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ON FRACTIONAL TRIGONOMETRIC FUNCTIONS AND THEIR GENERALIZATIONS

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ABSTRACT. In this paper, we obtain the definitions of fractional q cosine and fractional q sine functions as the solutions of the fractional harmonic equation. Further generalized fractional trigonometric like functions are defined through the solutions of the 3^{rd} order and higher order fractional differential equations. The properties of q-cosine and q-sine functions are obtained and the results are extended to generalized fractional trigonometric functions. This study is done parallel to the study of generalized trigonometric functions as solutions of higher order ordinary differential equations.

Key words. q cosine function, q sine function, fractional trigonometric like functions, fractional harmonic equation, f-Wronskian, Mittag-Leffler function, Caputo fractional differential equation

AMS (MOS) Subject Classification. 47G20

1. INTRODUCTION

The study of trigonometry is of special interest as many physical phenomena exhibit oscillatory behavior and hence the trigonometric functions are used to understand these systems [1, 2]. The introduction of the derivative of fractional order 300 years ago a paradox then, is now a vibrant field of research. The rich potential that the fractional differential equations (FDE) carry with them both in theory as well as mathematical models of many important physical phenomena is exciting and has opened a new area of research. See [3, 4, 5, 6].

In this context, the study of fractional trigonometry is interesting in itself and will provide an understanding of the basic structure of the physical phenomena modeled by fractional differential equations. The fractional derivatives are intuitively obtained as a generalization of the standard derivatives of integer order as can be observed from the evolution of fractional calculus [4].

In this paper, we propose to obtain the development of fractional trigonometric functions analytically, using the theory of FDE. We proceed to obtain fractional

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trigonometric-like functions as the solutions of 2^{nd} order and n^{th} order fractional differential equations employing this approach. Using the fractional counter part of the standard harmonic equation, we define the q-cosine and q-sine functions and develop their properties. Further generalizations are obtained using the n^{th} order FDE of similar type. Many types of fractional derivatives are defined by different scientists such as Grünwald-Letnikov fractional derivative, Riesz fractional derivative, Fourier fractional derivative, Riemann fractional derivative, Riemann-Liouville fractional derivative and Caputo fractional derivative. In this paper, we restrict ourselves to Caputo fractional derivative.

2. PRELIMINARIES

In order to investigate the solutions of $2nd$ order FDE and its generalizations we need to introduce definitions and concepts related to fractional derivatives. These definitions run parallel to the structure of solutions of ordinary differential equations. In this context, we begin with a generalization of the exponential function known as the Mittag-Leffler function which was discovered in 1903 [5].

Definition 2.1. The Mittag-Leffler function of one parameter, $E_q(z)$ is defined by

$$E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq+1)} \quad (z \in C, R(q) > 0). \tag{2.1}$$

Definition 2.2. The Mittag-Leffler function of two parameters, $E_{q,\beta}(z)$ is defined by

$$E_{q,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq+\beta)} \quad (z, \beta \in C, R(q) > 0). \tag{2.2}$$

The definitions of fractional derivatives for a series by Riemann and Caputo are given below.

Definition 2.3. Riemann-Liouville fractional derivative for series. If
$f(x) = x^{q-1} \sum_{k=0}^{\infty} a_k x^{kq}$ then

$$D^q f(x) = \frac{d^q(f(x))}{dx^q} = x^{q-1} \sum_{k=0}^{\infty} a_{k+1} \frac{\Gamma((k+2)q)}{\Gamma((k+1)q)} x^{kq}. \tag{2.3}$$

Definition 2.4. Caputo fractional derivative for series. If $f(x) = \sum_{k=0}^{\infty} a_k x^{kq}$ then

$$^cD^q f(x) = \frac{d^q(f(x))}{dx^q} = \sum_{k=0}^{\infty} a_{k+1} \frac{\Gamma(1+(k+1)q)}{\Gamma(1+kq)} x^{kq}. \tag{2.4}$$

Next we proceed to present the definitions of the fore mentioned derivatives in terms of the integrals.

Definition 2.5. Riemann-Liouville derivative of $x(t)$ is given by

$$D^q x(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} x(s) ds, (t \in R). \tag{2.5}$$

Definition 2.6. Caputo derivative of $x(t)$ is given by

$$^{c}D^q x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} x^{'}(s) ds. \tag{2.6}$$

The initial value problem for Riemann-Liouville fractional differential equation (RLFDE) and the initial value problem for Caputo fractional differential equation (CFDE) have a basic difference. The RLFDE has a singularity at the initial point and is given by

$$D^q x(t) = f(t, x(t)), \quad x^0 = x(t)(t-t_0)^{1-q}/t = t_0,$$

and the CFDE is given by

$$^{c}D^q x(t) = f(t, x(t)), \quad x(t_0) = x_0.$$

There exists a relation between the CFDE and RLFDE which is given by

$$^{c}D^q x(t) = D^q[x(t) - x_0].$$

It has been shown in [7] that the results which hold for the initial value problem of RLFDE are also true for CFDE. On basis of this result we give the existence and uniqueness results for linear n^{th} order RLFDE and for systems and propose that they can be naturally extended for linear CFDE. We now introduce the q-exponential function which is needed to define the solution of the linear Reimann Liouville fractional differential equation.

Definition 2.7. The q-exponential function $e_q^{\lambda z}$ is defined as

$$e_q^{\lambda z} = z^{q-1} E_{q,q}(\lambda z^q) \tag{2.7}$$

where ($z \in C\backslash\{0\}$, $R(q) > 0$) and $\lambda \in C$.

Definition 2.8. We define the function $e_{q,n}^{\lambda z}$ as

$$e_{q,n}^{\lambda z} = z^{q-1} \sum_{k=0}^{\infty} \frac{(k+n)!}{\Gamma[(k+n+1)q]} \frac{(\lambda z^q)^k}{k!}. \tag{2.8}$$

Consider the linear fractional differential equation (LFDE).

$$[L_{nq}(y)](t) := (D_{a^+}^{nq}) y(t) + \sum_{k=0}^{n-1} a_k (D_{a^+}^{kq}) y(t) = 0 \tag{2.9}$$

where the coefficients $\{a_j\}_{j=1}^{n-1}$ are real constants. Then we assume that the solution of the above RLFDE is of the form

$$y(t) = e_q^{\lambda(t-a)}, \quad \lambda \in C$$

and obtain the characteristic equation as

$$P_n(\lambda) = \lambda^n + \sum_{k=1}^{n-1} a_k \lambda^k, \quad \lambda \in C. \tag{2.10}$$

Please refer to [5] for lemmas and theorems that are necessary to obtain the existence and uniqueness result for LFDE (2.9).

We denote R^+ as the set of all non-negative real numbers.

3. TRIGONOMETRIC FUNCTIONS THROUGH SECOND ORDER FDES

In this section,we analytically obtain the trigonometric functions and discuss their properties by using the FDEs. We propose to show that the solutions of the α^{th} order fractional linear differential equation of the harmonic oscillator where $1 < \alpha < 2$ exhibit properties similar to those of cosine and sine functions, which are solutions of the 2^{nd} order ordinary harmonic differential equation. Before proceeding in that direction, for sake of completeness, we begin with the q^{th} order FDE, $0 < q < 1$ and state the following theorem from [5].

Theorem 3.1. *Consider the q^{th} order homogeneous Caputo fractional Initial value problem (IVP)*

$$^cD^q x(t) + x(t) = 0 \quad , \quad x(0) = 1 \tag{3.1}$$

where $0 < q < 1$, $t \geq 0$. Then the solution of (3.1) *is given in the infinite series of the form*

$$x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{kq}}{\Gamma(1+kq)} = E_q(-t^q), \quad t \geq 0.$$

The graph of the solution $x(t)$ is given below for different values of q in Fig-1.

Next we proceed to study the solutions of the fractional harmonic oscillator for $1 < \alpha < 2$ in the following theorem .

Theorem 3.2. *Consider the Initial value problem (IVP) of α^{th} order homogeneous fractional differential equation with Caputo derivative given by*

$$^cD^\alpha x(t) + x(t) = 0, \quad 1 < \alpha < 2, \quad t \geq 0, \tag{3.2}$$

$$x(0) = 1, \quad ^cD^q x(0) = 0, \quad \text{where } \alpha = 2q, \quad 0 < q < 1. \tag{3.3}$$

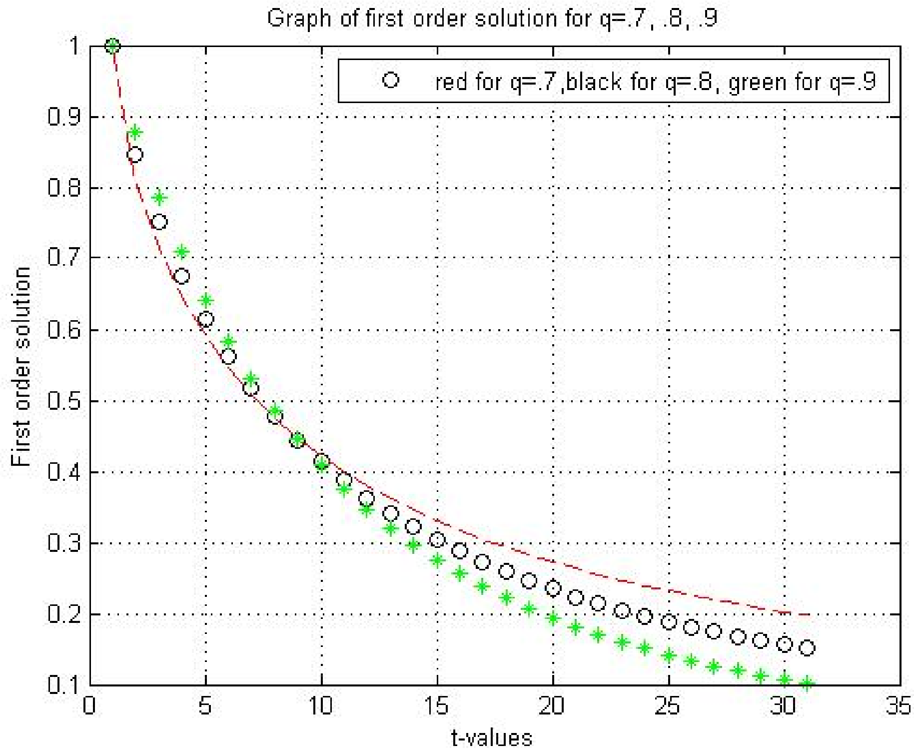

FIGURE 1

Then the general solution is given by $c_1x(t)+c_2y(t)$ (c_1, c_2 being arbitrary constants) where $x(t)$ and $y(t)$ are infinite series solutions of the form

$$x(t)=\sum_{k=0}^{\infty}\frac{(-1)^k t^{2kq}}{\Gamma(1+2kq)},\quad y(t)=\sum_{k=0}^{\infty}\frac{(-1)^k t^{(2k+1)q}}{\Gamma(1+(2k+1)q)},\quad t\geq 0,\quad 0<q<1.$$

Proof. We transform the given IVP to a system of equations of qth order, $0<q<1$ by taking $\alpha=2q$ and setting

$$^cD^qx(t)=-y(t) \text{ and } {}^cD^qy(t)=x(t),\quad x(0)=1, y(0)=0. \tag{3.4}$$

Now, let

$$x(t)=\sum_{k=0}^{\infty}a_kt^{kq},\quad y(t)=\sum_{k=0}^{\infty}b_kt^{kq} \tag{3.5}$$

be solutions of the system (3.2) where $a_k's$ and $b_k's$ are unknown constants and $t\geq 0$. We proceed to find $a_k's$ and $b_k's$ as follows. Using the initial conditions in (3.4), we obtain $a_0=1$, $b_0=0$. □

Utilizing the fact that $^cD^qx(t)=-y(t)$ and substituting (3.5) in this equation we get that

$$\sum_{k=0}^{\infty}a_{k+1}\frac{\Gamma(1+(k+1)q)}{\Gamma(1+kq)}t^{kq}=-\sum_{k=0}^{\infty}b_kt^{kq}.$$

which yields, $a_{k+1} = -\frac{\Gamma(1+kq)}{\Gamma(1+(k+1)q)} b_k$, for each $k = 0, 1, 2, \dots$ Similarly by using ${}^cD^q y(t) = x(t)$, $t \geq 0$, we get $b_{k+1} = \frac{\Gamma(1+kq)}{\Gamma(1+(k+1)q)} a_k$, for each $k = 0, 1, 2, \dots$. By substituting successively, we obtain the values of $a_1, a_2, \dots$ and $b_1, b_2, \dots$ and finally the solutions are given by

$$x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2kq}}{\Gamma(1+2kq)} = E_{2q}(-t^{2q}), \quad t \geq 0,$$

which is in view of (2.1) and

$$y(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(2k+1)q}}{\Gamma(1+(2k+1)q)} = t^q E_{2q,q+1}(-t^{2q}), \quad t \geq 0,$$

which is in view of (2.2). We designate these series by $\cos_q t$ and $sin_q t$ respectively. Thus $\cos_q t$ and $\sin_q t$ are defined as

$$\cos_q t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2kq}}{\Gamma(1+2kq)} = M^q_{2,0}(t) \quad \text{(say)}$$

$$\sin_q t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(2k+1)q}}{\Gamma(1+(2k+1)q)} = M^q_{2,1}(t) \quad \text{(say) respectively .}$$

We borrow this notation from the classical trigonometry since for $q = 1$, $\cos_q(t) = \cos t$ and $\sin_q(t) = \sin t$. The notation $M^q_{2,0}$, $M^q_{2,1}(t)$ is introduced for future convenience.

The graphs of the solutions $\cos_q(t)$ and $\sin_q(t)$ are given below for different values of q in Fig-2 and Fig-3 respectively.

Note: By assuming that the solution of the equation (3.2) is of the form $x(t) = E_q(\lambda t^q)$, we can show that $M^q_{2,0}(t) = \cos_q t$ and $M^q_{2,1}(t) = \sin_q t$ are two linearly independent solutions of the equation (3.2).

Definition 3.3 (Wronskian)**.** Suppose that $\phi_1, \phi_2, \dots, \phi_n$ are n real or complex valued functions defined on some nonempty interval I in R^+ and each having derivatives of order $\alpha = nq$. For $t \in I$, define the determinant

$$W(t) = W(\phi_1, \phi_2, \phi_3, \dots, \phi_n)(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \\ {}^cD^q\phi_1(t) & {}^cD^q\phi_2(t) & \cdots & {}^cD^q\phi_n(t) \\ \vdots & \vdots & \vdots & \vdots \\ {}^cD^{(n-1)q}\phi_1(t) & {}^cD^{(n-1)q}\phi_2(t) & \cdots & {}^cD^{(n-1)q}\phi_n(t) \end{vmatrix}.$$

The function $W(t)$ is called the fractional Wronskian of n-functions $\phi_1, \phi_2, \dots, \phi_n$.

We state and prove a result that relates the Wronskian and the solutions of the equation (3.2).

Theorem 3.4. *Let $x(t)$ and $y(t)$ be two solutions of the equation* (3.2) *on R^+. These two solutions are linearly independent on R^+ if and only if the f-Wronskian $W(t) \neq 0$ for every $t \geq 0$.*

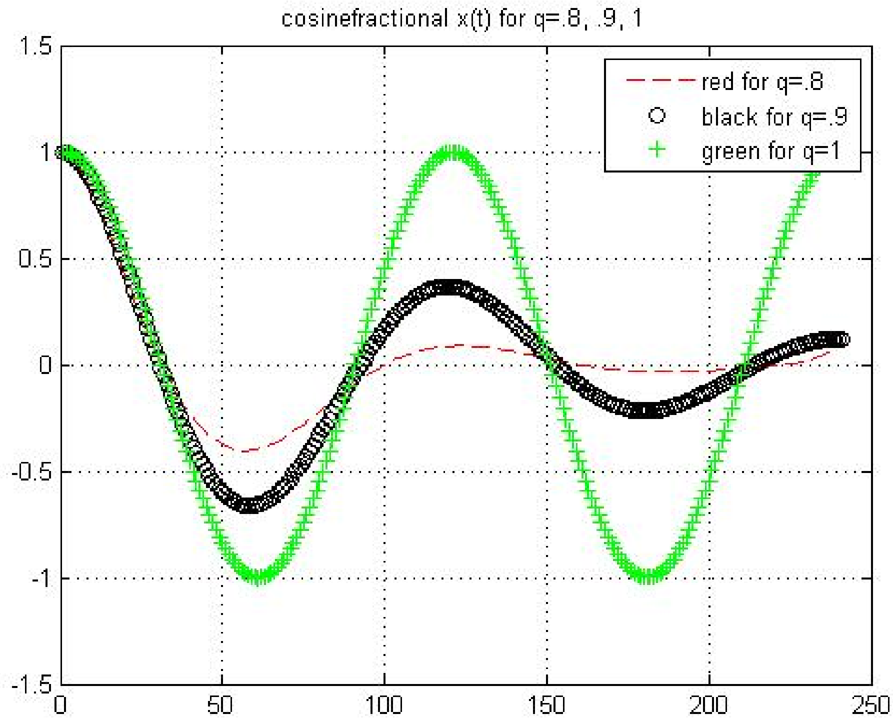

FIGURE 2

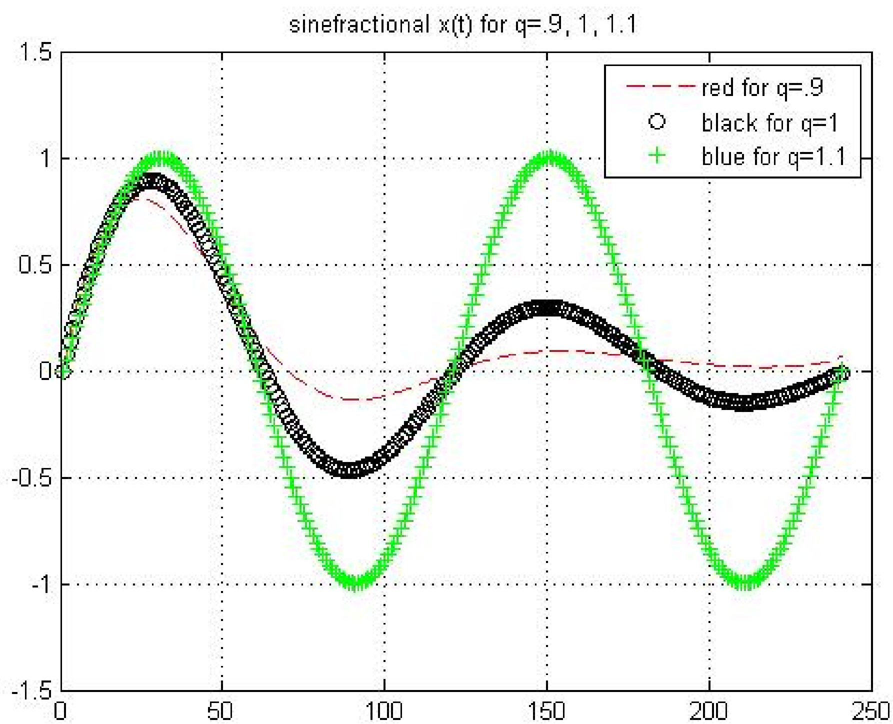

FIGURE 3

Proof. Suppose that $W(t) \neq 0$ of the solutions $x(t)$ and $y(t)$ of FDE (3.2). To show that $x(t)$ and $y(t)$ are linearly independent solutions. If possible assume that $x(t)$ and $y(t)$ are linearly dependent solutions. Then $x(t) = ky(t)$ where k is a constant. Now consider the Wronskian

$$W(t) = \begin{vmatrix} x(t) & y(t) \\ {}^cD^q x(t) & {}^cD^q y(t) \end{vmatrix} = \begin{vmatrix} ky(t) & y(t) \\ k\,{}^cD^q y(t) & {}^cD^q y(t) \end{vmatrix} = 0$$

which is a contradiction. Hence the solutions are linearly independent. To obtain a sufficient condition, assume that $x(t)$ and $y(t)$ are two linearly independent solutions of (3.2). Then the Wronskian for $t \geq 0$ is given by

$$W(t) = \begin{vmatrix} x(t) & y(t) \\ {}^cD^q x(t) & {}^cD^q y(t) \end{vmatrix}.$$

Differentiating W using the Caputo derivative definition for the determinant, we get

$${}^cD^q W(t) = 0.$$

Now this further implies that for Caputo derivative, $W(t) = c$ where c is constant, $t \geq 0$.

Noting that for t $= 0$, $W(0) = 1$. Then we get $c = 1$. This implies that $W(t) = 1 \neq 0$ for $t \geq 0$. The proof is complete. □

We now state and prove the following Corollary relating the solutions of (3.2).

Corollary 3.5. *If $x(t)$ and $y(t)$ are the two linearly independent solutions of the fractional harmonic equation* (3.2)*. Then $x^2(t) + y^2(t) = 1$, $t \geq 0$.*

Proof. Finding Caputo q-derivative of $W(t)$, $t \geq 0$, we get that ${}^cD^q W(t) = 0$, which implies that $W(t) = c = 1$ for $t \geq 0$ from the hypothesis. Thus we obtain $x^2(t) + y^2(t) = 1$, which is a basic property relating the solutions of (3.2), i.e., $\cos_q^2(t) + \sin_q^2(t) = (M_{2.0}^q(t))^2 + (M_{2,1}^q(t))^2 = 1$, $0 < q < 1$, $t \geq 0$. □

We now show that $\cos_q(t)$, $\sin_q(t)$, $t \geq 0$ possess oscillatory behavior.

Theorem 3.6. *The linearly independent solutions $x(t)$ and $y(t)$ of the FDE* (3.2) *have at-least one zero in R^+.*

Proof. Consider the FDE in (3.2) with initial condition.

Let ${}^cD^q x(t) = -y(t)$, ${}^cD^q y(t) = x(t)$ with initial conditions $x(0) = 1, y(0) = 0$. We now claim that there exists a positive number $t_0 \in R^+$ such that $x(t_0) = 0$. Suppose, if possible, we cannot find a $t_0 > 0$ such that $x(t_0) = 0$. Since $x(0) = 1$ and $x(t)$ is a continuous function, $x(t)$ must be positive for $t > 0$. Thus ${}^cD^q y(t) = x(t) > 0$

and ${}^cD^qy(t) > 0$ for $t > 0$. This further implies that $y(t)$ is non-negative for $t > 0$. Since $y(0) = 0$ and $-1 \le y(t) \le 1$ the solution is non-negative.

We have that $y(t)$ is increasing for a some small neighbourhood of zero.

Let $T > 0$ be any arbitrary number in the above neighbourhood of zero. Consider $0 < t < s < T$. Then,

$$y(t)(T-t)^{q-1} < y(s)(s-t)^{q-1} = \int_t^T y(s)(s-t)^{q-1}ds = \Gamma q I_T^q y(s)$$
$$= -\Gamma q I_T^{q\,c} D^q x(s) = -\Gamma q[x(s) - x(T)] < 2\Gamma q.$$

Because t, T, belong to the neighborhood of zero, $T - t < 1 \Rightarrow \frac{1}{T-t} > 1$. So we can choose T so small that $y(t)(T-t)^{q-1} > 2\Gamma q$. Hence the inequality leads to a contradiction. The conclusion is that there exists a positive number t_0 such that $x(t_0) = 0$. □

Theorem 3.7. *The zeros of the solutions $x(t)$ and $y(t)$ of second order CFDE interlace each other i.e., between any two consecutive zeros of $y(t)$ there exists one and only one zero of $x(t)$.*

Proof. Suppose that t_1 and t_2 are two consecutive zeros of $y(t)$ i.e., $y(t_1) = 0$, $y(t_2) = 0$ and $y(t) \neq 0$ for all $t \in (t_1, t_2)$. This implies that $y(t)$ is either positive or negative over the interval (t_1, t_2). Without loss of generality, let us assume that $y(t) > 0$ for $t \in (t_1, t_2)$. Then by Rolle's theorem, we have that $y'(\xi) = 0$ for some $\xi \in (t_1, t_2)$. This yields that $y'(s)$ is increasing in (t_1, ξ) and $y'(s)$ is decreasing in (ξ, t_2) or vice versa. This further implies that

$$x(\xi) = {}^cD^q y_{t_1}(\xi) = \frac{1}{\Gamma(1-q)} \int_{t_1}^{\xi} (\xi - s)^{-q} y'(s)ds \ge 0$$

and

$$x(\xi) = {}^cD^q y_{t_2}(\xi) = \frac{1}{\Gamma(1-q)} \int_{\xi}^{t_2} (s - \xi)^{-q} y'(s)ds \le 0,$$

which gives $x(\xi) = 0$. This follows from the fact that $x(t)$ is continuous at ξ and the fore mentioned property of $y'(s)$ in (t_1, t_2). Thus between two successive zeros of $y(t)$ there exists a zero of $x(t)$. We now proceed to show that the zero is unique. If not, suppose there exists $t_1 < \xi_1 < \xi_2 < t_2$ such that $x(\xi_1) = 0$ and $x(\xi_2) = 0$ and $x(s) \neq 0$ for $s \in (\xi_1, \xi_2)$. Now repeating the earlier proof by writing $x(t)$ in place of $y(t)$ in the interval (ξ_1, ξ_2) we obtain that there exists a $\eta \in (\xi_1, \xi_2)$ such that $y(\eta) = 0$. This contradicts the fact that t_1 and t_2 are two consecutive zeros of $y(t)$ and the proof is complete. □

The addition properties of the solutions of CFDE are discussed below.

Addition Formulae: We now show that the solution $(x(t), y(t))$ of (3.2) possesses the properties

$$y(t+\eta) = y(t)x(\eta) + y(\eta)x(t) \tag{3.6}$$

$$x(t+\eta) = x(t)x(\eta) - y(\eta)y(t), \quad (t \geq 0, \eta \geq 0). \tag{3.7}$$

To prove these properties we use the method of linear algebra.

It is known that if $(x(t), y(t))$ is a solution (3.2) then $(x(t+\eta), y(t+\eta))$, $\eta \geq 0$ also satisfies (3.2). Now these solutions can be expressed in terms of $x(t)$ and $y(t)$ in the following form

$$y(t+\eta) = c_1 y(t) + c_2 x(t) \tag{3.8}$$

$$x(t+\eta) = d_1 y(t) + d_2 x(t). \tag{3.9}$$

Here c_1, c_2, d_1 and d_2 are constants to be chosen appropriately for a given value of $\eta \geq 0$. Clearly, for $t = 0$ we have

$$y(\eta) = c_1 y(0) + c_2 x(0) = c_2$$

$$x(\eta) = d_1 y(0) + d_2 x(0) = d_2.$$

Further

$$x(t+\eta) =^c D^q(y(t+\eta)) = c_1 x(t) - c_2 y(t)$$

$$-y(t+\eta) =^c D^q(x(t+\eta)) = d_1 x(t) - d_2 y(t)$$

which yields, for $t = 0, x(\eta) = c_1$, $-y(\eta) = d_1$. Here we have used the initial conditions in (3.4). Substituting the values of c_1, d_1, c_2 and d_2 in (3.8) and (3.9) we get the relations (3.6) and (3.7).

The relations (3.6) and (3.7) can be described as addition formulae for the functions $x(t)$ and $y(t)$. These in turn give rise to several useful relations, for $\eta = t$ which are

$$y(2t) = 2x(t)y(t), \quad x(2t) = x^2(t) - y^2(t)$$

$$= 2x^2(t) - 1 = 1 - 2y^2(t), \quad (t \geq 0).$$

Further, let $\eta = 2t$. Then we obtain, for $t \geq 0$,

$$y(3t) = 3y(t) - 4y^3(t), \quad x(3t) = 4x^3(t) - 3x(t).$$

Even and Odd Functions: We now prove that $x(t)$ is an even function for $t \in R$. We know that

$$x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2kq}}{\Gamma(1+2kq)}$$

and for $t = -t$ we get

$$x(-t) = \sum_{k=0}^{\infty} \frac{(-1)^k (-t)^{2kq}}{\Gamma(1+2kq)} = \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^{2kq} t^{2kq}}{\Gamma(1+2kq)} = x(t). \tag{3.10}$$

This shows that $x(t)$ is an even function. Let us now consider the function $y(t)$ for $t \in R$.

$$y(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(2k+1)q}}{\Gamma(1+(2k+1)q)}$$

and for $t = -t$, we get

$$y(-t) = \sum_{k=0}^{\infty} \frac{(-1)^k (-t)^{(2k+1)q}}{\Gamma(1+(2k+1)q)} = \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^{(2k+1)q} t^{(2k+1)q}}{\Gamma(1+(2k+1)q)} = (-1)^q y(t)$$

Thus

$$y(-t) = (-1)^q y(t) \tag{3.11}$$

Now it is clear that for $q \neq 1$, $y(t)$ is not an odd function. Using (3.10) and (3.11) we show that for $t, \eta \in R^+$

$$x(t-\eta) = x(t)x(\eta) - (-1)^q y(t)y(\eta)$$

and

$$y(t-\eta) = y(t)x(\eta) + (-1)^q x(t)y(\eta).$$

Further we can obtain

$$y(t+\eta) + y(t-\eta) = 2y(t)x(\eta) + (1 + (-1)^q)x(t)y(\eta)$$

$$y(t+\eta) - y(t-\eta) = (1 - (-1)^q)x(t)y(\eta)$$

$$x(t+\eta) + x(t-\eta) = 2x(t)x(\eta) - ((-1)^q + 1)y(t)y(\eta)$$

$$x(t+\eta) - x(t-\eta) = ((-1)^q - 1)y(t)y(\eta).$$

When $q = 1$ the above formulae coincide with transformation formulae in classical trigonometry.

Euler's Formulae: The solutions of FDE (3.2) are $E_q(it^q)$ and $E_q(-it^q)$ where $\pm i$ are the roots of $\lambda^2 + 1 = 0$. $E_q(it^q)$ and $E_q(-it^q)$ can be expressed in terms of $M^q_{2,0}(t)$ and $M^q_{2,1}(t)$ as

(i) $E_q(it^q) = 1 - \frac{t^{2q}}{\Gamma(1+2q)} + \frac{t^{4q}}{\Gamma(1+4q)} - \cdots + i\left(\frac{t^q}{\Gamma(1+q)} - \frac{t^{3q}}{\Gamma(1+3q)} + \cdots\right) = M^q_{2,0}(t) + iM^q_{2,1}(t)$. and

(ii) $E_q(-it^q) = 1 - \frac{t^{2q}}{\Gamma(1+2q)} + \frac{t^{4q}}{\Gamma(1+4q)} - \cdots - i\left(\frac{t^q}{\Gamma(1+q)} - \frac{t^{3q}}{\Gamma(1+3q)} + \cdots\right) = M^q_{2,0}(t) - iM^q_{2,1}(t)$

which give

$$M_{2,0}^q(t) = \frac{1}{2}(E_q(it^q) + E_q(-it^q)) \tag{3.12}$$

and

$$M_{2,1}^q(t) = \frac{1}{2i}(E_q(it^q) - E_q(-it^q)), \quad t \in R^+. \tag{3.13}$$

4. EXTENSION OF TRIGONOMETRIC FUNCTIONS: 3^{rd} ORDER

We have already seen that the FDE (3.2) gives rise to trigonometric like functions $\cos_q(t)$ and $\sin_q(t)$, for $0 < q < 1$. Note that FDE (3.2) is of order two. We next consider the FDE of order three of the same family. Interestingly, the solutions of this equation also give trigonometric type of solutions having several similar properties. We call the solutions of the third order equation as extended trigonometric functions.

We now state some results corresponding to the third order CFDE. The proofs are parallel to the second order CFDE and hence are omitted.

We consider the α^{th} order $(2 < \alpha < 3)$ homogeneous Caputo fractional IVP,

$$^cD^\alpha x(t) + x(t) = 0, \quad x(0) = 1, \quad {}^cD^q x(0) = 0, \quad {}^cD^{2q}x(0) = 0 \tag{4.1}$$

where $\alpha = 3q$, $0 < q < 1$, $t \geq 0$. Then the general solution is given by $c_1 x(t) + c_2 y(t) + c_3 z(t)$ where $x(t), y(t)$ and $z(t)$ are infinite series solutions of the form

$$x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{3kq}}{\Gamma(1+3kq)} = M_{3,0}^q \text{ (say)}$$

$$y(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(3k+1)q}}{\Gamma(1+(3k+1)q)} = M_{3,1}^q i \text{ (say)}$$

$$z(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(3k+2)q}}{\Gamma(1+(3k+2)q)} = M_{3,2}^q \text{ (say)}, \quad t \geq 0.$$

The graphs of the solutions $x(t), y(t)$ and $z(t)$ are given below for different values of q in Fig-4, Fig-5 and Fig-6 respectively.

The Wronskian property in this setup is as follows.

Theorem 4.1. *Let $x(t)$, $y(t)$, and $z(t)$ be three solutions of the equation* (4.1)*. These three solutions are linearly independent on R^+ if and only if the Wronskian for every $t \geq 0$,*

$$W(t) = \begin{vmatrix} x(t) & y(t) & z(t) \\ ^cD^q x(t) & ^cD^q y(t) & ^cD^q z(t) \\ ^cD^{2q} x(t) & ^cD^{2q} y(t) & ^cD^{2q} z(t) \end{vmatrix} = \begin{vmatrix} x(t) & y(t) & z(t) \\ -z(t) & x(t) & y(t) \\ -y(t) & -z(t) & x(t) \end{vmatrix} \neq 0.$$

The following Corollary gives a relation between the solutions of the third order CFDE (4.1) *with initial conditions in* (4.1)*.*

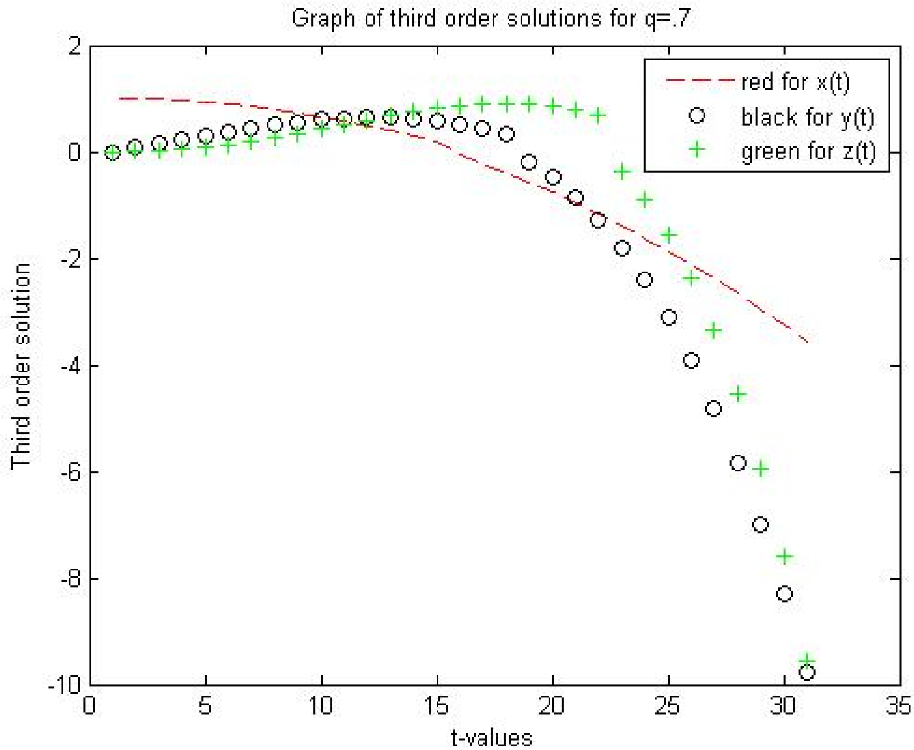

FIGURE 4

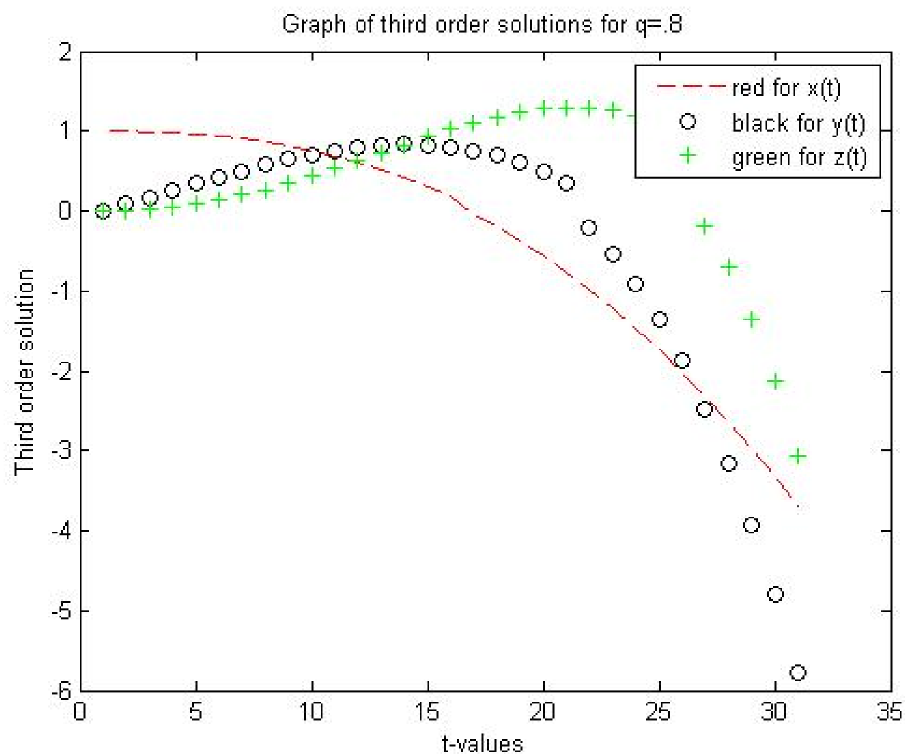

FIGURE 5

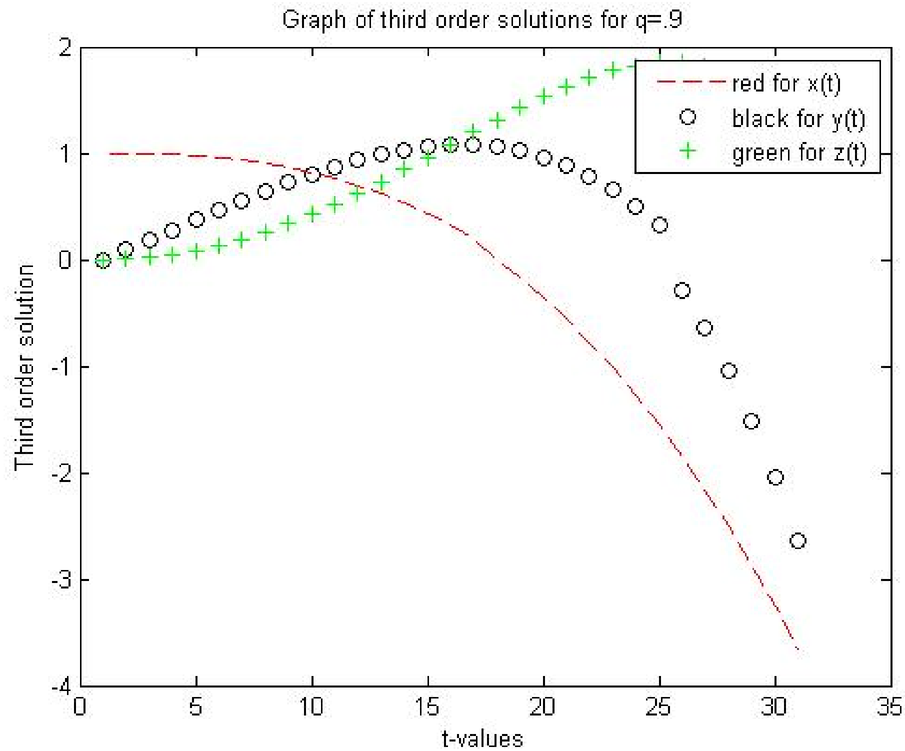

FIGURE 6

Corollary 4.2. *The linearly independent solutions of the third order CFDE satisfy the relation*

$$W(x(t), y(t), z(t)) = x^3(t) - y^3(t) + z^3(t) + 3x(t)y(t)z(t) = 1$$

where $x(t), y(t), z(t)$ *for* $t \geq 0$ *are the linearly independent solutions of the FDE* (4.1), $0 < q < 1$.

We present below the addition formulae for solutions of third order CFDE. The proofs can be obtained by following the technique used for solutions of second order CFDE.

Addition Formulae: Let $\eta \geq 0$ be arbitrary. Following the method of linear algebra again it follows that the solution $(x(t), y(t), z(t))$ of (4.1) possesses the properties

$$x(t+\eta) = x(t)x(\eta) - y(t)z(\eta) - z(t)y(\eta), \tag{4.2}$$

$$y(t+\eta) = x(t)y(\eta) + y(t)x(\eta) - z(t)z(\eta), \tag{4.3}$$

$$z(t+\eta) = x(t)z(\eta) + y(t)y(\eta) + z(t)x(\eta). \tag{4.4}$$

From these relations we derive, for $\eta = t$,

$$x(2t) = x^2(t) - 2y(t)z(t),$$

$$y(2t) = 2x(t)y(t) - z^2(t),$$

$$z(2t) = 2x(t)z(t) + y^2(t).$$

These results may be easily used to obtain the values of $x(3t)$, $y(3t)$, and $z(3t)$ and many similar relations.

Similar to the Euler's formulae for the second order FDE we can obtain the Euler's formulae for the third order FDE.

Euler's Formulae: The solutions of the FDE (4.1) are $E_q(-t^q)$, $E_q(\omega t^q)$ and $E_q(-\omega^2 t^q)$ where -1, ω, $-\omega^2$ ($\omega = \frac{1}{2} - \frac{\sqrt{3}i}{2}$) are the roots of the equation $\lambda^3 + 1 = 0$. We express $E_q(-t^q)$, $E_q(\omega t^q)$ and $E_q(-\omega^2 t^q)$ in terms of $M^q_{3,0}(t)$, $M^q_{3,1}(t)$ and $M^q_{3,2}(t)$ respectively as follows.

$$E_q(-t^q) = M^q_{3,0}(t) - M^q_{3,1}(t) + M^q_{3,2}(t), \tag{4.5}$$

$$E_q(\omega t^q) = M^q_{3,0}(t) + \omega M^q_{3,1}(t) + \omega^2 M^q_{3,2}(t), \tag{4.6}$$

$$E_q(-\omega^2 t^q) = M^q_{3,0}(t) - \omega^2 M^q_{3,1}(t) - \omega M^q_{3,2}(t), \tag{4.7}$$

We can also express $M^q_{3,0}(t)$, $M^q_{3,1}(t)$ and $M^q_{3,2}(t)$ in terms of $E_q(-t^q)$, $E_q(\omega t^q)$ and $E_q(-\omega^2 t^q)$. By solving (4.5), (4.6) and (4.7) we get

$$M^q_{3,0}(t) = \frac{1}{3}E_q(-t^q) - \frac{1}{3}E_q(\omega t^q) + \frac{1}{3}E_q(-\omega^2 t^q), \tag{4.8}$$

$$M^q_{3,1}(t) = -\frac{1}{3}E_q(-t^q) - \frac{\omega^2}{3}E_q(\omega t^q) + \frac{\omega}{3}E_q(-\omega^2 t^q), \tag{4.9}$$

$$M^q_{3,2}(t) = \frac{1}{3}E_q(-t^q) - \frac{\omega}{3}E_q(\omega t^q) + \frac{\omega^2}{3}E_q(-\omega^2 t^q). \tag{4.10}$$

5. EXTENSION OF TRIGONOMETRIC FUNCTIONS TO n^{th} ORDER FDE

It can be observed that, the results obtained in section three and section four can be generalized to n^{th} order CFDE. We proceed to do so in this section. As the proofs can be naturally extended to n^{th} order and are routine, we omit them.

Theorem 5.1. *Consider the n^{th} order fractional IVP of the form*

$$^cD^\alpha x(t) + x(t) = 0, \quad x(0) = 1, \quad ^cD^q x(0) = 0, \ldots, ^cD^{(n-1)q}x(0) = 0. \tag{5.1}$$

where $n < \alpha < n+1$, with $\alpha = nq$, $0 < q < 1$, n fixed.

The general solution of this equation is given by $c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t)$ where $x_1(t), x_2(t), \ldots, x_n(t)$ are infinite series solutions of the form

$$
\begin{aligned}
x_1(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{nkq}}{\Gamma(1+nkq)} = M^q_{n,o}(t) \quad \text{(say)} \\
x_2(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{(nk+1)q}}{\Gamma(1+(nk+1)q)} = M^q_{n,1}(t) \quad \text{(say)} \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
x_n(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{(nk+(n-1))q}}{\Gamma(1+(nk+(n-1))q)} = M^q_{n,n-1}(t) \quad \text{(say)} \qquad t \geq 0.
\end{aligned}
$$

At this stage, let us consider suitable notation to conveniently represent such infinite series. The notation is as follows.

$$
M^q_{n,r}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(nk+r)q}}{\Gamma(1+(nk+r)q)}, \quad 0 \leq r < n, \quad n \in N, \quad t \geq 0. \tag{5.2}
$$

These are the n linearly independent solutions of CFDE (5.1).

The Wronskian property for the n^{th} order CFDE is as follows.

Theorem 5.2. *Let $x_1(t), x_2(t), \ldots, x_n(t)$ be n solutions of the equation* (5.1). *These n solutions are linearly independent on R if and only if the Wronskian $W(t) \neq 0$ for every $t \in R^+$.*

Here

$$
W(t) = \begin{vmatrix} x_1 & x_2 \ldots & x_n & \\ -x_n & x_1 \ldots & x_{n-1} & \\ -x_{n-1} & -x_n & \ldots & x_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ -x_2 & -x_3 & \ldots & x_1 \end{vmatrix} (t).
$$

The following Corollary gives a relation between the solutions of the n^{th} order CFDE (5.1) with initial conditions in (5.1).

Corollary 5.3. *The linearly independent solutions of the n^{th} order CFDE* (5.1) *satisfy the relation*

$$
\begin{vmatrix} x_1 & x_2 \ldots & x_n & \\ -x_n & x_1 \ldots & x_{n-1} & \\ -x_{n-1} & -x_n & \ldots & x_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ -x_2 & -x_3 & \ldots & x_1 \end{vmatrix} (t) = 1.
$$

This formula includes the relations obtained in Corollary 3.5 *and Corollary* 4.2.

The addition formulae of the solutions of CFDE (5.1) are given below.

Addition Formulae: The method adopted for obtaining these addition formulae in section 3 and section 4 works smoothly for n^{th} order FDE also. Hence we only state these formulae, for $\eta \geq 0$, $t \geq 0$.

$$M^q_{n,r}(t+\eta) = \sum_{k=0}^{r} M^q_{n,k}(t) M^q_{n,r-k}(\eta) - \sum_{k=r+1}^{n-1} M^q_{n,k}(t) M^q_{n,n+r-k}(\eta). \tag{5.3}$$

All the addition formulae are particular cases of (5.3). Clearly for $(n, r) = (2, 0)$ and $(2, 1)$ and for $(n, r) = (3, 0)$, $(3, 1)$ and $(3, 2)$ we get the addition formulae obtained in (3.6), (3.7) and (4.2), (4.3), (4.4) respectively.

It is now easy to generate the formulae for multiple angles. The $M^q_{n,r}(t)$ in (5.2) are extended trigonometric like functions.

6. CONCLUSION

In this paper we have obtained the classical trigonometric functions $\cos_q t$ and $\sin_q t$ using 2^{nd} order FDE. Next using the 3^{rd} order FDE of the type ${}^cD^\alpha x(t)+x(t) = 0$ we obtained trigonometric like functions $M^q_{3,0}(t)$, $M^q_{3,1}(t)$ and $M^q_{3,2}(t)$. This approach has been generalized to obtain trigonometric like functions for n^{th} order. Further, using Matlab techniques we can formulate tables of trigonometric and extended trigonometric functions. It is also possible to use Laplace method to solve FDEs involved above.

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