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FUNDAMENTAL PROPERTIES OF SOLUTIONS OF NON LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS AND METHOD OF VARIATION OF PARAMETERS

TADESSE ZERIHUN AND G. S. LADDE

Department of Mathematics & Statistics, University of South Florida
Tampa, Florida, USA
tzerihun@mail.usf.edu

This work is dedicated to Professor V. Lakshmikantham.

ABSTRACT. A complex nonlinear nonstationary stochastic system of differential equations are decomposed into nonlinear systems of stochastic perturbed and unperturbed differential equations. Using this type of decomposition, the fundamental properties of solutions of nonlinear stochastic unperturbed systems of differential equations are investigated. The fundamental properties are used to find the representation of solution process of nonlinear stochastic perturbed system in terms of solution process of nonlinear stochastic unperturbed system.

Key words: Nonlinear stochastic differential equations.

1. INTRODUCTION

One of the most well known methods for investigating the nonlinear dynamic processes in sciences and engineering is the method of nonlinear variation of constant parameters [10, 11, 12, 14, 29].

Knowing the knowledge of the existence of solution process, the method of variation of parameters provides a very powerful tool for finding the solution representation of systems of differential equations [10, 11, 12, 14, 29]. The idea is to decompose a complex system of differential equations in to two parts in such a way that a system of differential equations corresponding to the simpler part is either easily solvable in a closed form or analytically analyzable. However, the over all complex system of differential equations are neither easily solvable in a closed form nor analytically analyzable [10, 11]. The method of variation of parameters provides a formula for a solution to the complex system in terms of the solution process of simpler system of differential equations.

In this paper, an attempt is made to find a representation of solutions of nonlinear and nonstationary Itô-Doob type stochastic system of differential equations in terms of solutions processes of smoother system of Itô-Doob type stochastic differentials.

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The organization is as follows: In section 2, the problem is formulated. In section 3, several auxiliary results are established for unperturbed system of nonlinear Itô-Doob type stochastic differential equations. In section 4, a variation of constants formula is established. In section 5, examples are given to illustrate the usefulness of the methods. The Developed results are a convenient tool in discussing the properties of solutions of the perturbed system.

2. PROBLEM FORMULATION

Let us formulate a problem. We consider a mathematical description of a nonlinear dynamic phenomenon under randomly varying environmental perturbations described by a complex system of nonlinear nonstationary Itô-Doob type systems of stochastic differential equations:

$$dy = c(t, y)dt + \Sigma(t, y)dw(t), \qquad y(t_0) = x_0, \tag{2.1}$$

where $y \in R^n$, $c \in C[J \times R^n, R^n]$, $\Sigma \in C[J \times R^n, R^{n\times m}]$, and $C[J \times R^n, R^n]$ ($C[J \times R^n, R^{n\times m}]$) stands for a class of continuous functions defined on $J \times R^n$ into R^n ($R^{n\times m}$); n and m are positive integers ; and $J = [t_0, t_0 + a)$ for some positive real number a; x_0 is an n-dimensional random variable defined on a complete probability space $(\Omega, \mathfrak{F}, P)$; $w(t) = (w_1(t), w_2(t), \ldots, w_m(t))^T$ is an m-dimensional normalized Wiener process with independent increments; x_0 and $w(t)$ are mutually independent for each $t \geq t_0$. We decompose complex system of stochastic differential equations (2.1) into two parts. The decomposition is based on the decomposition of its drift and diffusion rate functions as follows:

$$c(t, y) = f(t, y) + F(t, y)$$

and

$$\Sigma(t, y) = \sigma(t, y) + \Upsilon(t, y)$$

where the rate functions $f(t, y)$ and $\sigma(t, y)$ are considered to be simpler form in the sense of better structure and conceptually smooth. Thus, (2.1) can be rewritten as

$$\begin{aligned} dy &= [f(t, y) + F(t, y)]dt + [\sigma(t, y) + \Upsilon(t, y)]dw(t) \\ &= [f(t, y) + F(t, y)]dt + \textstyle\sum_{l=1}^{m}[\sigma^l(t, y) + \Upsilon^l(t, y)]dw_l(t), \qquad y(t_0) = x_0. \end{aligned} \tag{2.2}$$

The simpler form of mathematical model of dynamic process corresponding to (2.2) is described by

$$\begin{aligned} dx &= f(t, x)dt + \sigma(t, x)dw(t) \\ &= f(t, x)dt + \textstyle\sum_{l=1}^{m} \sigma^l(t, x)dw_l(t), \qquad x(t_0) = x_0. \end{aligned} \tag{2.3}$$

Systems (2.2) and (2.3) are considered to be perturbed and unperturbed systems of stochastic differential equations, respectively.

Remark 2.1. In the absence of any reasonable decomposition of the type (2.2), it is always possible to consider the above decompositions with $F(t, y) = c(t, y) - f(t, y)$ and $\Upsilon(t, y) = \Sigma(t, y) - \sigma(t, y)$ for any suitable choice of rate functions $f(t, y)$ and $\sigma(t, y)$.

3. AUXILIARY RESULTS

Our main objective is to develop the variation of constants formula with respect to (2.3) and its perturbed system (2.2). For this purpose, first we investigate the Itô-Doob stochastic partial differentials of solution process $x(t, t_0, x_0)$ of unperturbed system (2.3) with respect to initial conditions (t_0, x_0).

In the following, under certain smoothness assumption on the rate functions of unperturbed stochastic system of differential equations (2.3), we establish the second order differentials of the solution process of (2.3) with respect to (t_0, x_0). In this section, by recalling the existence of differential of solution process of unperturbed system of stochastic differential equations with respect to initial state, we first establish the existence of second order differential with respect to x_0. Moreover, as the byproduct, we show that the differentials satisfy Itô-Doob type of stochastic non homogeneous matrix differential equation.

Lemma 3.1. *Assume that σ and f in* (2.3) *are twice continuously differentiable with respect to x for fixed t, and f_{xx} , σ_{xx} are bounded with respect to x. Furthermore, the initial value problem* (2.3) *has a unique solution process $x(t, t_0, x_0)$ existing for $t \geq t_0$. Then*

$$\frac{\partial}{\partial x_0}\Phi(t, t_0, x_0) = \frac{\partial^2}{\partial x_0^2} x(t, t_0, x_0) \tag{3.1}$$

exists, and is the solution of the following Itô-Doob type nonhomogeneous stochastic matrix differential equation:

$$dY = [H(t, t_0, x_0)Y + P(t)]dt + \sum_{l=1}^{m}[\Gamma^l(t, t_0, x_0)Y + Q(t)]dw_l(t), \qquad Y(t_0) = 0; \tag{3.2}$$

where the $n\times n$ matrices $H(t, t_0, x_0) = f_x(t, x(t, t_0, x_0))$ and $\Gamma^l(t, t_0, x_0) = \sigma^l_x(t, x(t, t_0, x_0))$ are continuous;

$$P(t) = \left(\frac{\partial^2}{\partial x^2} f(t, x(t)) \otimes \sum_{k=1}^{n} \Phi(t, t_0, x_0)e_k\right) \Phi(t, t_0, x_0);$$

$$Q(t) = \left(\frac{\partial^2}{\partial x^2} \sigma^l(t, x(t)) \otimes \sum_{k=1}^{n} \Phi(t, t_0, x_0)e_k\right) \Phi(t, t_0, x_0),$$

$\Phi(t_0, t_0, x_0)$ is the $n \times n$ identity matrix and $\otimes$ is the tensor product of two matrices.

Proof. From the assumptions of the lemma, we conclude that $\Phi(t,t_0,x_0) = \frac{\partial}{\partial x_0}x(t,t_0,x_0)$ exists and is the solution of the Itô-Doob type stochastic matrix differential equations along the solution process $x(t,t_0,x_0)$ of (2.3)[11, 23]:

$$dY = H(t,t_0,x_0)Y dt + \Gamma(t,t_0,x_0)Y dw(t), \qquad Y(t_0) = I_{n\times n}. \tag{3.3}$$

In the following, we show that $\frac{\partial^2}{\partial x_0^2}x(t,t_0,x_0)$ exists and it satisfies the stochastic differential equation (3.2). For this purpose, we consider the following: For small $\lambda > 0$, let $\Delta x_0 = \sum_{k=1}^{n} \lambda e_k$; where $e_k = (0,0,\ldots,1,\ldots,0)^T$ whose k-th component is 1. Moreover, let $\Phi(t,\lambda) = \Phi(t,t_0,x_0+\Delta x_0)$ and $\Phi(t) = \Phi(t,t_0,x_0)$ be solutions of (3.3) through $(t_0,x_0+\Delta x_0)$ and (t_0,x_0), respectively, and $x(t,\lambda) = x(t,t_0,x_0+\Delta x_0)$ and $x(t) = x(t,t_0,x_0)$ be solutions of (2.3) through $(t_0,x_0+\Delta x_0)$ and (t_0,x_0) respectively. Under the assumptions of Lemma 3.1 and applying Lemma 6.1[11], we conclude that

$$\lim_{\lambda\to 0} \Phi(t,\lambda) = \Phi(t) \text{ uniformly on J.} \tag{3.4}$$

We set

$$\Delta\Phi(t,\lambda) = \Phi(t,\lambda) - \Phi(t), \qquad \Delta\Phi(t_0,\lambda) = 0. \tag{3.5}$$

Let $R(\theta) = \frac{\partial}{\partial x} f(t, x(t,t_0,x_0+\theta\Delta x_0))$ with $0 \le \theta \le 1$. From the assumptions, we note that R is continuously differentiable with respect to θ, and hence

$$\frac{d}{d\theta}R(\theta) = f_{xx}(t, x(t,t_0,x_0+\theta\Delta x_0)) \otimes (\Phi(t,t_0,x_0+\theta\Delta x_0)\Delta x_0). \tag{3.6}$$

By integrating both sides of (3.6) with respect to θ over an interval [0,1], we have

$$R(1) - R(0) = \int_0^1 f_{xx}(t, x(t,t_0,x_0+\theta\Delta x_0)) \otimes (\Phi(t,t_0,x_0+\theta\Delta x_0)\Delta x_0) d\theta.$$

This together with the fact that $R(1) = \frac{\partial}{\partial x} f(t, x(t,\lambda))$ and $R(0) = \frac{\partial}{\partial x} f(t, x(t))$ yields

$$\frac{\partial}{\partial x} f(t, x(t,\lambda)) - \frac{\partial}{\partial x} f(t, x(t)) = J(t, x(t,\lambda), \Phi(t,\lambda)),$$

where

$$J(t,x(t,\lambda),\Phi(t,\lambda)) = \int_0^1 f_{xx}(t,x(t,t_0,x_0+\theta\Delta x_0))\otimes(\Phi(t,t_0,x_0+\theta\Delta x_0)\Delta x_0)d\theta \tag{3.7}$$

Similarly, by setting

$$G(\theta) = \sum_{l=1}^{m} \frac{\partial}{\partial x}\sigma^l(t, x(t,t_0,x_0+\theta\Delta x_0))$$

and using the continuous differentiability of G with respect to θ and chain rule, we have

$$\frac{d}{d\theta}G(\theta) = \sum_{l=1}^{m} \sigma^l_{xx}(t, x(t,t_0,x_0+\theta\Delta x_0)) \otimes (\Phi(t,t_0,x_0+\theta\Delta x_0)\Delta x_0). \tag{3.8}$$

By integrating both sides of (3.8) with respect to θ over an interval [0,1], we get

$$G(1) - G(0) = \sum_{l=1}^{m} \int_0^1 \sigma_{xx}^l(t, x(t, t_0, x_0 + \theta\Delta x_0)) \otimes (\Phi(t, t_0, x_0 + \theta\Delta x_0)\Delta x_0) d\theta.$$

This together with the fact that $G(1) = \sum_{l=1}^m \frac{\partial}{\partial x}\sigma^l(t, x(t, \lambda))$ and $G(0) = \sum_{l=1}^m \frac{\partial}{\partial x}\sigma^l(t, x(t))$ yields

$$\sum_{l=1}^{m}[\frac{\partial}{\partial x}\sigma^l(t, x(t, \lambda)) - \frac{\partial}{\partial x}\sigma^l(t, x(t))] = \sum_{l=1}^{m} \Lambda^l(t, x(t, \lambda), \Phi(t, \lambda)),$$

where

$$\Lambda^l(t, x(t), \Phi(t, \lambda)) = \int_0^1 \sigma_{xx}^l(t, x(t, t_0, x_0+\theta\Delta x_0)) \otimes (\Phi(t, t_0, x_0+\theta\Delta x_0)\Delta x_0) d\theta. \tag{3.9}$$

Note that the integrals in (3.7) and (3.9) are cauchy-Riemann integrals. Using the hypotheses of the Lemma, $n \times n$ matrices $J(t, x(t, \lambda), \Phi(t, \lambda))$ and $\Lambda^l(t, x(t, \lambda), \Phi(t, \lambda))$ are continuous in (t, x, λ) for $l = 1, 2, 3, \dots, m$. Furthermore, from (3.7), (3.9) and the bounded convergence theorem[24], we obtain

$$\lim_{\lambda \to 0} \frac{J(t, x(t, \lambda), \Phi(t, \lambda))}{\lambda} = f_{xx}(t, x(t, t_0, x_0)) \otimes (\sum_{k=1}^{n} \Phi(t, t_0, x_0)e_k) \tag{3.10}$$

and

$$\lim_{\lambda \to 0} \frac{\Lambda^l(t, x(t, \lambda), \Phi(t, \lambda))}{\lambda} = \sigma_{xx}^l(t, x(t, t_0, x_0)) \otimes (\sum_{k=1}^{n} \Phi(t, t_0, x_0)e_k). \tag{3.11}$$

From (3.5), using the fact that $\Phi(t, \lambda)$ and $\Phi(t)$ are solutions of (3.3), we obtain

$$\begin{aligned} d(\Phi(t, \lambda) - \Phi(t)) &= d\Phi(t, \lambda) - d\Phi(t) \\ &= f_x(t, x(t, \lambda))\Phi(t, \lambda)dt + \sum_{l=1}^{m} \sigma_x^l(t, x(t, \lambda))\Phi(t, \lambda)dw_l(t) \\ &\quad - [f_x(t, x(t))\Phi(t)dt + \sum_{l=1}^{m} \sigma_x^l(t, x(t))\Phi(t)dw_l(t)] \\ &= [f_x(t, x(t, \lambda))\Phi(t, \lambda) - f_x(t, x(t))\Phi(t)]dt \\ &\quad + \sum_{l=1}^{m}[\sigma_x^l(t, x(t, \lambda))\Phi(t, \lambda) - \sigma_x^l(t, x(t))\Phi(t)]dw_l(t). \end{aligned} \tag{3.12}$$

By adding and subtracting $f_x(t, x(t))\Phi(t, \lambda)dt$ and $\sum_{l=1}^m \sigma_x^l(t, x(t))\Phi(t, \lambda)dw_l(t)$ in (3.12), we obtain

$$\begin{aligned} d(\Phi(t, \lambda) - \Phi(t)) &= [f_x(t, x(t, \lambda))\Phi(t, \lambda) - f_x(t, x(t))\Phi(t, \lambda) \\ &\quad + f_x(t, x(t))\Phi(t, \lambda) - f_x(t, x(t))\Phi(t)]dt \\ &\quad + [\sum_{l=1}^{m} \sigma_x^l(t, x(t, \lambda))\Phi(t, \lambda)dw_l(t) - \sum_{l=1}^{m} \sigma_x^l(t, x(t))\Phi(t, \lambda)dw_l(t) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{m}\sigma_x^l(t,x(t))\Phi(t,\lambda)dw_l(t)-\sum_{l=1}^{m}\sigma_x^l(t,x(t))\Phi(t)dw_l(t)] \\
= \;& [f_x(t,x(t))(\Phi(t,\lambda)-\Phi(t))+(f_x(t,x(t,\lambda))-f_x(t,x(t)))\Phi(t,\lambda)]dt \\
&+\sum_{l=1}^{m}[\sigma_x^l(t,x(t))(\Phi(t,\lambda)-\Phi(t)) \\
&+(\sigma_x^l(t,x(t,\lambda))-\sigma_x^l(t,x(t)))\Phi(t,\lambda)]dw_l(t).
\end{aligned}
$$

This, together with (3.7), (3.9) and the definitions of $\Delta\Phi(t,\lambda)$ in (3.5), yields

$$
\begin{aligned}
d\left(\frac{\Delta\Phi(t,\lambda)}{\lambda}\right) = \;& [f_x(t,x(t))\frac{\Delta\Phi(t,\lambda)}{\lambda}+\frac{J(t,x(t,\lambda),\Phi(t,\lambda))}{\lambda}\Phi(t,\lambda)]dt \\
&+\sum_{l=1}^{m}[\sigma_x^l(t,x(t))\frac{\Delta\Phi(t,\lambda)}{\lambda}+\frac{\Lambda^l(t,x(t,\lambda),\Phi(t,\lambda))}{\lambda}\Phi(t,\lambda)]dw_l(t).
\end{aligned}
\tag{3.13}
$$

(3.14)

From (3.10) and (3.11), system (3.2) can be considered as the nominal system corresponding to (3.13) with initial data $Y(t_0)=0$. It is obvious that the initial value problem (3.13) satisfies all the hypothesis of Lemma 6.1 [11], and hence by its application, we have

$$
\lim_{\lambda\to 0}\frac{\Delta\Phi(t,\lambda)}{\lambda}=Y(t) \text{ uniformly on } J, \tag{3.15}
$$

where $Y(t)$ is the solution process of (3.13). Because of (3.4) and (3.5), we note that the limit of $\frac{\Delta\Phi(t,\lambda)}{\lambda}$ in (3.13) is equivalent to $\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)$. Thus $\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)$ is the solution process of (3.2). Moreover, $\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)=\frac{\partial^2}{\partial x_0^2}x(t,t_0,x_0)$. $\square$

Example 3.1. Let us consider a scalar nonlinear unperturbed stochastic differential equation:

$$
dx=\alpha x(\rho-x)dt+\beta x dw(t), \qquad x(t_0)=x_0. \tag{3.16}
$$

where α, β and ρ are any constant. Find $\frac{\partial}{\partial x_0}x(t,t_0,x_0)$ and $\frac{\partial^2}{\partial x_0^2}x(t,t_0,x_0)$.

Solution: We note that $f(t,x)=\alpha x(\rho-x)$ and $\sigma(t,x)=\beta x$ are continuously differentiable with respect to x. In fact $\frac{\partial}{\partial x}f(t,x)=\alpha(\rho-2x)$, $\frac{\partial^2}{\partial x^2}f(t,x)=-2\alpha x$, $\frac{\partial}{\partial x}\sigma(t,x)=\beta$, and $\frac{\partial^2}{\partial x^2}\sigma(t,x)=0$. The closed form solution of (3.16) is

$$
x(t,t_0,x_0)=\left[\Phi(t,t_0)x_0^{-1}+\alpha\int_{t_0}^{t}\Phi(t,s)ds)\right]^{-1},
$$

where $\Phi(t,t_0)=exp[-(\alpha\rho-\frac{1}{2}\beta^2)(t-t_0)-\beta(w(t)-w(t_0))]$. The partial derivative of solution processes $x(t,t_0,x_0)$ with respect to x_0 is

$$
\frac{\partial}{\partial x_0}x(t,t_0,x_0)=\frac{\Phi(t,t_0)}{(\Phi(t,t_0)+\alpha x_0\int_{t_0}^{t}\Phi(t,s)ds)^2} \tag{3.17}
$$

and

$$\frac{\partial^2}{\partial x_0^2} x(t,t_0,x_0) = \frac{-2\alpha\Phi(t,t_0)\int_{t_0}^t \Phi(t,s)ds}{(\Phi(t,t_0)+\alpha x_0\int_{t_0}^t \Phi(t,s)ds)^3}. \tag{3.18}$$

Moreover, $\frac{\partial^2}{\partial x_0^2} x(t,t_0,x_0)$ satisfies the following matrix differential equation:

$$\begin{aligned} dY = {} & \Big[\Big[\alpha(\rho - 2\Big[\Phi(t,t_0)x_0^{-1} + \alpha\int_{t_0}^t \Phi(t,s)ds\Big]^{-1}\Big)\Big] \\ & \times Y - 2\alpha\Phi^2(t,t_0)]dt + \beta Y dw(t), \qquad Y(t_0) = 0; \end{aligned} \tag{3.19}$$

The following result shows the existence of partial differential of solution process of (2.3) with respect to t_0.

Lemma 3.2. *Let us assume that all the hypothesis of Lemma* 3.1 *be satisfied. Let* $x(t,t_0,x_0)$ *be the solution process of* (2.3) *existing for* $t \geq t_0$. *Then*

$$\partial_{t_0} x(t,t_0,x_0)$$

exists and:

$$\begin{aligned} \partial_{t_0} x(t,t_0,x_0) = {} & \frac{1}{2}\left(\sum_{j=1}^n \sum_{l=1}^m \frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma_j^l(t_0,x_0)\right)_{n\times 1} dt_0 \\ & +\Phi(t,t_0,x_0)[\sum_{l=1}^m \sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0) - f(t_0,x_0)]dt_0 \\ & -\sum_{l=1}^m \Phi(t,t_0,x_0)\sigma^l(t_0,x_0)dw_l(t_0) \end{aligned} \tag{3.20}$$

with

$$\begin{aligned} \partial_{t_0} x(t_0,t_0,x_0) = {} & \left[\sum_{l=1}^m \sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0) - f(t_0,x_0)\right] dt_0 \\ & -\sum_{l=1}^m \sigma^l(t_0,x_0)dw_l(t_0) \end{aligned} \tag{3.21}$$

Proof. Let $\Delta t_0 = \lambda > 0$ be a positive increment to t_0, and define

$$\Delta x(t,\lambda) = x(t,t_0+\lambda,x_0) - x(t,t_0,x_0) \tag{3.22}$$

where $x(t,t_0+\lambda,x_0)$ and $x(t,t_0,x_0)$ are solution processes of (2.3) through $(t_0+\lambda,x_0)$ and (t_0,x_0), respectively. Let

$$\Delta x(t_0) = x(t_0+\lambda,t_0,x_0) - x(t_0,t_0,x_0).$$

Set $R(\theta) = x(t,t_0+\lambda,x_0+\theta\Delta x(t_0))$. It is obvious that R is continuously differentiable with respect to θ, and hence

$$\frac{d}{d\theta}R(\theta) = \frac{\partial}{\partial x_0} x(t,t_0+\lambda,x_0+\theta\Delta x(t_0))\Delta x(t_0) \tag{3.23}$$

$$= \Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0))\Delta x(t_0).$$

By integrating both sides of (3.23) with respect to θ over an interval [0,1], we have

$$R(1) - R(0) = \int_0^1 \Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0))\Delta x(t_0) d\theta.$$

This, together with the fact that $R(1) = x(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))$ and $R(0) = x(t, t_0 + \lambda, x_0)$, yields

(3.24)
$$x(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) - x(t, t_0 + \lambda, x_0) = \int_0^1 \Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0))\Delta x(t_0) d\theta.$$

Because of the uniqueness of solution of (2.3) we have $x(t, t_0, x_0) = x(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))$ and equation (3.24) can be written as

$$x(t, t_0 + \lambda, x_0) - x(t, t_0, x_0) = -\int_0^1 \Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0))\Delta x(t_0) d\theta. \tag{3.25}$$

By adding and subtracting $\Phi(t, t_0 + \lambda, x_0)\Delta x(t_0)$, $\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))\Delta x(t_0)$ and $\Phi(t, t_0, x_0)\Delta x(t_0)$ in (3.25) and using the fact that

$$\Phi(t, t_0, x_0) = \Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))\Phi(t_0 + \lambda, t_0, x_0), \tag{3.26}$$

we have

(3.27)
$$\begin{aligned}
\Delta x(t, \lambda) = &-\int_0^1 [\Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0)) - \Phi(t, t_0 + \lambda, x_0)]\Delta x(t_0) d\theta \\
&+ [\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) - \Phi(t, t_0 + \lambda, x_0)]\Delta x(t_0) \\
&+ [\Phi(t, t_0, x_0) - \Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))]\Delta x(t_0) - \Phi(t, t_0, x_0)\Delta x(t_0) \\
= &-\int_0^1 [\Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0)) - \Phi(t, t_0 + \lambda, x_0)]\Delta x(t_0) d\theta \\
&+ [\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) - \Phi(t, t_0 + \lambda, x_0)]\Delta x(t_0) \\
&+ [\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))\Phi(t_0 + \lambda, t_0, x_0) \\
&- \Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))]\Delta x(t_0) \\
&- \Phi(t, t_0, x_0)\Delta x(t_0) \\
= &-\int_0^1 [\Phi(t, t_0 + \lambda, x_0 + \theta \Delta x(t_0)) - \Phi(t, t_0 + \lambda, x_0)]\Delta x(t_0) d\theta \\
&+ [\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0)) - \Phi(t, t_0 + \lambda, x_0)]\Delta x(t_0) \\
&+ [\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))(\Phi(t_0 + \lambda, t_0, x_0) - \Phi(t_0, t_0, x_0))]\Delta x(t_0) \\
&- \Phi(t, t_0, x_0)\Delta x(t_0).
\end{aligned}$$

We set $G(\psi) = \Phi(t, t_0 + \lambda, x_0 + \psi\theta\Delta x(t_0))$ for $0 \le \psi \le 1$. It is obvious that G is continuously differentiable with respect to ψ, and hence

$$\frac{d}{d\psi} G(\psi) = \frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0 + \psi\theta\Delta x(t_0)) \otimes (\theta\Delta x(t_0)) \tag{3.28}$$

By integrating both sides of (3.28) with respect to ψ over an interval [0,1], we have

$$G(1) - G(0) = \int_0^1 \frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0 + \psi\theta\Delta x(t_0)) \otimes (\theta\Delta x(t_0)) d\psi.$$

This together with $G(1) = \Phi(t, t_0 + \lambda, x_0 + \theta\Delta x(t_0))$ and $G(0) = \Phi(t, t_0 + \lambda, x_0)$ yields

$$\Phi(t, t_0+\lambda, x_0+\theta\Delta x(t_0)) - \Phi(t, t_0+\lambda, x_0) = \int_0^1 \frac{\partial}{\partial x_0}\Phi(t, t_0+\lambda, x_0+\psi\theta\Delta x(t_0)) \otimes (\theta\Delta x(t_0)) d\psi \tag{3.29}$$

Similarly, by setting $g(\beta) = \Phi(t, t_0 + \lambda, x_0 + \beta\Delta x(t_0))$ for $0 \le \beta \le 1$, and repeating the previous argument, we obtain

$$\frac{d}{d\beta} g(\beta) = \frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0 + \beta\Delta x(t_0)) \otimes \Delta x(t_0). \tag{3.30}$$

This together with $g(1) = \Phi(t, t_0 + \lambda, x_0 + \Delta x(t_0))$ and $g(0) = \Phi(t, t_0 + \lambda, x_0)$ yields

$$\Phi(t, t_0+\lambda, x_0+\Delta x(t_0)) - \Phi(t, t_0+\lambda, x_0) = \int_0^1 \frac{\partial}{\partial x_0}\Phi(t, t_0+\lambda, x_0+\beta\Delta x(t_0)) \otimes \Delta x(t_0) d\beta. \tag{3.31}$$

Using (3.29) and (3.31), (3.27) reduces to

$$\begin{aligned} \Delta x(t, \lambda) = &-\int_0^1\int_0^1 \frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0 + \psi\theta\Delta x(t_0)) \\ &\otimes (\theta\Delta x(t_0))\Delta x(t_0) d\psi d\theta \\ &+ \int_0^1 \frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0 + \beta\Delta x(t_0)) \otimes \Delta x(t_0)\Delta x(t_0) d\beta \\ &+ [\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))(\Phi(t_0 + \lambda, t_0, x_0) - \Phi(t_0, t_0, x_0))]\Delta x(t_0) \\ &- \Phi(t, t_0, x_0)\Delta x(t_0). \end{aligned} \tag{3.32}$$

Adding and subtracting $\frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0) \otimes (\theta\Delta x(t_0))\Delta x(t_0)$ and $\frac{\partial}{\partial x_0}\Phi(t, t_0 + \lambda, x_0) \otimes (\Delta x(t_0))\Delta x(t_0)$ in (3.32) yields,

$$\begin{aligned} \Delta x(t, \lambda) = &-\int_0^1\int_0^1 \frac{\partial}{\partial x_0}[\Phi(t, t_0 + \lambda, x_0 + \psi\theta\Delta x(t_0)) \\ &- \Phi(t, t_0 + \lambda, x_0)] \otimes (\theta\Delta x(t_0))\Delta x(t_0) d\psi d\theta \\ &+ \int_0^1 \frac{\partial}{\partial x_0}[\Phi(t, t_0 + \lambda, x_0 + \beta\Delta x(t_0)) - \Phi(t, t_0 + \lambda, x_0)] \otimes \Delta x(t_0)\Delta x(t_0) d\beta \\ &+ [\Phi(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))(\Phi(t_0 + \lambda, t_0, x_0) - \Phi(t_0, t_0, x_0))]\Delta x(t_0) \end{aligned} \tag{3.33}$$

$$+\frac{1}{2}\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0)\otimes\Delta x(t_0)\Delta x(t_0)-\Phi(t,t_0,x_0)\Delta x(t_0).$$

Using the bounded convergence theorem[28], the concept of Itô-Doob type differential and sufficiently small increment Δt_0 to t_0, (3.33) reduces to

$$\begin{aligned}\partial_{t_0}x(t,t_0,x_0) &= [\Phi(t,t_0,x_0)d\Phi(t_0)]dx(t_0)\\ &\quad+\frac{1}{2}\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)\otimes dx(t_0)dx(t_0)-\Phi(t,t_0,x_0)dx(t_0)\\ &= \Phi(t,t_0,x_0)\sum_{l=1}^{m}\sigma_x^l(t_0,x_0)dw_l(t_0)\sum_{l=1}^{m}\sigma^l(t_0,x_0)dw_l(t_0)\\ &\quad+\frac{1}{2}\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)\otimes\sum_{l=1}^{m}\sigma^l(t_0,x_0)dw_l(t_0)\sum_{l=1}^{m}\sigma^l(t_0,x_0)dw_l(t_0)\\ &\quad-\Phi(t,t_0,x_0)[f(t_0,x_0)dt_0+\sum_{l=1}^{m}\sigma^l(t_0,x_0)dw_l(t_0)]\\ &= \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma_j^l(t_0,x_0)\right)_{n\times 1}dt_0\\ &\quad+\Phi(t,t_0,x_0)[\sum_{l=1}^{m}\sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0)-f(t_0,x_0)]dt_0\\ &\quad-\sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma^l(t_0,x_0)dw_l(t_0)\end{aligned}\tag{3.34}$$

This shows that $\partial_{t_0}x(t,t_0,x_0)$ exists and it is represented as in (3.20). This together with $t=t_0$ and (3.2) yields (3.21). □

Example 3.2. Let us consider a scalar linear unperturbed stochastic differential equation:

$$dx=f(t)xdt+\sigma(t)xdw(t),\qquad x(t_0)=x_0,\tag{3.35}$$

where f and σ are any differentiable functions defined on $J=[t_0,t_0+a]$ into R, where $a>0$. Find $\partial_{t_0}x(t,t_0,x_0)$.

Solution: Note that $f(t,x)=f(t)x$ and $\sigma(t,x)=\sigma(t)x$ are continuously differentiable with respect to x. Moreover, $\frac{\partial}{\partial x}f(t,x)=f(t)$ and $\frac{\partial}{\partial x}\sigma(t,x)=\sigma(t)$. The closed form solution of (3.35) is given by

$$x(t,t_0,x_0)=\Phi(t,t_0)x_0.$$

Let us consider the following:

$$\begin{aligned}&x(t,t_0+\lambda,x_0)-x(t,t_0,x_0)\\ &\quad= x(t,t_0+\lambda,x_0)-x(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\\ &\quad= \Phi(t,t_0+\lambda)x_0-\Phi(t,t_0+\lambda)x(t_0+\lambda,t_0,x_0)\end{aligned}\tag{3.36}$$

$$\begin{aligned}
&= -\Phi(t, t_0 + \lambda)[x(t_0 + \lambda, t_0, x_0) - x_0] \\
&= -\Phi(t, t_0 + \lambda)\Delta x(t_0) \\
&= -[\Phi(t, t_0 + \lambda) - \Phi(t, t_0) + \Phi(t, t_0)]\Delta x(t_0) \\
&= -[\Phi(t, t_0 + \lambda) - \Phi(t, t_0)]\Delta x(t_0) - \Phi(t, t_0)\Delta x(t_0) \\
&= [\Phi(t, t_0) - \Phi(t, t_0 + \lambda)]\Delta x(t_0) - \Phi(t, t_0)\Delta x(t_0) \\
&= [\Phi(t, t_0 + \lambda)\Phi(t_0 + \lambda, t_0) - \Phi(t, t_0 + \lambda)]\Delta x(t_0) - \Phi(t, t_0)\Delta x(t_0) \\
&= \Phi(t, t_0 + \lambda)[\Phi(t_0 + \lambda, t_0) - I]\Delta x(t_0) - \Phi(t, t_0)\Delta x(t_0) \\
&= \Phi(t, t_0 + \lambda)\Delta\Phi(t_0, t_0)\Delta x(t_0) - \Phi(t, t_0)\Delta x(t_0).
\end{aligned}$$

Using the bounded convergence theorem [28], the concept of Itô-Doob type differential and sufficiently small increment λ to t_0, (3.36) reduces to

$$\begin{aligned}
\partial_{t_0} x(t, t_0, x_0) &= \Phi(t, t_0) d\Phi(t_0, t_0) dx(t_0) - \Phi(t, t_0) dx(t_0) \\
&= \Phi(t, t_0)[f(t_0)\Phi(t_0, t_0)dt_0 + \sigma(t_0)\Phi(t_0, t_0)dw(t_0)] \\
&quad \times [f(t_0)x_0 dt_0 + \sigma(t_0)x_0 dw(t_0)] \\
&\quad - \Phi(t, t_0)[f(t_0)x_0 dt_0 + \sigma(t_0)x_0 dw(t_0)] \\
&= \Phi(t, t_0)\sigma(t_0)\sigma(t_0)x_0 dt_0 - \Phi(t, t_0)[f(t_0)x_0 dt_0 + \sigma(t_0)x_0 dw(t_0)] \\
&= \Phi(t, t_0)[\sigma^2(t_0) - f(t_0)]x_0 dt_0 - \Phi(t, t_0)\sigma(t_0)x_0 dw(t_0).
\end{aligned} \tag{3.37}$$

Example 3.3. Let us consider a scalar linear perturbed stochastic differential equation:

$$dx = [f(t)x + p(t)]dt + [\sigma(t)x + q(t)]dw(t), \qquad x(t_0) = x_0, \tag{3.38}$$

where f, σ, p and q are any differentiable functions defined on $J = [t_0, t_0 + a]$ into R, where $a > 0$. Find $\partial_{t_0} x(t, t_0, x_0)$.

Solution: Note that $f(t, x) = f(t)x + p(t)$ and $\sigma(t, x) = \sigma(t)x + q(t)$ are continuously differentiable with respect to x. Moreover, $\frac{\partial}{\partial x} f(t, x) = f(t)$ and $\frac{\partial}{\partial x}\sigma(t, x) = \sigma(t)$. Using the application of lemma (3.2) we obtain

$$\begin{aligned}
\partial_{t_0} x(t, t_0, x_0) &= \Phi(t, t_0)[(\sigma^2(t_0) - f(t_0))x_0 + \sigma(t_0)q(t_0) - p(t_0)]dt_0 \\
&\quad -\Phi(t, t_0)[\sigma(t_0)x_0 + q(t_0)]dw(t_0).
\end{aligned} \tag{3.39}$$

In the following, we state and prove the existence of Itô-Doob type mixed partial differentials of solution process of (2.3).

Lemma 3.3. *Assume that all the hypothesis of Lemma* 3.1 *hold. Let* $x(t, t_0, x_0)$ *be the solution process of* (2.3) *existing for* $t \geq t_0$*. Then the mixed Itô-Doob type partial differentials* $\partial_{x_0}(\partial_{t_0} x(t, t_0, x_0))$ *and* $\partial_{t_0}(\partial_{x_0} x(t, t_0, x_0))$ *exists and they are equal.*

Moreover,

$$\partial_{x_0}(\partial_{t_0}x(t,t_0,x_0)) = -\Big[\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma^l_j(t_0,x_0)\right)_{n\times 1} + \sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma^l_x(t_0,x_0)\sigma^l(t_0,x_0)\Big]dt_0 \tag{3.40}$$

With initial condition:

$$\partial_{x_0}(\partial_{t_0}x(t_0)) = -\sum_{l=1}^{m}\sigma^l_x(t_0,x_0)\sigma^l(t_0,x_0)dt_0 \tag{3.41}$$

Proof. Let $\Delta x(t_0) = x(t_0+\lambda,t_0,x_0) - x(t_0,t_0,x_0)$. Using (3.26), Lemma (3.1) and the continuous dependence of solution process of (3.1), we examine the following differential:

$$\begin{aligned}
&\partial_{x_0}x(t,t_0+\lambda,x_0) - \partial_{x_0}x(t,t_0,x_0)\\
&= \frac{\partial}{\partial x_0}x(t,t_0+\lambda,x_0)dx_0 + \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0)\otimes dx_0)dx_0\\
&\quad -[\frac{\partial}{\partial x_0}x(t,t_0,x_0)dx_0 + \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,t_0,x_0)\otimes dx_0)dx_0]\\
&= [\Phi(t,t_0+\lambda,x_0) - \Phi(t,t_0,x_0)]dx_0\\
&\quad +\frac{1}{2}[\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0) - \Phi(t,t_0,x_0))\otimes dx_0]dx_0.
\end{aligned} \tag{3.42}$$

Since $\Phi(t,t_0,x_0) = \Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\Phi(t_0+\lambda,t_0,x_0)$, by adding and subtracting $\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))dx_0$ in (3.42), using generalized mean value theorem and algebraic manipulations, we get

(3.43)

$$\begin{aligned}
&\partial_{x_0}x(t,t_0+\lambda,x_0) - \partial_{x_0}x(t,t_0,x_0)\\
&= [\Phi(t,t_0+\lambda,x_0) - \Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\Phi(t_0+\lambda,t_0,x_0)]dx_0\\
&\quad + \frac{1}{2}[\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0) - \Phi(t,t_0,x_0))\otimes dx_0]dx_0\\
&= [\Phi(t,t_0+\lambda,x_0) - \Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0)) + \Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\\
&\quad - \Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\Phi(t_0+\lambda,t_0,x_0)]dx_0\\
&\quad + \frac{1}{2}[\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0) - \Phi(t,t_0,x_0))\otimes dx_0]dx_0\\
&= -[\int_0^1 \frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0+\theta\Delta x(t_0))\otimes\Delta x(t_0)d\theta]dx_0\\
&\quad - \Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))(\Phi(t_0+\lambda,t_0,x_0) - I_{n\times n})dx_0\\
&\quad + \frac{1}{2}[\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0) - \Phi(t,t_0,x_0))\otimes dx_0]dx_0
\end{aligned}$$

Again by adding and subtracting $\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0)\otimes\Delta x(t_0)dx_0$ in (3.43), we get

(3.44)

$$\begin{aligned}
&\partial_{x_0}x(t,t_0+\lambda,x_0)-\partial_{x_0}x(t,t_0,x_0)\\
&=-[\int_0^1\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0+\theta\Delta x(t_0))-\Phi(t,t_0+\lambda,x_0))\otimes\Delta x(t_0)d\theta]dx_0\\
&\quad-\Phi(t,t_0+\lambda,x(t_0+\lambda,t_0,x_0))\Delta\Phi(t_0)dx_0-\frac{\partial}{\partial x_0}\Phi(t,t_0+\lambda,x_0))\otimes\Delta x(t_0)dx_0\\
&\quad+\frac{1}{2}[\frac{\partial}{\partial x_0}(\Phi(t,t_0+\lambda,x_0)-\Phi(t,t_0,x_0))\otimes dx_0]dx_0
\end{aligned}$$

For sufficiently small $\Delta t_0=\lambda>0$, uniform convergence theorem, solution process of Itô-Doob type stochastic differential equations (2.3) and (3.3), Itô-Doob calculus and continuous dependence of solutions with respect to initial conditions, we obtain

$$\begin{aligned}
\partial_{t_0}(\partial_{x_0}x(t,t_0,x_0))=-\Big[&\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma^l_j(t_0,x_0)\right)_{n\times 1}\\
&+\sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma^l_x(t_0,x_0)\sigma^l(t_0,x_0)\Big]dt_0.
\end{aligned}\tag{3.45}$$

On the other hand, using (3.20) we examine the following differential

(3.46)

$$\begin{aligned}
&\partial_{t_0}x(t,t_0,x_0+\Delta x_0)-\partial_{t_0}x(t,t_0,x_0)\\
&=[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)\sigma^l_j(t_0,x_0+\Delta x_0)\right)_{n\times 1}\\
&\quad+\Phi(t,t_0,x_0+\Delta x_0)[\sum_{l=1}^{m}\sigma^l_x(t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)-f(t_0,x_0+\Delta x_0)]]dt_0\\
&\quad-\sum_{l=1}^{m}\Phi(t,t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)dw_l(t_0)\\
&\quad-[[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma^l_j(t_0,x_0)\right)_{n\times 1}\\
&\quad+\Phi(t,t_0,x_0)[\sum_{l=1}^{m}\sigma^l_x(t_0,x_0)\sigma^l(t_0,x_0)-f(t_0,x_0)]]dt_0-\sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma^l(t_0,x_0)dw_l(t_0)]\\
&=[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)\sigma^l_j(t_0,x_0+\Delta x_0)\right)_{n\times 1}\\
&\quad-\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma^l_j(t_0,x_0)\right)_{n\times 1}]dt_0
\end{aligned}$$

$$+ \Phi(t,t_0,x_0+\Delta x_0)[\sum_{l=1}^{m} \sigma_x^l(t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0) - f(t_0,x_0+\Delta x_0)]dt_0$$
$$- \Phi(t,t_0,x_0)[\sum_{l=1}^{m} \sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0) - f(t_0,x_0)]dt_0$$
$$- \sum_{l=1}^{m} \Phi(t,t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)dw_l(t_0) + \sum_{l=1}^{m} \Phi(t,t_0,x_0)\sigma^l(t_0,x_0)dw_l(t_0).$$

By adding and subtracting $\Phi(t,t_0,x_0)\sum_{l=1}^{m}\sigma^l(t_0,x_0+\Delta x_0)dw_l(t_0)$ in (3.46) yields

(3.47)

$$\begin{aligned}
&\partial_{t_0}x(t,t_0,x_0+\Delta x_0) - \partial_{t_0}x(t,t_0,x_0)\\
&= [\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0)\sigma_j^l(t_0,x_0+\Delta x_0)\right)_{n\times 1}\\
&\quad - \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma_j^l(t_0,x_0)\right)_{n\times 1}]dt_0\\
&\quad + \Phi(t,t_0,x_0+\Delta x_0)[\sum_{l=1}^{m}\sigma_x^l(t_0,x_0+\Delta x_0)\sigma^l(t_0,x_0+\Delta x_0) - f(t_0,x_0+\Delta x_0)]dt_0\\
&\quad - \Phi(t,t_0,x_0)[\sum_{l=1}^{m}\sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0) - f(t_0,x_0)]dt_0\\
&\quad - [\Phi(t,t_0,x_0+\Delta x_0) - \Phi(t,t_0,x_0)]\sum_{l=1}^{m}\sigma^l(t_0,x_0+\Delta x_0)dw_l(t_0)\\
&\quad - \Phi(t,t_0,x_0)\sum_{l=1}^{m}(\sigma^l(t_0,x_0+\Delta x_0) - \sigma^l(t_0,x_0))dw_l(t_0)
\end{aligned}$$

From the continuity of rate coefficient matrices and the continuous dependence of solution process, we have

$$\begin{aligned}
\partial_{x_0}(\partial_{t_0}x(t,t_0,x_0)) = -[&\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,t_0,x_0)\sigma^l(t_0,x_0)\sigma_j^l(t_0,x_0)\right)_{n\times 1}\\
&+ \sum_{l=1}^{m}\Phi(t,t_0,x_0)\sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0)]dt_0
\end{aligned} \tag{3.48}$$

This establishes the proof of (3.40). Since $\frac{\partial}{\partial x_0}\Phi(t_0,t_0,x_0) = 0$ and $\Phi(t_0,t_0,x_0) = I_{n\times n}$ at $t = t_0$, we have

$$\partial_{x_0}(\partial_{t_0}x(t_0)) = -\sum_{l=1}^{m}\sigma_x^l(t_0,x_0)\sigma^l(t_0,x_0)dt_0. \tag{3.49}$$

This completes the proof of the Lemma. □

Example 3.4. Let us consider a scalar linear unperturbed stochastic differential equation:

$$dx = f(t)xdt + \sigma(t)xdw(t), \qquad x(t_0) = x_0, \tag{3.50}$$

where f and σ are any differentiable functions defined on $J = [t_0, t_0 + a]$ into R, where $a > 0$. Find $\partial_{x_0}(\partial_{t_0} x(t, t_0, x_0))$.

Solution: Note that $f(t, x) = f(t)x$ and $\sigma(t, x) = \sigma(t)x$ are continuously differentiable with respect to x. Moreover, $\frac{\partial}{\partial x} f(t, x) = f(t)$ and $\frac{\partial}{\partial x}\sigma(t, x) = \sigma(t)$. The closed form solution of (3.50) is given by

$$x(t, t_0, x_0) = \Phi(t, t_0)x_0.$$

Using (3.26), Lemma (3.1) and the continuous dependence of solution process of (3.1), we examine the following differential:

$$\begin{aligned}
&\partial_{x_0} x(t, t_0 + \lambda, x_0) - \partial_{x_0} x(t, t_0, x_0) \\
&= [\frac{\partial}{\partial x_0} x(t, t_0 + \lambda, x_0) - \frac{\partial}{\partial x_0} x(t, t_0 + \lambda, x(t_0 + \lambda, t_0, x_0))]dx_0 \\
&= [\Phi(t, t_0 + \lambda) - \Phi(t, t_0)]dx_0 \\
&= [\Phi(t, t_0 + \lambda) - \Phi(t, t_0 + \lambda)\Phi(t_0 + \lambda, t_0)]dx_0 \\
&= -\Phi(t, t_0 + \lambda)[\Phi(t_0 + \lambda, t_0) - \Phi(t_0, t_0)]dx_0 \\
&= -\Phi(t, t_0 + \lambda)\Delta\Phi(t_0, t_0)dx_0.
\end{aligned} \tag{3.51}$$

Using the bounded convergence theorem[28], the concept of Itô-Doob type differential and sufficiently small increment λ to t_0, (3.51) reduces to

(3.52)

$$\begin{aligned}
&\partial_{t_0}(\partial_{x_0} x(t, t_0, x_0)) \\
&= -\Phi(t, t_0)d\Phi(t_0, t_0)dx_0 \\
&= -\Phi(t, t_0)[f(t_0)\Phi(t_0, t_0)dt_0 + \sigma(t_0)\Phi(t_0, t_0)dw(t_0)][f(t_0)x_0dt_0 + \sigma(t_0)x_0dw(t_0)] \\
&= -\Phi(t, t_0)\sigma^2(t_0)x_0dt_0.
\end{aligned}$$

4. METHOD OF VARIATION OF CONSTANTS FORMULA

In this section we shall establish the method of variation of constants formula with respect to (2.3) and its perturbed system (2.2).

Theorem 4.1. *Let the assumption of Lemma* 3.1 *be satisfied. Let* $y(t, t_0, x_0)$ *and* $x(t, t_0, x_0)$ *be solution processes of* (2.2) *and* (2.3)*, respectively, through the same* *initial data* (t_0, x_0), *for all* $t \geq t_0$. *Then*

(4.1)

$$y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t} \Big[-\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Upsilon_j^l(s,y(s))\right)_{n\times 1}$$
$$+\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1}$$
$$+\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\Upsilon_j^l(s,y(s))\right)_{n\times 1}$$
$$+\Phi(t,s,y(s))[F(s,y(s)) - \sum_{l=1}^{m}\sigma_x^l(s,y(s))\Upsilon^l(s,y(s))]\Big]ds$$
$$+\sum_{l=1}^{m}\int_{t_0}^{t}\Phi(t,s,y(s))\Upsilon^l(s,y(s))dw_l(s).$$

Proof. From the application of Lemmas 3.1, 3.2, 3.3 and the Itô-Doob differential formula with respect to s for $t_0 \leq s \leq t$, we have

(4.2)

$$d_s x(t,s,y(s)) = \partial_{t_0}x(t,s,y(s)) + \partial_{x_0}x(t,s,y(s)) + \partial_{x_0}(\partial_{t_0}x(t,s,y(s)))$$
$$+\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes dy)dy$$
$$=\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1} ds$$
$$+\Phi(t,s,y(s))(\sum_{l=1}^{m}\sigma_x^l(s,y(s))\sigma^l(s,y(s)) - f(s,y(s)))ds$$
$$-\Phi(t,s,y(s))\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s) + \Phi(t,s,y(s))dy(s)$$
$$-\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))(\sigma_j^l(s,y(s)) + \Upsilon_j^l(s,y(s)))\right)_{n\times 1} ds$$
$$-\Phi(t,s,y(s))\sum_{l=1}^{m}\sigma_x^l(s,y(s))(\sigma^l(s,y(s)) + \Upsilon^l(s,y(s)))ds$$
$$+\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)$$
$$+\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes\sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)$$

$$+\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)$$

$$+\frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes\sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\Upsilon^l(s,y(s))dw_l(s)$$

By simplifying (4.2), we get

$$\begin{aligned} d_s x(t,s,y(s)) = \Big[&-\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Upsilon^l_j(s,y(s))\right)_{n\times 1} \\ &+\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\sigma^l_j(s,y(s))\right)_{n\times 1} \\ &+\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\Upsilon^l_j(s,y(s))\right)_{n\times 1} \\ &+\Phi(t,s,y(s))[F(s,y(s))-\sum_{l=1}^{m}\sigma^l_x(s,y(s))\Upsilon^l(s,y(s))]\Big]ds \\ &+\sum_{l=1}^{m}\Phi(t,s,y(s))\Upsilon^l(s,y(s))dw_l(s). \end{aligned} \tag{4.3}$$

Since the right hand side of (4.3) is continuous with respect to s, we integrate from t_0 to t, and obtain the variation of constant formula (4.1). □

Corollary 4.2. *Let the assumption of Lemma* 3.1 *be satisfied, except that only* $c(t,y)$ *in* (2.2) *can be decomposed. Let* $y(t,t_0,x_0)$ *and* $x(t,t_0,x_0)$ *be solution processes of* (2.2) *and* (2.3)*, respectively, through the same initial data* (t_0,x_0)*, for all* $t\ge t_0$*. Then*

(4.4)

$$\begin{aligned} y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t}\Big[&\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma^l_j(s,y(s))\right)_{n\times 1} \\ &-\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} \\ &+\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} \\ &+\Phi(t,s,y(s))(\sum_{l=1}^{m}[\sigma^l_x(s,y(s))\sigma^l(s,y(s))-\sigma^l_x(s,y(s))\Sigma^l(s,y(s))] \\ &+F(s,y(s)))\Big]ds \\ &+\sum_{l=1}^{m}\int_{t_0}^{t}\Phi(t,s,y(s))[\Sigma^l(s,y(s))-\sigma^l(s,y(s))]dw_l(s) \end{aligned}$$

Proof. From the application of Lemmas 3.1, 3.2, 3.3 and the Itô-Doob differential formula with respect to s for $t_0 \leq s \leq t$, we have

(4.5)

$$\begin{aligned}
d_s x(t,s,y(s)) &= \partial_{t_0} x(t,s,y(s)) + \partial_{x_0} x(t,s,y(s)) + \partial_{x_0}(\partial_{t_0} x(t,s,y(s))) \\
&\quad + \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes dy)dy \\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma^l_j(s,y(s))\right)_{n\times 1} ds \\
&\quad + \Phi(t,s,y(s))(\sum_{l=1}^{m}\sigma^l_x(s,y(s))\sigma^l(s,y(s)) - f(s,y(s)))ds \\
&\quad - \Phi(t,s,y(s))\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s) \\
&\quad + \Phi(t,s,y(s))[(f(s,y(s)) + F(s,y(s)))ds + \sum_{l=1}^{m}\Sigma^l(s,y(s))dw_l(s)] \\
&\quad - \left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} ds \\
&\quad - \Phi(t,s,y(s))\sum_{l=1}^{m}\sigma^l_x(s,y(s))\Sigma^l(s,y(s))ds \\
&\quad + \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes \sum_{l=1}^{m}\Sigma^l(s,y(s))dw_l(s)\sum_{l=1}^{m}\Sigma^l(s,y(s))dw_l(s)
\end{aligned}$$

By simplifying (4.5), we get

(4.6)

$$\begin{aligned}
d_s x(t,s,y(s)) &= \Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma^l_j(s,y(s))\right)_{n\times 1} \\
&\quad - \left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} \\
&\quad + \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Sigma^l(s,y(s))\Sigma^l_j(s,y(s))\right)_{n\times 1} \\
&\quad + \Phi(t,s,y(s))(\sum_{l=1}^{m}[\sigma^l_x(s,y(s))\sigma^l(s,y(s)) - \sigma^l_x(s,y(s))\Sigma^l(s,y(s))] \\
&\quad + F(s,y(s)))\Big]ds \\
&\quad + \sum_{l=1}^{m}\Phi(t,s,y(s))[\Sigma^l(s,y(s)) - \sigma^l(s,y(s))]dw_l(s)
\end{aligned}$$

Since the right hand side of (4.6) is continuous with respect to s, we integrate from t_0 to t, and obtain the variation of constant formula (4.4). □

Corollary 4.3. *Let the assumption of Lemma* 3.1 *be satisfied, except that only* $\Sigma(t, y)$ *in* (2.2) *can be decomposed. Let* $y(t, t_0, x_0)$ *and* $x(t, t_0, x_0)$ *be solution processes of* (2.2) *and* (2.3)*, respectively, through the same initial data* (t_0, x_0)*, for all* $t \geq t_0$*. Then*

(4.7)

$$\begin{aligned}
y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t} \Big[\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1} \\
-\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Upsilon_j^l(s,y(s))\right)_{n\times 1} \\
+\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\Upsilon_j^l(s,y(s))\right)_{n\times 1} \\
+\Phi(t,s,y(s))[c(s,y(s)) - f(s,y(s)) - \sum_{l=1}^{m}\sigma_x^l(s,y(s))\Upsilon^l(s,y(s))]\Big]ds \\
+\sum_{l=1}^{m}\int_{t_0}^{t}\Phi(t,s,y(s))\Upsilon^l(s,y(s))dw_l(s).
\end{aligned}$$

Proof. From the application of Lemmas 3.1, 3.2, 3.3 and the Itô-Doob differential formula with respect to s for $t_0 \leq s \leq t$, we have

(4.8)

$$\begin{aligned}
d_s x(t,s,y(s)) &= \partial_{t_0}x(t,s,y(s)) + \partial_{x_0}x(t,s,y(s)) + \partial_{x_0}(\partial_{t_0}x(t,s,y(s))) \\
&\quad + \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s))\otimes dy)dy \\
&= \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1} ds \\
&\quad + \Phi(t,s,y(s))(\sum_{l=1}^{m}\sigma_x^l(s,y(s))\sigma^l(s,y(s)) - f(s,y(s)))ds \\
&\quad - \Phi(t,s,y(s))\sum_{l=1}^{m}\sigma^l(s,y(s))dw_l(s) \\
&\quad + \Phi(t,s,y(s))[c(s,y(s))ds + \sum_{l=1}^{m}(\sigma^l(s,y(s)) + \Upsilon^l(s,y(s)))dw_l(s)] \\
&\quad - \left(\sum_{j=1}^{n}\sum_{l=1}^{m}\frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))(\sigma_j^l(s,y(s)) + \Upsilon_j^l(s,y(s)))\right)_{n\times 1} ds
\end{aligned}$$

$$
\begin{aligned}
&- \Phi(t,s,y(s)) \sum_{l=1}^{m} \sigma_x^l(s,y(s))(\sigma^l(s,y(s)) + \Upsilon^l(s,y(s)))ds \\
&+ \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes \sum_{l=1}^{m} \sigma^l(s,y(s))dw_l(s) \sum_{l=1}^{m} \sigma^l(s,y(s))dw_l(s) \\
&+ \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes \sum_{l=1}^{m} \Upsilon^l(s,y(s))dw_l(s) \sum_{l=1}^{m} \sigma^l(s,y(s))dw_l(s) \\
&+ \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes \sum_{l=1}^{m} \sigma^l(s,y(s))dw_l(s) \sum_{l=1}^{m} \Upsilon^l(s,y(s))dw_l(s) \\
&+ \frac{1}{2}(\frac{\partial}{\partial x_0}\Phi(t,s,y(s)) \otimes \sum_{l=1}^{m} \Upsilon^l(s,y(s))dw_l(s) \sum_{l=1}^{m} \Upsilon^l(s,y(s))dw_l(s)
\end{aligned}
$$

By simplifying (4.8), we get

(4.9)

$$
\begin{aligned}
d_s x(t,s,y(s)) = \Big[\frac{1}{2}&\left(\sum_{j=1}^{n}\sum_{l=1}^{m} \frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1} \\
&- \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m} \frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\Upsilon_j^l(s,y(s))\right)_{n\times 1} \\
&+ \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m} \frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Upsilon^l(s,y(s))\Upsilon_j^l(s,y(s))\right)_{n\times 1} \\
&+ \Phi(t,s,y(s))[c(s,y(s)) - f(s,y(s)) - \sum_{l=1}^{m} \sigma_x^l(s,y(s))\Upsilon^l(s,y(s))]\Big]ds \\
&+ \sum_{l=1}^{m} \Phi(t,s,y(s))\Upsilon^l(s,y(s))dw_l(s).
\end{aligned}
$$

Since the right hand side of (4.9) is continuous with respect to s, we integrate from t_0 to t, and obtain the variation of constant formula (4.7). □

Corollary 4.4. *Let the assumption of Lemma* 3.1 *be satisfied, except that* $c(t,y)$ *in* (2.2) *and* $\Sigma(t,y)$ *in* (2.3) *cannot be decomposed. Let* $y(t,t_0,x_0)$ *and* $x(t,t_0,x_0)$ *be solution processes of* (2.2) *and* (2.3)*, respectively, through the same initial data* (t_0,x_0)*, for all* $t \geq t_0$*. Then*

(4.10)

$$
\begin{aligned}
y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t} \Big[\frac{1}{2}&\left(\sum_{j=1}^{n}\sum_{l=1}^{m} \frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\sigma^l(s,y(s))\sigma_j^l(s,y(s))\right)_{n\times 1} \\
&+ \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{l=1}^{m} \frac{\partial}{\partial x_0}\Phi_{ij}(t,s,y(s))\Sigma^l(s,y(s))\Sigma_j^l(s,y(s))\right)_{n\times 1}
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial}{\partial x_0} \Phi_{ij}(t,s,y(s)) \sigma^l(s,y(s)) \Sigma_j^l(s,y(s)) \right)_{n\times 1} \\
&+ \Phi(t,s,y(s)) \Big[\sum_{l=1}^{m} \sigma_x^l(s,y(s)) (\sigma^l(s,y(s)) \\
&- \Sigma^l(s,y(s))) + c(s,y(s)) - f(s,y(s)) \Big] \Big] ds \\
&+ \sum_{l=1}^{m} \int_{t_0}^{t} \Phi(t,s,y(s)) [\Sigma^l(s,y(s)) - \sigma^l(s,y(s))] dw_l(s).
\end{aligned}
$$

Proof. From the application of Lemmas 3.1, 3.2, 3.3 and the Itô-Doob differential formula with respect to s for $t_0 \le s \le t$, we have

(4.11)

$$
\begin{aligned}
d_s x(t,s,y(s)) &= \partial_{t_0} x(t,s,y(s)) + \partial_{x_0} x(t,s,y(s)) + \partial_{x_0}(\partial_{t_0} x(t,s,y(s))) \\
&\quad + \frac{1}{2} (\frac{\partial}{\partial x_0} \Phi(t,s,y(s)) \otimes dy) dy \\
&= \frac{1}{2} \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial}{\partial x_0} \Phi_{ij}(t,s,y(s)) \sigma^l(s,y(s)) \sigma_j^l(s,y(s)) \right)_{n\times 1} ds \\
&\quad + \Phi(t,s,y(s)) [(\sum_{l=1}^{m} \sigma_x^l(s,y(s)) \sigma^l(s,y(s)) - f(s,y(s))) ds \\
&\quad - \sum_{l=1}^{m} \sigma^l(s,y(s)) dw_l(s)] \\
&\quad + \Phi(t,s,y(s)) [c(s,y(s)) ds + \sum_{l=1}^{m} \Sigma^l(s,y(s)) dw_l(s)] \\
&\quad - \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial}{\partial x_0} \Phi_{ij}(t,s,y(s)) \sigma^l(s,y(s)) \Sigma_j^l(s,y(s)) \right)_{n\times 1} ds \\
&\quad - \Phi(t,s,y(s)) \sum_{l=1}^{m} \sigma_x^l(s,y(s)) \Sigma^l(s,y(s)) ds \\
&\quad + \frac{1}{2} (\frac{\partial}{\partial x_0} \Phi(t,s,y(s)) \otimes \sum_{l=1}^{m} \Sigma^l(s,y(s)) dw_l(s) \sum_{l=1}^{m} \Sigma^l(s,y(s)) dw_l(s)
\end{aligned}
$$

By simplifying (4.11), we get

$$
\begin{aligned}
d_s x(t,s,y(s)) = \Big[&\frac{1}{2} \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial}{\partial x_0} \Phi_{ij}(t,s,y(s)) \sigma^l(s,y(s)) \sigma_j^l(s,y(s)) \right)_{n\times 1} \\
&+ \frac{1}{2} \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial}{\partial x_0} \Phi_{ij}(t,s,y(s)) \Sigma^l(s,y(s)) \Sigma_j^l(s,y(s)) \right)_{n\times 1}
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial}{\partial x_0} \Phi_{ij}(t,s,y(s)) \sigma^l(s,y(s)) \Sigma_j^l(s,y(s)) \right)_{n\times 1} \\
&+ \Phi(t,s,y(s)) \Big[\sum_{l=1}^{m} \sigma_x^l(s,y(s))(\sigma^l(s,y(s)) \\
&- \Sigma^l(s,y(s))) + c(s,y(s)) - f(s,y(s)) \Big]\Big] ds \\
&+ \sum_{l=1}^{m} \Phi(t,s,y(s))[\Sigma^l(s,y(s)) - \sigma^l(s,y(s))] dw_l(s).
\end{aligned}
$$

Since the right hand side of (4.12) is continuous with respect to s, we integrate from t_0 to t, and obtain the variation of constant formula (4.10). □

Remark 4.1. In Corollary (4.4), if

1. $\sigma(t,x) = 0$ and $c(t,x) = f(t,x)$, then the variation of constant formula reduces to

$$
\begin{aligned}
y(t,t_0,x_0) &= x(t,t_0,x_0) \\
&+ \int_{t_0}^{t} \frac{1}{2} \left(\sum_{j=1}^{n} \sum_{l=1}^{m} \frac{\partial}{\partial x_0} \Phi_{ij}(t,s,y(s)) \Sigma^l(s,y(s)) \Sigma_j^l(s,y(s)) \right)_{n\times 1} ds \\
&+ \sum_{l=1}^{m} \int_{t_0}^{t} \Phi(t,s,y(s)) \Sigma^l(s,y(s)) dw_l(s).
\end{aligned} \tag{4.12}
$$

2. $f(t,x) = 0$ and $\Sigma(t,x) = \sigma(t,x)$, then the variation of constant formula reduces to

$$
y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,s,y(s)) c(s,y(s)) d(s). \tag{4.13}
$$

5. EXAMPLES

Example 5.1. Consider a scalar linear unperturbed and perturbed stochastic differential equations:

$$
dx = f(t)xdt + \sigma(t)xdw(t), \qquad x(t_0) = x_0, \tag{5.1}
$$

and

$$
dy = [f(t)y + p(t)]dt + [\sigma(t)y + q(t)]dw(t), \qquad y(t_0) = x_0, \tag{5.2}
$$

where f, σ, p and q are any differentiable functions defined on $J = [t_0, t_0 + a]$ into R, where $a > 0$. Then

$$
y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,s)[p(s) - \sigma(s)q(s)]d(s) + \int_{t_0}^{t} \Phi(t,s)q(s)dw(s). \tag{5.3}
$$

Solution: Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$ be solution processes of (5.1) and (5.2) through (t_0, x_0), respectively. Since $x(t) = x(t, t_0, x_0) = \Phi(t, t_0)x_0$ [11], the partial derivative of $x(t, t_0, x_0)$ with respect to x_0 will be $\frac{\partial}{\partial x_0}x(t, t_0, x_0) = \Phi(t, t_0)$. Moreover, $\frac{\partial^2}{\partial x_0^2}x(t, t_0, x_0) = 0$. From Lemma (3.2), the partial differential of $x(t, t_0, x_0)$ with respect to t_0 is given by

$$\partial_{t_0}x(t, t_0, x_0) = \Phi(t, t_0)[\sigma^2(t_0) - f(t_0)]x_0dt_0 - \Phi(t, t_0)\sigma(t_0)x_0dw(t_0) \tag{5.4}$$

At $t = t_0$, we have

$$\partial_{t_0}x(t_0, t_0, x_0) = [\sigma^2(t_0) - f(t_0)]x_0dt_0 - \sigma(t_0)x_0dw(t_0) \tag{5.5}$$

Moreover, from Lemma (3.3), the corresponding Itô-Doob mixed partial differential of solution process $x(t_0, t_0, x_0)$ of (5.1) is given by

$$\partial_{t_0x_0}x(t, t_0, x_0) = -\Phi(t, t_0)\sigma^2(t_0)x_0dt_0 \tag{5.6}$$

Using the method of variational constants, Theorem 4.1, the solution of (5.2) is given by (5.3).

Example 5.2. Consider a scalar nonlinear unperturbed and perturbed stochastic differential equations:

$$dx = f(t)xdt + \sigma(t)xdw(t), \qquad x(t_0) = x_0, \tag{5.7}$$

and

$$dy = [f(t)y - p(t)\frac{1}{2}y^3]dt + \sigma(t)ydw(t), \qquad y(t_0) = x_0, \tag{5.8}$$

where f, σ and p are any differentiable functions defined on J into R. Then

$$y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^{t} \Phi(t, s)[\frac{1}{2}p(t)y^3(s)]ds. \tag{5.9}$$

Solution: Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$ be solution processes of (5.7) and (5.8) through (t_0, x_0), respectively. The partial differential of $x(t, t_0, x_0)$ with respect to initial data is given in Example 1. The closed form solution of (5.8) is

$$y(t, t_0, x_0) = \frac{\Phi(t, t_0)|x_0|}{\sqrt{1 + x_0^2 \int_{t_0}^{t} p(s)\Phi^2(t, s)ds}}. \tag{5.10}$$

The partial differential of $y(t, t_0, x_0)$ with respect to t_0 is

$$\partial_{t_0}y(t, t_0, x_0) = \Phi(t, t_0)[(f(t_0) + \sigma^2(t_0))x_0 - \frac{1}{2}p(t){x_0}^3]dt_0 - \Phi(t, t_0)\sigma(t_0)x_0dw(t_0). \tag{5.11}$$

At $t = t_0$, we have

$$\partial_{t_0}y(t_0, t_0, x_0) = [(f(t_0) + \sigma^2(t_0))x_0 - \frac{1}{2}p(t){x_0}^3]dt_0 - \sigma(t_0)x_0dw(t_0) \tag{5.12}$$

Moreover, the corresponding Itô-Doob mixed partial differential of solution process $y(t_0, t_0, x_0)$ of (5.7) is given by

$$\partial_{t_0 x_0} y(t_0, t_0, x_0) = -\Phi(t, t_0)\sigma^2(t_0)x_0 dt_0 \tag{5.13}$$

Using the method of variational constants, Theorem 4.1, the solution of (5.8) is given by (5.9).

Example 5.3. Consider a nonlinear unperturbed and perturbed stochastic differential equation:

$$dx = \alpha x(n - x)dt + \beta x dw(t), \qquad x(t_0) = x_0, \tag{5.14}$$

and

$$dy = [\alpha y(n - y) + g(t, y)]dt + [\beta y + \sigma(t, y)]dw(t), \qquad y(t_0) = x_0, \tag{5.15}$$

where α, β and n are any constant, g and σ are differentiable functions. Then

(5.16)

$$\begin{aligned} y(t, t_0, x_0) &= x(t, t_0, x_0) \\ &+ \int_{t_0}^{t} \left[\frac{1}{2}\frac{\partial}{\partial x_0}\Phi(t, s, y(s))\sigma^2(s, y(s)) + \Phi(t, s, y(s))[g(s, y(s)) - \beta\sigma(s, y(s))]\right] ds \\ &+ \int_{t_0}^{t} \Phi(t, s)\sigma(s, y(s))dw(s). \end{aligned}$$

Solution: Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$ be solution processes of (5.14) and (5.15) through (t_0, x_0), respectively. The partial derivative of solution processes $x(t, t_0, x_0)$ with respect to x_0 is

$$\frac{\partial}{\partial x_0} x(t, t_0, x_0) = \frac{\Phi(t, t_0)}{(\Phi(t, t_0) + \alpha x_0 \int_{t_0}^{t} \Phi(t, s)ds)^2} \tag{5.17}$$

and

$$\frac{\partial^2}{\partial x_0^2} x(t, t_0, x_0) = \frac{-2\alpha\Phi(t, t_0)\int_{t_0}^{t} \Phi(t, s)ds}{(\Phi(t, t_0) + \alpha x_0 \int_{t_0}^{t} \Phi(t, s)ds)^3}. \tag{5.18}$$

Moreover, the partial differential of $x(t, t_0, x_0)$ with respect to t_0 is

$$\begin{aligned} \partial_{t_0} x(t, t_0, x_0) = &\frac{-\alpha\beta^2 x_0^2 \Phi(t, t_0)\int_{t_0}^{t} \Phi(t, s)ds}{(\Phi(t, t_0) + \alpha x_0 \int_{t_0}^{t} \Phi(t, s)ds)^3} \\ &+ \beta^2\Phi(t, t_0)x_0 - \Phi(t, t_0)\alpha x_0(n - x_0)dt_0 - \Phi(t, t_0)\beta x_0 dw(t_0). \end{aligned} \tag{5.19}$$

The corresponding Itô-Doob mixed partial differential of solution process $x(t, t_0, x_0)$ of (5.14) is given by

$$\partial_{t_0 x_0} x(t, t_0, x_0) = \left[\frac{2\alpha\beta^2 x_0^2 \Phi(t, t_0)\int_{t_0}^{t} \Phi(t, s)ds}{(\Phi(t, t_0) + \alpha x_0 \int_{t_0}^{t} \Phi(t, s)ds)^3} + \Phi(t, t_0)\beta^2 x_0\right] dt_0. \tag{5.20}$$

Using the method of variational constants, Theorem (4.1) with $f(t,x) = \alpha x(n-x)$, $\sigma(t,x) = \beta x$ the solution of (5.15) is given by (5.16).

Example 5.4. Consider a scalar linear unperturbed and perturbed stochastic differential equation as:

$$dx = A(t)xdt + \sum_{l=1}^{m} \sigma^l(t)xdw_l(t), \qquad x(t_0) = x_0, \tag{5.21}$$

and

$$dy = [A(t)y + P(t)]dt + \sum_{l=1}^{m} (\sigma^l(t)y + \Upsilon^l(t))dw_l(t), \qquad y(t_0) = x_0, \tag{5.22}$$

where A and σ^l are any differentiable functions defined on J into $R^{n\times n}$ and P, Υ^l are any differentiable functions defined on J into R^n.Then

(5.23)

$$y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,s)[P(s) - \sum_{l=1}^{m} \sigma^l(s)\Upsilon^l(s)]ds + \sum_{l=1}^{m} \int_{t_0}^{t} \Phi(t,s)\Upsilon^l(s)dw_l(s).$$

Solution: Let $x(t) = x(t,t_0,x_0)$ and $y(t) = y(t,t_0,x_0)$ be solution processes of (5.21) and (5.22) through (t_0, x_0), respectively. The partial differential of $x(t,t_0,x_0)$ with respect to initial data is given in Example 1. Using the method of variational constants, Theorem 4.1 with $f(t,x) = A(t)x$, $\sigma(t,x) = \sum_{l=1}^{m} \sigma^l(t)x$ the solution of (5.22) is given by (5.23)

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