s), s, \alpha(s))|^2}{2|X(s)|^2} \right] ds + M(t). \qquad (4.3)
\end{aligned}$$

where

$$M(t) = \int_0^t \frac{x'(s) g(X(s), s, \alpha(s))}{|X(s)|^2} dW(s)$$

is a martingale term.

Step 2. By a standard result in stochastic processes and condition (4.2), the quadratic variation of $M(t)$ is given by

$$\begin{aligned}
\langle M, M\rangle(t) &= \int_0^t \frac{|x'(s) g(X(s), s, \alpha(s))|^2}{|X(s)|^4} ds \\
&\leq \int_0^t \frac{|X(s)|^2 |g(X(s), s, \alpha(s))|^2}{|X(s)|^4} ds \\
&\leq \int_0^t K^g(s, \alpha(s)) ds
\end{aligned}$$

$$\leq \max_{(s,i)\in[0,\infty)\times\mathcal{M}} K^g(s,i)t$$

Thus, an application of the strong law of large numbers for local martingales yields that $M(t)/t \to 0$ with probability 1 as $t \to \infty$.

Step 3. We work with the rest of the terms in (4.3). By condition (4.2), for any $\varepsilon > 0$, there exists a constant T, whenever $t > T$ we have

$$K^f(t,i) \leq \widehat{K}^f(i) + \frac{\varepsilon}{3}, \quad K^g(t,i) \leq \widehat{K}^g(i) + \frac{\varepsilon}{3}, \quad K^d(t,i) \geq \widehat{K}^d(i) - \frac{\varepsilon}{3}.$$

In view of the argument in Yin and Xi [18], since $Q(x) = \widehat{Q}$ is irreducible, the associated Markov chain is ergodic with the associated stationary distribution given by $\nu = (\nu_1, \ldots, \nu_M)$. It is well known that the finite-state Markov chain $\alpha(t)$ is φ-mixing with exponential mixing rate. In fact $|E(\mathbf{I}_{\{\alpha(t)=i\}} - \nu_i|\alpha(s))| \leq K\exp(-\kappa_0(t-s))$ for $t \geq s$ and for some $\kappa_0 \geq 0$. Moreover, for each $i \in \mathcal{M}$,

$$\frac{1}{t}\int_0^t \left[\mathbf{I}_{\{\alpha(s)=i\}} - \nu_i\right] ds \to 0 \ \text{ a.s. as } \ t \to \infty,$$

Then we can deduce that

$$\begin{aligned}
&\limsup_{t\to\infty}\left|\frac{1}{t}\int_0^t \frac{x'(s)f(X(s),s,\alpha(s))}{|X(s)|^2}ds\right| \\
&\leq \limsup_{t\to\infty}\left|\frac{1}{t}\sum_{i=1}^m\int_0^t \frac{x'(s)f(X(s),s,i)}{|X(s)|^2}\mathbf{I}_{\{\alpha(s)=i\}}ds\right| \\
&\leq \limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^m\int_0^t K^f(s,i)\mathbf{I}_{\{\alpha(s)=i\}}ds \\
&\leq \limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^m\left[\int_0^T \left(K^f(s,i) - \widehat{K}^f(i) + \frac{\varepsilon}{3}\right)\mathbf{I}_{\{\alpha(s)=i\}}ds\right. \\
&\quad\left. + \int_0^t \left(\widehat{K}^f(i) + \frac{\varepsilon}{3}\right)\mathbf{I}_{\{\alpha(s)=i\}}ds\right] \\
&\leq \limsup_{t\to\infty}\sum_{i=1}^m \left(\widehat{K}^f(i) + \frac{\varepsilon}{3}\right)\frac{1}{t}\int_0^t \mathbf{I}_{\{\alpha(s)=i\}}ds \\
&\leq \sum_{i=1}^m \nu_i\widehat{K}^f(i) + \frac{\varepsilon}{3} \qquad (4.4)
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
&\limsup_{t\to\infty}\left|\frac{1}{t}\int_0^t\left[\frac{|g(X(s),s,\alpha(s))|^2}{2|X(s)|^2} - \frac{|x'(s)g(X(s),s,\alpha(s))|^2}{|X(s)|^4}\right]ds\right| \\
&\leq \limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^m\int_0^t\left(\frac{1}{2}K^g(s,i) - K^d(s,i)\right)\mathbf{I}_{\{\alpha(s)=i\}}ds \\
&\leq \sum_{i=1}^m \nu_i\left(\frac{1}{2}\widehat{K}^g(i) - \widehat{K}^d(i)\right) + \frac{2\varepsilon}{3} \qquad (4.5)
\end{aligned}$$

Hence, by (4.4) and (4.5) we have

$$\begin{aligned}
&\limsup_{t\to\infty}\Bigg|\frac{1}{t}\int_0^t\Bigg[\frac{x'(s)f(X(s),s,\alpha(s))}{|X(s)|^2}-\frac{|x'(s)g(X(s),s,\alpha(s))|^2}{|X(s)|^4}\\
&\qquad\qquad+\frac{|g(X(s),s,\alpha(s))|^2}{2|X(s)|^2}\Bigg]ds\Bigg|\\
&\le\sum_{i=1}^m\nu_i\left(\widehat{K}^f(i)+\frac{1}{2}\widehat{K}^g(i)-\widehat{K}^d(i)\right)+\varepsilon. \qquad (4.6)
\end{aligned}$$

Step 4. By Definition 3.1 and condition (4.1), we have

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{\log\lambda(t)}=\limsup_{t\to\infty}\frac{\log|X(t)|}{t}\frac{t}{\log\lambda(t)}\le\overline{\lambda}\limsup_{t\to\infty}\frac{\log|X(t)|}{t}.$$

Finally, using the already obtained $M(t)/t\to 0$ w.p.1 as $t\to\infty$ and (4.6), and letting $\varepsilon\to 0$, the desired result follows. □

Remark 4.2. We have obtained sufficient conditions ensuring almost sure decay rates for Markovian switching diffusions. For the corresponding x-dependent counterpart, stronger conditions are needed due to the x-dependence of $Q(x)$. There are a couple of possibilities.

Observe that under the conditions of the theorem, the trivial solution 0 is stable in probability. That is, for any $\eta>0$ and $\widetilde{\eta}>0$, there is a $\delta=\delta(\eta)>0$ such that whenever $|X(0)|=|x_0|<\delta$, $\mathbb{P}\left\{\sup_{t\ge 0}|X^{x_0,i_0}(t)|>\widetilde{\eta}\right\}<\eta$. This indicates that with probability $1-\eta$, we have $|X^{x_0,i_0}(t)|\le\widetilde{\eta}$, which suggests that we work on the set $S_{\widetilde{\eta}}:=\{\sup_{t\ge 0}|X^{x_0,i_0}(t)|\le\widetilde{\eta}\}$. As in Yin and Xi [18], using $Q(x)=\widehat{Q}+o(1)$, $Q(x)$is locally like $\widehat{Q}$. Then on the set $S_{\widetilde{\eta}}$, we can replace $Q(x)$ by its approximation $\widehat{Q}$. The corresponding stability result may be obtained but the set $S_{\widetilde{\eta}}$ will be needed.

Using similar approach as in Theorem 4.1, we can derive the following result, which is more general than Theorem 3.3 under some special conditions.

Corollary 4.1. *Let the switching process $\alpha(t)$ be independent of the Brownnian motion and $Q(x)=\widehat{Q}$ be irreducible. Assume the solution of* (2.1) *satisfies that $X(t,x_0)\neq 0$ for all $t>0$, a.s. provided $x_0\neq 0$ a.s.. Suppose $V(\cdot,\cdot,i)\in C^{2,1}[(\mathbb{R}^r-\{0\})\times[0,\infty);\mathbb{R}_+]$ and $\forall i\in\mathcal{M}$, $\psi_1(t,i)$ be real continuous functions, $\psi_3(t,i)$ be continuous nonnegative functions, and $c(t)$, $\psi_2(t,i)$ be continuous positive functions. Assume that for all $x\neq 0$ and $t\ge 0$, there exist constants $p>0$, $\overline{\lambda}>0$, $0<\beta<1$, $\varpi\in\mathbb{R}$, and $\forall i\in\mathcal{M}$, $\rho_i\ge 0$, $\vartheta_i\ge 0$, $\theta_i\in\mathbb{R}$, such that*

(a) $c(t)|x|^p\le V(x,t,i)$;
(b) $\mathcal{L}V(x,t,i)\le\psi_1(t,i)V(x,t,i)$;
(c) $\mathcal{Q}_{\mathcal{B}}V(x,t,i)\ge\psi_2(t,i)V^2(x,t,i)$;
(d) $\mathcal{Q}_{\mathcal{M}}V(x,t,i)\le\psi_3(t,i)\min_{j\in\mathcal{M}}V^2(x,t,j)$;

(e) *the following conditions hold:*

$$\liminf_{t\to\infty}\frac{\log c(t)}{t}\ge\varpi,\quad \limsup_{t\to\infty}\frac{\int_0^t\psi_1(s,i)ds}{t}\le\theta_i,\quad \liminf_{t\to\infty}\frac{\int_0^t\psi_2(s,i)ds}{t}\ge 2\rho_i,$$
$$\limsup_{t\to\infty}\frac{\int_0^t\psi_3(s,i)ds}{t}\le\vartheta_i,\quad \limsup_{t\to\infty}\frac{t}{\log\lambda(t)}\le\overline{\lambda}.$$

Then $X(t)$, the solution of (2.1) *satisfies*

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{\log\lambda(t)}\le-\frac{\overline{\lambda}\left[\varpi+\sum_{i=1}^m\nu_i\left(\rho_i-\theta_i-\vartheta_i\right)\right]}{p}\quad a.s.$$

Proof. As equation (3.5) in the proof of Theorem 3.3, we can derive that

$$\begin{aligned}&\log V(X(t),t,\alpha(t))\\ &\le\log V(x_0,0,i_0)+2\beta^{-1}\log\frac{k-1}{2^N}+2\log\frac{k-1}{2^N}+\int_0^t\psi_1(s,\alpha(s))ds\\ &\quad-\frac{1}{2}(1-\beta)\int_0^t\psi_2(s,\alpha(s))ds+\int_0^t\psi_3(s,\alpha(s))ds\ \ \text{a.s.}\end{aligned}$$

Using similar approach as in Step 3 in the proof of Theorem 4.1, by condition (e), we can show that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\psi_1(s,\alpha(s))ds=\limsup_{t\to\infty}\frac{1}{t}\sum_{i=1}^m\int_0^t\psi_1(s,i)\mathbf{I}_{\{\alpha(s)=i\}}ds\le\sum_{i=1}^m\nu_i\theta_i\ \ \text{a.s.}$$

It is also not difficult to obtain that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t\psi_2(s,\alpha(s))ds\ge 2\sum_{i=1}^m\nu_i\rho_i\ \ \text{a.s.}$$
$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\psi_3(s,\alpha(s))ds\le\sum_{i=1}^m\nu_i\vartheta_i\ \ \text{a.s.}$$

Since $\lim_{t\to\infty}\log t/t=0$, by Step 4 in the proof of Theorem 4.1 and letting $\beta\to 0$, we derive the desired result immediately. □

Remark 4.3. A comparison of Theorem 3.3 and Corollary 4.1 shows that for the situation treated in Theorem 3.3 if $\limsup_{t\to\infty}t/\log\lambda(t)\le\overline{\lambda}$, where $\overline{\lambda}>0$, and $Q(x)=\widehat{Q}+o(1)$ with $\widehat{Q}$ being irreducible, we can replace the functions $\psi_j(t)$ by $\psi_j(t,i)$ (for $j=1,2,3$) and proceed as in the previous theorems to get some more precise results such as Theorem 4.1 and Corollary 4.1.

5. EXAMPLES

This section provides several simple examples to illustrate the theorems obtained.

Example 5.1. Consider a regime-switching diffusion

$$dX(t)=-\frac{p}{1+t}X(t)dt+(1+t)^{-p}\sigma_{\alpha(t)}dW(t),\quad t\ge 0,\tag{5.1}$$

with initial data $x(0) = x_0 \in \mathbb{R}$, $\alpha(0) = i_0 \in \mathcal{M} = \{1, \ldots, m\}$, where $p > \frac{1}{2}$ is a constant. We assume $W(t)$ is a one-dimensional standard Brownian motion and $\forall i \in \mathcal{M}$, $\sigma_i \in \mathbb{R}$. We also assume that $\alpha(t)$ is irreducible (as defined in Yin and Zhang [19]). Then there exists an associated stationary distribution given by $\nu = (\nu_1, \ldots, \nu_m)$.

We construct a Lyapunov function

$$V(x,t,i) = (1+t)^{2p}x^2, \quad (x,t,i) \in \mathbb{R} \times [0,\infty) \times \mathcal{M}$$

It is easy to deduce that for any $\delta > 1$, if we define $K = \max_{j\in\mathcal{M}} \sigma_j^2$, then

$$\begin{aligned}
&\mathcal{L}V(x,t,i) = \sigma_i^2 \le K \\
&\mathcal{Q}_{\mathcal{B}}V(x,t,i) = 4\sigma_i^2 V(x,t,i) \le 4KV(x,t,i) \\
&\mathcal{Q}_{\mathcal{M}}V(x,t,i) = 0 \\
&\mathcal{L}V(x,t,i) + \frac{1}{4K(1+t)^\delta}\mathcal{Q}_{\mathcal{B}}V(x,t,i) \le K + \frac{1}{(1+t)^\delta}V(x,t,i)
\end{aligned}$$

Using Theorem 3.1 and letting $\delta \to 1$, we can obtain that whenever $p > \frac{1}{2}$, we have

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} \le -\left(p - \frac{1}{2}\right) \quad \text{a.s.}$$

That is the solution is almost sure polynomially stable with order $p - \frac{1}{2}$.

Remark 5.1. In fact, it is easy to obtain the explicit solution of equation (5.1)

$$X(t) = (x_0 + M(t))\,(1+t)^{-p}, \quad t \ge 0,$$

where

$$M(t) = \int_0^t \sigma_{\alpha(s)} dW(s)$$

is a martingale. Since the quadratic variation of $M(t)$ is given by

$$\langle M(t) \rangle = \int_0^t \sigma_{\alpha(s)}^2 ds$$

Similar to the treatment in [18],

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \sigma_{\alpha(s)}^2 ds = \lim_{t\to\infty} \sum_{i=1}^m \frac{1}{t}\int_0^t \sigma_i^2 \mathbf{I}_{\{\alpha(s)=i\}} ds = \sum_{i=1}^m \nu_i \sigma_i^2 \quad \text{a.s.}$$

Note that

$$\limsup_{t\to\infty} \frac{\log\langle M(t)\rangle}{t} = \limsup_{t\to\infty} \frac{\log\left(\frac{1}{t}\int_0^t \sigma_{\alpha(s)}^2 ds\right) + \log t}{t} = 0 \quad \text{a.s.}$$

$$\limsup_{t\to\infty} \frac{\log\langle M(t)\rangle}{\log t} = \limsup_{t\to\infty} \frac{\log\left(\frac{1}{t}\int_0^t \sigma_{\alpha(s)}^2 ds\right) + \log t}{\log t} = 1 \quad \text{a.s.}$$

Noting the law of the iterated logarithm

$$\limsup_{t\to\infty} \frac{M(t)}{\sqrt{2\langle M(t)\rangle \log\log\langle M(t)\rangle}} = 1 \quad \text{a.s.}$$

Therefore,

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{t} = 0 \quad \text{a.s.}$$

However, we have

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} \le -\left(p - \frac{1}{2}\right) \quad \text{a.s.}$$

That is, the solution is not exponentially stable but polynomially stable with probability one.

Example 5.2. Consider

$$\begin{aligned} dX(t) = &-\left(\frac{1}{2(1+t)} + \frac{q}{(1+t)\log(1+t)}\right) X(t)dt \\ &+ \sigma_{\alpha(t)}(1+t)^{-\frac{1}{2}}(\log(1+t))^{-q} dW(t), \quad t \ge 0 \end{aligned}$$

with initial data $x(0) = x_0 \in \mathbb{R}$, $\alpha(0) = i_0 \in \mathcal{M} = \{1, \ldots, m\}$, $q > 0$ is a positive constant. We assume $W(t)$ is a one-dimensional standard Brownian motion and $\forall i \in \mathcal{M}$, $\sigma_i \in \mathbb{R}$. As in Example 5.1, we also assume that $\alpha(t)$ is irreducible with the associated stationary distribution $\nu = (\nu_1, \ldots, \nu_m)$.

It is easy to obtain the explicit solution

$$X(t) = (x_0 + M(t))\,(1+t)^{-1/2}[\log(1+t)]^{-q}, \quad t \ge 0$$

where

$$M(t) = \int_0^t \sigma_{\alpha(s)} dW(s)$$

is a martingale as Example 5.1. Therefore, it can be shown that

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} = 0 \quad \text{a.s.}$$

So the solution of this example is not polynomially stable. However, we can obtain that whenever $q > 0$,

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log\log t} \le -q \quad \text{a.s.}$$

that is the solution is logarithmically stable with probability one.

In fact, we can simply construct a Lyapunov function as follows:

$$V(x,t,i) = (\log(1+t))^{2q+\delta} x^2, \quad (x,t,i) \in \mathbb{R} \times [0,\infty) \times \mathcal{M}$$

where $\delta > -1$, and

$$\begin{aligned}
\mathcal{L}V(x,t,i) &= \frac{\sigma_i^2(\log(1+t))^\delta}{1+t} + \frac{\delta}{(1+t)\log(1+t)}V(x,t,i) - \frac{1}{1+t}V(x,t,i),\\
\mathcal{Q}_{\mathcal{B}}V(x,t,i) &= \frac{4\sigma_i^2(\log(1+t))^\delta}{1+t}V(x,t,i),\\
\mathcal{Q}_{\mathcal{M}}V(x,t,i) &= 0.
\end{aligned}$$

Taking $K = \max_{j\in\mathcal{M}} \sigma_j^2$, we have

$$\begin{aligned}
&\mathcal{L}V(x,t,i) + \frac{1}{4K(\log(1+t))^\delta}\mathcal{Q}_{\mathcal{B}}V(x,t,i)\\
&\leq \frac{K(\log(1+t))^\delta}{1+t} + \frac{\delta}{(1+t)\log(1+t)}V(x,t,i).
\end{aligned}$$

Using Theorem 3.1 and letting $\delta \to -1$, we can obtain the same result.

Example 5.3. Consider a regime-switching diffusion

$$\begin{aligned}
dX(t) = &\left(-\frac{p}{1+t}X(t) + \frac{\mu_{\alpha(t)}X(t)^{2\beta-1}}{2(1+t)^{2p(1-\beta)}}\right)dt\\
&+ (1+t)^{-p}g(X(t),t,\alpha(t))dW(t), \quad t \geq 0
\end{aligned}$$

with initial data $x(0) = x_0 \in \mathbb{R}, \alpha(0) = i_0 \in \mathcal{M}$, $W(t)$ is a one-dimensional standard Brownian motion, where $0 \leq \beta < 1$, $p > \frac{1}{2(1-\beta)}$ are two constants. We assume that $\forall i \in \mathcal{M}$, $\mu_i > 0$ and there exist positive constants $K, N > 0$ such that $N = \max_{j\in\mathcal{M}} \mu_j$ and $\forall (x,t,i) \in \mathbb{R}\times[0,\infty)\times\mathcal{M}$, $|g(x,t,i)|^2 \leq K$.

We construct a Lyapunov function as follows:

$$V(x,t,i) = (1+t)^{2p}x^2, \quad (x,t,i) \in \mathbb{R}\times[0,\infty)\times\mathcal{M}$$

It is easy to deduce that for any $\delta > 1$,

$$\begin{aligned}
&\mathcal{L}V(x,t,i) \leq K + \mu_i V^\beta(x,t,i) \leq K + NV^\beta(x,t,i)\\
&\mathcal{Q}_{\mathcal{B}}V(x,t,i) \leq 4KV(x,t,i)\\
&\mathcal{Q}_{\mathcal{M}}V(x,t,i) = 0\\
&\mathcal{L}V(x,t,i) + \frac{1}{4K(1+t)^\delta}\mathcal{Q}_{\mathcal{B}}V(x,t,i)\\
&\leq K + \frac{1}{(1+t)^\delta}V(x,t,i) + NV^\beta(x,t,i)
\end{aligned}$$

Using Theorem 3.2 and letting $\delta \to 1$, we can obtain that whenever $p > \frac{1}{2(1-\beta)}$, we have

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} \leq -\left(p - \frac{1}{2(1-\beta)}\right) \quad \text{a.s.}$$

That is the solution is almost sure polynomially stable with order $p - \frac{1}{2(1-\beta)}$.

Example 5.4. Consider a regime-switching diffusion that is linear in the x variable,

$$dX(t) = \mu_{\alpha(t)}\left(1+\frac{1}{t+1}\right)X(t)dt + \sigma_{\alpha(t)}\left(1+\frac{1}{t+1}\right)X(t)dW(t), \quad t \geq 0$$

with initial data $x(0) = x_0 \in \mathbb{R}$, $W(t)$ is a one-dimensional standard Brownian motion, $\alpha(t) \in \mathcal{M} = \{1,2\}$, and

$$Q = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$\mu_1 = -\frac{1}{2}, \quad \mu_2 = 4, \quad \sigma_1 = 1, \quad \sigma_2 = 2.$$

Then the stationary distribution associated with Q is given by $\nu = (3/4, 1/4)$.

Associated with the switching diffusion, there are two diffusions that interact and switch back and forth. These diffusions are given by

$$dX(t) = -\frac{1}{2}\left(1+\frac{1}{t+1}\right)X(t)dt + \left(1+\frac{1}{t+1}\right)X(t)dW(t), \quad t \geq 0$$
$$dX(t) = 4\left(1+\frac{1}{t+1}\right)X(t)dt + 2\left(1+\frac{1}{t+1}\right)X(t)dW(t), \quad t \geq 0$$

From the classical stability results of diffusions (see [8]), the equilibrium solution of the first equation is almost sure exponentially stable but that of the second equation is not stable. However,

$$x'f(x,t,i) = -\mu_i\left(1+\frac{1}{t+1}\right)|x|^2,$$
$$|g(x,t,i)|^2 = \sigma_i^2\left(1+\frac{1}{t+1}\right)^2|x|^2,$$
$$|x'g(x,t,i)| = |\sigma_i|\left(1+\frac{1}{t+1}\right)|x|^2,$$

then by Theorem 4.1, we can obtain

$$\limsup_{t\to\infty}\frac{\log|X(t)|}{t} \leq \sum_{i=1}^{2}\nu_i\left(\mu_i - \frac{\sigma_i^2}{2}\right) = -\frac{1}{4} \quad \text{a.s.}$$

That is the switching diffusion is almost sure exponentially stable with order $1/4$.

In fact, we can construct a Lyapunov function as follows:

$$V(x,t,i) = x^2, \quad (x,t,i) \in \mathbb{R}\times[0,\infty)\times\mathcal{M}$$

It is easy to deduce that

$$\mathcal{L}V(x,t,i) = \left(2\mu_i\left(1+\frac{1}{t+1}\right) + \sigma_i^2\left(1+\frac{1}{t+1}\right)^2\right)V(x,t,i)$$

$$\mathcal{Q}_{\mathcal{B}}V(x,t,i) = 4\sigma_i^2\left(1+\frac{1}{t+1}\right)^2V^2(x,t,i)$$

$$\mathcal{Q}_{\mathcal{M}}V(x,t,i)=0.$$

And

$$\lim_{t\to\infty}\frac{\int_0^t\left(2\mu_i\left(1+\frac{1}{s+1}\right)+\sigma_i^2\left(1+\frac{1}{s+1}\right)^2\right)ds}{t}=2\mu_i+\sigma_i^2,$$
$$\lim_{t\to\infty}\frac{\int_0^t 4\sigma_i^2\left(1+\frac{1}{s+1}\right)^2 ds}{t}=4\sigma_i^2,$$

Hence, taking

$$\varpi=0,\theta_i=2\mu_i+\sigma_i^2,\rho_i=2\sigma_i^2,\vartheta_i=0,\overline{\lambda}=1,p=2$$

in Corollary 4.1, we can obtain the same result as above. But we can see that Theorem 3.3 does not work. This example explain Remark 4.3 to some extent.

6. HYBRID SWITCHING DIFFUSIONS WITH DELAYS

Nowadays, systems with delays have been extensively studied. For the discussion of general decay rates of stochastic systems with delays without hybrid switching, we refer the reader to [1, 3, 5, 7, 10, 11] and the references therein. In this section, we will concentrate ourselves on the behavior of hybrid switching diffusions with delays.

Let $l>0$ and denote by $C([-l,0],\mathbb{R}^r)$ the space of all continuous functions defined on $[-l,0]$ with values in $\mathbb{R}^r$. And a norm over this space is given by $\|u\|=\max\{u(s):-l\le s\le 0\}$, $u\in C([-l,0],\mathbb{R}^r)$. Addition, let $L^2(\Omega,\mathscr{F}_0,\mathbb{P};C([-l,0],\mathbb{R}^r))$ denote the family of all $\mathscr{F}_0$-measurable $C([-l,0],\mathbb{R}^r)$-valued random variable $\varphi(t)$ with $E\|\varphi\|^2<\infty$.

Consider a regime-switching diffusion with delays:

$$\left\{\begin{array}{ll} dX(t)=f(X(t),X(t-\delta(t)),t,\alpha(t))dt & \\ \qquad\qquad +g(X(t),X(t-\delta(t)),t,\alpha(t))dW_t, & t\ge 0;\\ X(t)=\varphi(t),\alpha(t)=i_0 & t\in[-l,0].\end{array}\right. \tag{6.1}$$

with initial data $X(t)=\varphi(t)\in L^2(\Omega,\mathscr{F}_0,\mathbb{P};C([-l,0],\mathbb{R}^r))$, $i_0\in\mathcal{M}$. Here $f:\mathbb{R}^r\times\mathbb{R}^r\times[0,\infty)\times\mathcal{M}\mapsto\mathbb{R}^r$, $g:\mathbb{R}^r\times\mathbb{R}^r\times[0,\infty)\times\mathcal{M}\mapsto\mathbb{R}^{r\times d}$ are measurable mappings and $\delta(\cdot):[0,\infty)\to[0,l]$ is a continuous function which shall play the role of variable delays. We also assume that the equation (6.1) we considered has a unique global solution which is denoted by $X(t,\varphi)\in\mathbb{R}^r$.

The process $(X(t,\varphi),\alpha(t))$ has a generator $\mathcal{L}$ as follows. For each $i\in\mathcal{M}$, $V(\cdot,\cdot,i)\in C^{2,1}(\mathbb{R}^r\times[0,\infty);\mathbb{R}_+)$, define the function $\mathcal{L}V(x,y,t,i)$ as follows. For arbitrary $x, y \in \mathbb{R}^r$, $t \in [0, \infty)$, $i \in \mathcal{M}$, we set

$$\begin{aligned}\mathcal{L}V(x,y,t,i) := {}& \frac{\partial}{\partial t}V(x,t,i) + \sum_{i=1}^{r} f^i(x,y,t,i)\frac{\partial}{\partial x_i}V(x,t,i)\\ &+\frac{1}{2}\sum_{i,j=1}^{r}\sum_{k=1}^{m} g^{ik}(x,y,t,i)g^{jk}(x,y,t,i)\frac{\partial^2}{\partial x_i x_j}V(x,t,i)\\ &+\sum_{j\in\mathcal{M}} q_{ij}(x)V(x,t,j),\end{aligned}$$

and define

$$\begin{aligned}\mathcal{Q}_{\mathcal{B}}V(x,y,t,i) &:= \sum_{i,j=1}^{r}\sum_{k=1}^{m} g^{ik}(x,y,t,i)g^{jk}(x,y,t,i)\frac{\partial}{\partial x_i}V(x,t,i)\frac{\partial}{\partial x_j}V(x,t,i)\\ \mathcal{Q}_{\mathcal{M}}V(x,y,t,i) &:= \sum_{j\neq i, j\in\mathcal{M}} q_{ij}(x)|V(x,t,j) - V(x,t,i)|^2.\end{aligned}$$

Definition 6.1. Suppose that $\lambda(t)$ is a positive function satisfying $\lambda(t) \uparrow +\infty$ as $t \to +\infty$. Assume there exists a sufficiently large $T > 0$ such that

(1) $\log \lambda(t)$ is uniformly continuous over $t \geq T$.
(2) For all $s, t \geq T$, $\lambda(s)\lambda(t) \geq \lambda(s+t)$.
(3) There exists a nonnegative constant $\tau \geq 0$ such that

$$\limsup_{t\to\infty} \frac{\log\log t}{\log \lambda(t)} \leq \tau.$$

The solution $X(t, \varphi)$ of equation (6.1) is said to be *almost surely stable with rate function* $\lambda(t)$ *of order* $\gamma > 0$ if and only if

$$\limsup_{t\to\infty} \frac{\log |X(t,\varphi)|}{\log \lambda(t)} \leq -\gamma \quad \text{a.s.}$$

for initial conditions $X(t) = \varphi(t), \alpha(t) = i_0, t \in [-l, 0]$. If in addition 0 is solution to (6.1), then we call the zero solution is almost surely stable with rate function $\lambda(t)$ of order at least γ.

In order to obtain our main conclusion, we need the following lemma.

Lemma 6.1. *Assume $h(t)$ is a continuous, nonnegative function defined on $[-l, T]$ where $0 < l < T$. Let $w(t)$ be a continuous, nonnegative, nondecreasing function and $u_1(t), v_1(t), u_2(t), v_2(t)$ be four continuous nonnegative functions, which are all defined on $[0, T]$. Assume $\delta(t)$ is defined as above and $0 \leq \beta < 1$, such that*

$$\begin{aligned} h(t) \leq {}& w(t) + \int_0^t u_1(s)h(s)ds + \int_0^t u_2(s)h^{\beta}(s)ds\\ &+ \int_0^t v_1(s)h(s-\delta(s))ds + \int_0^t v_2(s)h^{\beta}(s-\delta(s))ds \end{aligned} \tag{6.2}$$

Then

$$
\begin{aligned}
h(t) \le & \left(M(t)^{1-\beta} + (1-\beta) \int_0^t \big(u_2(s) + 2^\beta v_2(s)\big)\, ds \right)^{\frac{1}{1-\beta}} \\
& \times \exp\left\{ \left(\frac{1}{1-\beta} \right) \int_0^t (u_1(s) + v_1(s))\, ds \right\}
\end{aligned}
\tag{6.3}
$$

where

$$
M(t) = w(t) + \left[\sup_{-l \le r \le 0} h(r) \right] \int_0^l v_1(s) ds + \left[2 \sup_{-l \le r \le 0} h(r) \right]^\beta \int_0^l v_2(s) ds
$$

for all $t \in [0, T]$.

Remark 6.1. In fact, Lemma 6.1 contains the results in Liu and Mao [11]. For example, if we take $u_2(\cdot) \equiv 0$, $v_2(\cdot) \equiv 0$ we get Lemma 1.4, and if $u_2(\cdot) \equiv 0$, $v_1(\cdot) \equiv 0$ we get Lemma 1.5 in Liu and Mao [11], respectively.

Proof. We first define a nondecreasing $z(t)$ on $[0, T]$ as

$$
\begin{aligned}
z(t) = & \, w(t) + \int_0^t u_1(s) h(s) ds + \int_0^t u_2(s) h^\beta(s) ds \\
& + \int_0^t v_1(s) h(s - \delta(s)) ds + \int_0^t v_2(s) h^\beta(s - \delta(s)) ds
\end{aligned}
$$

By (6.2) we have

$$
h(t) \le z(t), \quad 0 \le t \le T. \tag{6.4}
$$

Therefore

$$
\begin{cases}
h(s - \delta(s)) \le \left[\sup_{-l \le r \le 0} h(r) \right] + z(s), & 0 \le s \le l; \\
h(s - \delta(s)) \le z(s), & l \le s \le T.
\end{cases}
$$

By the elementary inequality $(|a| + |b|)^c \le 2^c(|a|^c + |b|^c)$ $(\forall a, b \in \mathbb{R}, \forall c \in \mathbb{R}_+)$ we have

$$
\begin{aligned}
z(t) \le & \, w(t) + \int_0^t u_1(s) z(s) ds + \int_0^t u_2(s) z^\beta(s) ds \\
& + \int_0^l v_1(s) \left[\sup_{-l \le r \le 0} h(r) \right] ds + \int_0^t v_1(s) z(s) ds \\
& + 2^\beta \int_0^l v_2(s) \left[\sup_{-l \le r \le 0} h(r) \right]^\beta ds + 2^\beta \int_0^t v_2(s) z^\beta(s) ds \\
\le & \, w(t) + \left[\sup_{-l \le r \le 0} h(r) \right] \int_0^l v_1(s) ds + \left[2 \sup_{-l \le r \le 0} h(r) \right]^\beta \int_0^l v_2(s) ds \\
& + \int_0^t (u_1(s) + v_1(s)) z(s) ds + \int_0^t (u_2(s) + 2^\beta v_2(s)) z^\beta(s) ds
\end{aligned}
$$

Letting

$$
M(t) = w(t) + \left[\sup_{-l \le r \le 0} h(r) \right] \int_0^l v_1(s) ds + \left[2 \sup_{-l \le r \le 0} h(r) \right]^\beta \int_0^l v_2(s) ds,
$$

by Gronwall's inequality, we have

$$z(t) \le \left(M(t) + \int_0^t (u_2(s) + 2^\beta v_2(s)) z^\beta(s) ds\right) \exp\left\{\int_0^t (u_1(s) + v_1(s)) ds\right\}$$

Using Lemma 3.1, we can deduce that

$$\begin{aligned} z(t) &\le \Bigg(\left[M(t) \exp\left\{\int_0^t (u_1(s) + v_1(s)) ds\right\}\right]^{1-\beta} \\ &\quad + (1-\beta) \int_0^t (u_2(s) + 2^\beta v_2(s)) ds \exp\left\{\int_0^t (u_1(s) + v_1(s)) ds\right\}\Bigg)^{\frac{1}{1-\beta}} \\ &\le \left(M(t)^{1-\beta} + (1-\beta) \int_0^t \left(u_2(s) + 2^\beta v_2(s)\right) ds\right)^{\frac{1}{1-\beta}} \\ &\quad \times \exp\left\{\left(\frac{1}{1-\beta}\right) \int_0^t (u_1(s) + v_1(s))\, ds\right\} \end{aligned} \tag{6.5}$$

Hence by (6.4) and (6.5), we get the desired conclusion. □

Now we are in a position to prove our main result.

Theorem 6.1. *Let $V(\cdot, \cdot, i) \in C^{2,1}(\mathbb{R}^r \times [0, \infty); \mathbb{R}_+)$ for each $i \in \mathcal{M}$, and $\psi_j(t)$ for $j = 1, \ldots, 5$ be continuous and nonnegative functions. Assume that for all $x, y \in \mathbb{R}^r$, $t \ge 0$ and $i \in \mathcal{M}$, there exist positive constants $c_1 > 0$, $c_2 > 0$, $p > 0$, $0 \le \beta < 1$, real numbers ϖ, θ, ϑ, γ, ρ, ϱ, μ, η, and two continuous non-increasing positive functions $\xi(t) > 0$, $\zeta(t) > 0$ such that*

(a) $c_1|x|^p \lambda(t)^\varpi \le V(x, t, i) \le c_2|x|^p \lambda(t)^\varpi, \quad (x, t, i) \in \mathbb{R}^r \times \mathbb{R}_+ \times \mathcal{M}$;

(b) $\mathcal{L}V(x, y, t, i) + \xi(t)\mathcal{Q}_\mathcal{B} V(x, y, t, i) + \zeta(t)\mathcal{Q}_\mathcal{M} V(x, t, i)$
$\le \psi_1(t) + \psi_2(t)V(x, t, i) + \psi_3(t)V(y, t, i) + \psi_4(t)V(x, t, i)^\beta$
$+ \psi_5(t)V(y, t, i)^\beta, \quad (x, y, t, i) \in \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R}_+ \times \mathcal{M}$;

(c) *the following conditions hold:*

$$\limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_1(s) ds\right)}{\log \lambda(t)} \le \theta, \quad \limsup_{t\to\infty} \frac{\int_0^t \psi_2(s) ds}{\log \lambda(t)} \le \rho(1-\beta),$$
$$\limsup_{t\to\infty} \frac{\int_0^t \psi_3(s) ds}{\log \lambda(t)} \le \varrho(1-\beta), \quad \limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_4(s) ds\right)}{\log \lambda(t)} \le \vartheta(1-\beta),$$
$$\limsup_{t\to\infty} \frac{\log\left(\int_0^t \psi_5(s) ds\right)}{\log \lambda(t)} \le \gamma(1-\beta), \quad \liminf_{t\to\infty} \frac{\log \xi(t)}{\log \lambda(t)} \ge -\mu,$$
$$\liminf_{t\to\infty} \frac{\log \zeta(t)}{\log \lambda(t)} \ge -\eta.$$

Then the solution $X(t, \varphi)$ of equation (6.1) satisfies

$$\limsup_{t\to\infty} \frac{\log |X(t, \varphi)|}{\log \lambda(t)} \le -\frac{\varpi - [\tau + (c_2/c_1)(\rho + \lambda(l)^m \varrho) + \mu \vee \eta \vee \theta \vee \vartheta \vee \gamma]}{p} \quad a.s.$$

Proof. Step 1. By the generalized Itô formula and the definition of $\mathcal{L}$, we can derive that

$$\begin{aligned} V(X(t),t,\alpha(t)) = V(x_0,0,i_0) + \int_0^t \mathcal{L}V(X(s),X(s-\delta(s)),s,\alpha(s))ds \\ + M_1(t) + M_2(t) \end{aligned} \tag{6.6}$$

where

$$\begin{aligned} M_1(t) &= \int_0^t \sum_{i=1}^r \sum_{k=1}^d g^{ik}(X(s),X(s-\delta(s)),s,\alpha(s))\frac{\partial}{\partial x_i}V(X(s),s,\alpha(s))dW_s^k \\ M_2(t) &= \int_0^t \int_{\mathbb{R}} \left[V(X(s^-),s^-,\alpha(s^-)+h(X(s^-),\alpha(s^-),z)) \right. \\ &\quad \left. -V(X(s^-),s^-,\alpha(s^-))\right] \widetilde{N}_1(ds,dz). \end{aligned}$$

are two martingale terms.

Step 2. As in *Step 2* of the proof of Theorem 3.1, we use the uniform continuity of $\log\lambda(t)$, for any $\varepsilon > 0$ there exist two positive integers $N = N(\varepsilon)$ and $k_0 = k_0(\varepsilon)$ such that,

$$\left|\log\lambda\left(\frac{k}{2^N}\right) - \log\lambda(t)\right| \leq \varepsilon \tag{6.7}$$

if $\frac{k-1}{2^N} \leq t \leq \frac{k}{2^N}, k \geq k_0(\varepsilon)$.

By the exponential martingale inequality, the definition of $\mathcal{Q}_{\mathcal{B}}$ and $\mathcal{Q}_{\mathcal{M}}$, for any u_i, v_i, and $w_i > 0$ with $i = 1, 2$,

$$P\left\{\omega : \sup_{0\leq t\leq w_1}\left[M_1(t) - \int_0^t \frac{u_1}{2}\mathcal{Q}_{\mathcal{B}}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\right] > v_1\right\} \leq e^{-u_1v_1},$$

$$P\left\{\omega : \sup_{0\leq t\leq w_2}\left[M_2(t) - \int_0^t \frac{u_2}{2}\mathcal{Q}_{\mathcal{M}}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\right] > v_2\right\} \leq e^{-u_2v_2}.$$

In particular, for $k = 2, 3, \ldots$, taking

$$\begin{aligned} u_1 = 2\xi\left(\frac{k}{2^N}\right), \quad v_1 = \xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}, \quad w_1 = \frac{k}{2^N}; \\ u_2 = 2\zeta\left(\frac{k}{2^N}\right), \quad v_2 = \zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}, \quad w_2 = \frac{k}{2^N}; \end{aligned}$$

By virtue of the Borel-Cantelli lemma, there exist two integers $k_1(\varepsilon,\omega) > 0$ and $k_2(\varepsilon,\omega) > 0$, for almost all $\omega \in \Omega$ such that

$$\begin{aligned} M_1(t) \leq \xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N} + \xi\left(\frac{k}{2^N}\right)\int_0^t \mathcal{Q}_{\mathcal{B}}V(X(s),X(s-\delta(s)),s,\alpha(s))ds \\ M_2(t) \leq \zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N} + \zeta\left(\frac{k}{2^N}\right)\int_0^t \mathcal{Q}_{\mathcal{M}}V(X(s),X(s-\delta(s)),s,\alpha(s))ds \end{aligned}$$

for all $0 \leq t \leq \frac{k}{2^N}$, $k \geq k_0(\varepsilon) \vee k_1(\varepsilon,\omega) \vee k_2(\varepsilon,\omega)$.

Step 3. Substituting these equations into (6.6) and using the condition (b),

$$
\begin{aligned}
&V(X(t),t,\alpha(t))\\
&\le V(x_0,0,i_0)+\int_0^t \mathcal{L}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\\
&\quad+\xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}+\zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}\\
&\quad+\xi\left(\frac{k}{2^N}\right)\int_0^t \mathcal{Q}_{\mathcal{B}}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\\
&\quad+\zeta\left(\frac{k}{2^N}\right)\int_0^t \mathcal{Q}_{\mathcal{M}}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\\
&\le V(x_0,0,i_0)+\int_0^t \mathcal{L}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\\
&\quad+\xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}+\zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}\\
&\quad+\int_0^t \xi(s)\mathcal{Q}_{\mathcal{B}}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\\
&\quad+\int_0^t \zeta(s)\mathcal{Q}_{\mathcal{M}}V(X(s),X(s-\delta(s)),s,\alpha(s))ds\\
&\le V(x_0,0,i_0)+\xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}+\zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}\\
&\quad+\int_0^t\big[\psi_1(s)+\psi_2(s)V(X(s),s,\alpha(s))+\psi_3(s)V(X(s-\delta(s)),s,\alpha(s))\\
&\quad+\psi_4(s)V(X(s),s,\alpha(s))^{\beta}+\psi_5(s)V(X(s-\delta(s)),s,\alpha(s))^{\beta}\big]\,ds\quad\text{a.s.}
\end{aligned}
$$

for all $0\le t\le \frac{k}{2^N}$, $k\ge k_0(\varepsilon)\vee k_1(\varepsilon,\omega)\vee k_2(\varepsilon,\omega)$. Then, from condition (a) and condition (2) of Definition 6.1, we have

$$
\begin{aligned}
&c_1|X(t)|^p\lambda(t)^{\varpi}\\
&\le\Bigg\{\Bigg(V(x_0,0,i_0)+\xi\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}+\zeta\left(\frac{k}{2^N}\right)^{-1}\log\frac{k-1}{2^N}\\
&\quad+\int_0^t\psi_1(s)ds+\left[\sup_{-l\le r\le 0}\varphi(r)\right]\frac{c_2}{c_1}\lambda(l)^{\varpi}\int_0^l\psi_3(s)ds\\
&\quad+\left[2\sup_{-l\le r\le 0}\varphi(r)\right]^{\beta}\left[\frac{c_2}{c_1}\right]^{\beta}\lambda(l)^{\beta\varpi}\int_0^l\psi_5(s)ds\Bigg)^{1-\beta}\\
&\quad+\left[\frac{c_2}{c_1}\right]^{\beta}\int_0^l\psi_4(s)ds+\left[\frac{c_2}{c_1}\right]^{\beta}2^{\beta}\lambda(l)^{\beta\varpi}\int_0^l\psi_5(s)ds\Bigg\}^{\frac{1}{1-\beta}}\\
&\quad\times\exp\left\{\left(\frac{1}{1-\beta}\right)\left(\frac{c_2}{c_1}\int_0^t\psi_2(s)ds+\frac{c_2}{c_1}\lambda(l)^{\varpi}\int_0^t\psi_3(s)ds\right)\right\}\quad\text{a.s.}
\end{aligned}
$$

for all $0 \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon) \vee k_1(\varepsilon,\omega) \vee k_2(\varepsilon,\omega)$.

Step 4. By virtue of condition (c) and inequality (6.7), there exists a positive integer $k_3(\varepsilon,\omega)$ such that

$$\begin{aligned}
&\log\left(c_1|X(t)|^p\lambda(t)^{\varpi}\right)\\
&\le \log\Bigg\{\Bigg(V(x_0,0,i_0)+\lambda(t)^{(\mu+\varepsilon)(1+\varepsilon)}\log\frac{k-1}{2^N}+\lambda(t)^{(\eta+\varepsilon)(1+\varepsilon)}\log\frac{k-1}{2^N}\\
&\quad+\lambda(t)^{\theta+\varepsilon}+\left[\sup_{-l\le r\le 0}\varphi(r)\right]\frac{c_2}{c_1}\lambda(l)^{\varpi}\int_0^l\psi_3(s)ds\\
&\quad+\left[2\sup_{-l\le r\le 0}\varphi(r)\right]^{\beta}\left[\frac{c_2}{c_1}\right]^{\beta}\lambda(l)^{\beta\varpi}\int_0^l\psi_5(s)ds\Bigg)^{1-\beta}\\
&\quad+\left[\frac{c_2}{c_1}\right]^{\beta}\lambda(t)^{(1-\beta)(\vartheta+\varepsilon)}+\left[\frac{c_2}{c_1}\right]^{\beta}2^{\beta}\lambda(l)^{\beta\varpi}\lambda(t)^{(1-\beta)(\gamma+\varepsilon)}\Bigg\}^{\frac{1}{1-\beta}}\\
&\quad+\frac{c_2}{c_1}(\rho+\varepsilon)\log\lambda(t)+\frac{c_2}{c_1}\lambda(l)^{\varpi}(\varrho+\varepsilon)\log\lambda(t)\quad\text{a.s.}
\end{aligned}$$

for all $\frac{k-1}{2^N} \le t \le \frac{k}{2^N}$, $k \ge k_0(\varepsilon) \vee k_1(\varepsilon,\omega) \vee k_2(\varepsilon,\omega) \vee k_3(\varepsilon,\omega)$. Then, by condition (3) of Definition 6.1, it implies that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\log\left(c_1|X(t)|^p\lambda(t)^{\varpi}\right)}{\log\lambda(t)}\\
&\le(\tau+\varepsilon)+\frac{c_2}{c_1}\left[(\rho+\varepsilon)+\lambda(l)^{\varpi}(\varrho+\varepsilon)\right]\\
&\quad+(\mu+\varepsilon)(1+\varepsilon)\vee(\eta+\varepsilon)(1+\varepsilon)\vee(\theta+\varepsilon)\vee(\vartheta+\varepsilon)\vee(\gamma+\varepsilon)\quad\text{a.s.}
\end{aligned}$$

Finally, letting $\varepsilon \to 0$ yields

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\log|X(t)|}{\log\lambda(t)}\\
&\le\limsup_{t\to\infty}\frac{1}{p}\frac{\log\left[\lambda(t)^{-\varpi}\left(c_1|X(t)|^p\lambda(t)^{\varpi}\right)\right]}{\log\lambda(t)}\\
&\le-\frac{1}{p}\left(\varpi-\left[\tau+\frac{c_2}{c_1}(\rho+\lambda(l)^{\varpi}\varrho)+\mu\vee\eta\vee\theta\vee\vartheta\vee\gamma\right]\right)\quad\text{a.s.}
\end{aligned}$$

□

Remark 6.2. Similar to Lemma 6.1, Theorem 6.1 is more general than that of the results in Liu and Mao [11]. For example, if we take $\psi_4(\cdot)\equiv 0$, $\psi_5(\cdot)\equiv 0$ or $\psi_3(\cdot)\equiv 0$, $\psi_4(\cdot)\equiv 0$ in Theorem 6.1, we get the similar results in the regime-switching case as Theorem 4.1 and 4.2 in Liu and Mao [11], respectively.

As in Section 4, if we put more conditions to the generator $Q(x)$ and the decay rate $\lambda(t)$, we may replace the functions $\psi_j(t)$ by $\psi_j(t,i)$ in Theorem 6.1 to get more precise results such as Remark 4.3.

We close this section with the following example.

Example 6.1. Consider a regime-switching diffusion with constant time delay

$$\begin{aligned} dX(t) &= \left(-\frac{p}{1+t}X(t) + \frac{1}{(1+t)}X(t-l)\right)dt \\ &\quad + \left(\frac{\mu_{\alpha(t)}}{2(1+t)^{2p(1-\beta)}}X(t)^{2\beta-1} + \frac{\vartheta_{\alpha(t)}}{2(1+t)^{2p(1-\beta)}}\frac{X(t-l)^{2\beta}}{X(t)}\right)dt \\ &\quad + (1+t)^{-p}g(X(t), X(t-l), t, \alpha(t))dW(t), \quad t \geq 0 \end{aligned}$$

with initial data $X(t) = \varphi(t)$, $t \in [-l, 0]$. Here p, l are two positive numbers and $0 \leq \beta < 1$. $W(t)$ is a one-dimensional standard Brownian motion.

We assume that there exists a stationary distribution $\nu = (\nu_1, \ldots, \nu_m)$ associated with Q, and $\mu_i > 0, \vartheta_i > 0, \forall i \in \mathcal{M}$. We also assume there exist positive constants $K, N > 0$ such that $N = \max_j(\mu_i \vee \vartheta_i)$ and $|g(x,y,t,i)|^2 \leq K$, $\forall (x,y,t,i) \in \mathbb{R} \times \mathbb{R} \times [0,\infty) \times \mathcal{M}$.

We construct a Lyapunov function as follows:

$$V(x,t,i) = (1+t)^{2p}x^2, \quad (x,t,i) \in \mathbb{R} \times [0,\infty) \times \mathcal{M}$$

It is easy to deduce that for any $\delta > 1$,

$$\begin{aligned} \mathcal{L}V(x,y,t,i) &\leq K + \frac{1}{1+t}V(x,t,i) + \frac{1}{1+t}V(y,t,i) \\ &\qquad + \mu_i V^\beta(x,t,i) + \vartheta_i V^\beta(y,t,i) \\ \mathcal{Q}_{\mathcal{B}}V(x,y,t,i) &\leq 4KV(x,t,i) \\ \mathcal{Q}_{\mathcal{M}}V(x,y,t,i) &= 0 \\ \mathcal{L}V(x,t,i) &+ \frac{1}{4K(1+t)^\delta}\mathcal{Q}_{\mathcal{B}}V(x,t,i) \\ &\leq K + \left(\frac{1}{(1+t)^\delta} + \frac{1}{1+t}\right)V(x,t,i) + \frac{1}{1+t}V(y,t,i) \\ &\quad + NV^\beta(x,t,i) + NV^\beta(y,t,i). \end{aligned}$$

Thus, the conditions of Theorem 6.1 are verified. Letting $\delta \to 1$, we can obtain that whenever $p > \frac{1+l^{2p}/2}{1-\beta}$, we have

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} \leq -\left(p - \frac{1+l^{2p}/2}{1-\beta}\right) \quad \text{a.s.}$$

That is the solution is almost sure polynomially stable with order $p - \frac{1+l^{2p}/2}{1-\beta}$.

7. CONCLUDING REMARKS

This work focused on stability of switching diffusions that decay slower than exponential. If we assume that the switching diffusion evolves on a compact set, and the switching process jump changes much faster than the diffusion part (for example, assume the generator is given by $Q(x)/\varepsilon$), then the switching process can be treated as a noise. In this case, it may be possible to get the desired results by means of two-time-scale approach; see [6] for a possible such model. Nevertheless, much care and thoughts have to be given.

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