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# EXISTENCE AND UNIQUENESS OF SOLUTIONS OF THREE-POINT BOUNDARY VALUE PROBLEMS FOR SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

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**ABSTRACT.** In this article, we establish the results on the existence and uniqueness of positive solutions of the three-point boundary value problem for the nonlinear singular multi-term fractional differential equation with the lower derivative $\mu \in (0, \alpha)$. Our analysis rely on the well known Schauder's fixed point theorem.

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## 1. INTRODUCTION

Fractional differential equations have excited in recent years a considerable interest both in mathematics and in applications. They were used in modeling of many physical and chemical processes and in engineering. In its turn, mathematical aspects of fractional differential equations and methods of their solutions were discussed by many authors, see the text books [4,7,9], the survey paper [2] and papers [1, 5, 6, 8, 10] and the references therein.

In this paper, we discuss the three point boundary value problem (BVP for short) of the nonlinear singular fractional differential equation of the form

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t), D_{0^+}^{\mu} u(t)) = 0, \quad 0 < t < 1, 1 < \alpha < 2, \\ \lim_{t \to 0} t^{2-\alpha} u(t) = 0, \\ u(1) = \beta u(\eta)\,, \end{cases} \tag{1.1}$$

where $\beta \in [0, \infty)$, $\eta \in (0, 1)$, $D_{0^+}^{\alpha}$ (or $D_{0^+}^{\mu}$) is the Riemann-Liouville fractional derivative of order $\alpha$ (or $\mu$), $\mu \in (0, \alpha)$, $f$ defined on $(0, 1) \times R \times R$ and $f$ may be singular at $t = 0$ and $t = 1$.

We obtain the existence and uniqueness results for BVP(1.1) by using the well known Schauder's fixed point theorem. The novelty of this paper is as follows: (i) $0 < \mu < \alpha$, (ii) $f$ may be singular both at $t = 0$ and $t = 1$, $f$ may be a non-Caratheodory function.

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This paper is motivated by [6] in which the authors studied the following boundary value problem for fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, u(1) = 0\,, \end{cases} \tag{1.2}$$

where $1 < \alpha < 2$ is a real number, $D_{0+}^{\alpha}$ is the standard Riemann-Liouville differentiation, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. By means of some fixed-point theorems on cone, some existence and multiplicity results of positive solutions of BVP(1.2) were obtained.

This paper is also motivated by [11]. E. R. Kaufmann and E. Mboumi studied the following boundary value problem for the fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) + a(t)f(t,u(t)) = 0, \quad 0 < t < 1, 1 < \alpha < 2, \\ u(0) = 0, u'(1) = 0\,, \end{cases} \tag{1.3}$$

by using the properties of the Green's function of the corresponding BVP, the Leggett-Williams fixed point theorem and the Krasnoselskii fixed point theorem. Under the assumptions:

**(A1)** $f : [0,1] \times [0,+\infty) \to [0,\infty)$ is continuous,

**(A2)** $a \in L^{\infty}[0,1]$,

**(A3)** there exists a constant $m > 0$ such that $a(t) \geq m$ a.e. $t \in [0,1]$,

the authors in [11] proved that BVP(1.3) has at least one or three positive solutions.

One notes that $f$ in [6, 11] is continuous on $[0,1] \times [0,\infty)$. While the existence and uniqueness of positive solutions for the nonlinear singular fractional differential equations have not been studied well.

In recent paper [12], the authors studied the following boundary value problem for singular fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t),D_{0+}^{\mu}u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, u(1) = 0\,, \end{cases} \tag{1.4}$$

where $1 < \alpha < 2, 0 < \mu \leq \alpha - 1$, $D_{0+}^{\alpha}$ is the standard Riemann-Liouville differentiation, $f : [0,1] \times (0,\infty) \times R \to (0,\infty)$ is continuous and $f(t,x,y)$ may be singular at $x = 0$. The existence of positive solutions of BVP(1.4) is obtained based upon the fixed point theorem on cone, the regularization and sequential techniques. In [12], $\mu \in (0, \alpha - 1]$ is supposed. But in (1.1), it is supposed that $\mu \in (0,\alpha)$.

The paper is divided into three sections. In Section 2 we give some basic definitions in Riemann-Liouville fractional calculus. The corresponding Green's function and its positivity is argued. In Section 3, by means of the contraction map principle, the uniqueness results of positive solution are obtained, then some existence results of positive solution are obtained by the use of the Schauder's fixed point theorem.

## 2. PRELIMINARY RESULTS

For the convenience of the readers, we present here the necessary definitions from the fixed point theory and the fractional calculus theory. These definitions and results can be found in the literatures [3, 4, 6, 7, 9].

**Definition 2.1.** Let $X$ be a real Banach space. The nonempty convex closed subset $P$ of $X$ is called a cone in $X$ if $ax \in P$ for all $x \in P$ and $a \geq 0$, $x \in X$ and $-x \in X$ imply $x = 0$.

**Definition 2.2.** An operator $T : X \to X$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

**Definition 2.3.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side exists.

**Definition 2.4.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to R$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n-1 \leq \alpha < n$, provided that the right-hand side is point-wise defined on $(0, \infty)$.

The Gamma and beta functions $\Gamma(\alpha)$ and $\mathbf{B}(p, q)$ be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx, \quad \mathbf{B}(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$$

and

$$\|m\|_1 = \int_0^1 |m(s)| ds \text{ for } m \in L^1(0, 1).$$

**Lemma 2.1.** *Let $n-1 \leq \alpha < n$, $u \in C^0(0, 1) \bigcap L^1(0, 1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

*where $C_i \in R$, $i = 1, 2, \ldots, n$.*

**Lemma 2.2.** *Let $n-1 \leq \alpha < n$, $u \in C^0(0, 1) \bigcap L^1(0, 1)$. Then $D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$.*

**Lemma 2.3** (Leray-Schauder Nonlinear Alternative [3])**.** *Let $X$ be a Banach space and $T : X \to X$ be a completely continuous operator. Suppose $\Omega$ is a nonempty open subset of $X$ centered at zero. Then either there exists $x \in \partial\Omega$ and $\lambda \in (0, 1)$ such that $x = \lambda T x$ or there exists $x \in \overline{\Omega}$ such that $x = Tx$.*

**Lemma 2.4.** *Suppose that $1 \neq \beta\eta^{\alpha-1}$. Given $h \in C(0,1)$ satisfying that*

**(H1)** *there exist constants $M \geq 0$, $k > -1$ and $l \in (\mu-\alpha, 0]$ such that $\alpha + l + k \geq 0$ and $|h(t)| \leq Mt^k(1-t)^l$ for all $t \in (0,1)$.*

*Then $u$ is a solution of*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \\ \lim_{t\to 0} t^{2-\alpha} u(t) = 0, \\ u(1) = \beta u(\eta). \end{cases} \tag{2.1}$$

*if and only if*

$$u(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.2}$$

*where*

$$G(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} - \frac{\beta t^{\alpha-1}(\eta-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & s \leq t, s \leq \eta, \\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & \eta \leq s \leq t, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} - \frac{\beta t^{\alpha-1}(\eta-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & t \leq s \leq \eta, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & t \leq s, s \geq \eta. \end{cases} \tag{2.3}$$

*Proof.* By the assumption, for $t \in (0,1)$, we have

$$\begin{aligned} \left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right| &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |h(s)|ds \\ &\leq M\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(1-s)^l ds \leq M\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(t-s)^l ds \\ &= Mt^{\alpha+l+k}\int_0^1 (1-w)^{\alpha+l-1} w^k dw = Mt^{\alpha+l+k}\mathbf{B}(\alpha+l, k+1) < \infty. \end{aligned}$$

We may apply Lemma 2.1 to reduce BVP(2.1) to an equivalent integral equation

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$$

for some $c_i \in R$, $i = 1, 2$.

From $\lim_{t\to 0} t^{2-\alpha}u(t) = 0$, one has $c_2 = 0$.

By $u(1) = \beta u(\eta)$, we get

$$c_1 = \frac{1}{1-\beta\eta^{\alpha-1}} \left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - \beta\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds\right).$$

Therefore, the unique solution of BVP(5) is

$$\begin{aligned} u(t) \;=\; & -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ & -\frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds. \end{aligned}$$

For $\eta < t$, we can express $u(t)$ by

$$\begin{aligned} u(t) &= \int_0^\eta \left( -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} - \frac{\beta t^{\alpha-1}(\eta-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} \right) h(s)ds \\ &\quad + \int_\eta^t \left( -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} \right) h(s)ds \\ &\quad + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

For $\eta \geq t$, we can do it similarly

$$\begin{aligned} u(t) &= \int_0^t \left( -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} - \frac{\beta t^{\alpha-1}(\eta-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} \right) h(s)ds \\ &\quad + \int_t^\eta \left( \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} - \frac{\beta t^{\alpha-1}(\eta-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} \right) h(s)ds \\ &\quad + \int_\eta^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

Here $G$ is defined by (2.3).

Reciprocally, let $u$ satisfy (2.2). Then by

$$t^{2-\alpha}\left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \right| \leq M t^{2+l+k} \int_0^1 (1-w)^{\alpha-1} w^k dw.$$

Since $\alpha + l + k \geq 0$, we have $2 + l + k > 0$. Then $\lim_{t\to 0} t^{2-\alpha}u(t) = 0$, $u(1) = \beta u(\eta)$, furthermore, use Lemma 2.2, we have $D_0^\alpha u(t) = -h(t)$. The proof is complete. □

**Remark 2.1.** If $\alpha = \frac{7}{4}$, $\mu = \frac{1}{2}$, $k = -\frac{1}{2}$ and $l = -1$, $h(t) = t^{-\frac{3}{4}}(1-t)^{-1}$, then $h$ satisfies (H1). But $h \notin L^1(0,1)$.

**Remark 2.2.** Suppose that $\beta \geq 0$, $1 > \beta\eta^{\alpha-1}$. Then

$$G(t,s) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha) - \beta\eta^{\alpha-1}}, \text{ for all } s,t \in [0,1],$$

and

$$G(t,s) \geq 0 \text{ for all } t \in [0,1], s \in [0,1].$$

In fact, one sees from (2.3) that

$$G(t,s) \leq \begin{cases} \frac{t^{\alpha-1}}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & s \leq t, s \leq \eta, \\ \frac{t^{\alpha-1}}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & \eta \leq s \leq t, \\ \frac{t^{\alpha-1}}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & t \leq s \leq \eta, \\ \frac{t^{\alpha-1}}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & t \leq s, \ s \geq \eta \end{cases}$$

$$\leq \quad \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}.$$

On the other hand, we have from (2.3) that

$$G(t,s) \geq \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(1-\beta\eta^{\alpha-1})(t-s)^{\alpha-1}-\beta t^{\alpha-1}(\eta-s\eta)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & s\leq t, s\leq \eta,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(1-\beta\eta^{\alpha-1})(t-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & \eta\leq s\leq t,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}-\beta t^{\alpha-1}(\eta-s\eta)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & t\leq s\leq \eta,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & t\leq s, s\geq \eta. \end{cases}$$

$$= \begin{cases} \frac{(1-\beta\eta^{\alpha-1})[t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & s\leq t, s\leq \eta,\\ \frac{\beta\eta^{\alpha-1}(t-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & \eta\leq s\leq t,\\ \frac{(1-\beta\eta^{\alpha-1})t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & t\leq s\leq \eta,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, & t\leq s, s\geq \eta. \end{cases}$$

$$\geq \quad 0.$$

For our construction, we let

$$X = \left\{ x:(0,1]\to R \begin{array}{c} x\in C^0(0,1], D^{\mu}_{0^+}x\in C^0(0,1] \\ \text{there exist the limits } \lim_{t\to 0} t^{2-\alpha}x(t) \\ \lim_{t\to 0} t^{\mu+2-\alpha}D^{\mu}_{0^+}x(t) \end{array} \right\}$$

with the norm

$$\|u\| = \max\left\{ \sup_{0<t\leq 1} t^{2-\alpha}|u(t)|, \sup_{t\in(0,1]} t^{\mu+2-\alpha}|D^{\mu}_{0^+}u(t)| \right\}, \quad u\in X.$$

Then $X$ is a Banach space. We seek solutions of BVP(1.1) that lie in the cone

$$P = \{u\in X : u(t)\geq 0,\ 0<t\leq 1\}.$$

Define the operator $T$ on $P$ by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s),D^{\mu}_{0^+}u(s))ds\,.$$

**Lemma 2.5.** *Suppose that* $1\neq \beta\eta^{\alpha-1}$ *and* $0<\mu<\alpha-1$, *and*

**(H2)** $f$ *satisfies that*

- **(i)** $t\to f(t,t^{\alpha-2}u,t^{\alpha-\mu-2}v)$ *is continuous on* $(0,1)$ *for each* $u\in R, v\in R$;
- **(ii)** $(u,v)\to f(t,t^{\alpha-2}u,t^{\alpha-\mu-2}v)$ *is continuous on* $R\times R$ *for each* $t\in(0,1)$;
- **(iii)** *for each* $r>0$, *there exist constants* $M\geq 0$, $k>-1$ *and* $l\in(\mu-\alpha,0]$ *such that* $\alpha+l+k\geq 0$

$$|f(t,t^{\alpha-2}u,t^{\alpha-\mu-2}v)|\leq Mt^k(1-t)^l, t\in(0,1), |u|, \quad |v|\leq r.$$

*Then* $x$ *is a solution of BVP*(1.1) *if and only if* $x$ *is a fixed point of* $T$, $T:P\to P$ *is well defined and completely continuous.*

*Proof.* By Lemma 2.4, we see that $x$ is a solution of BVP(1.1) if and only if $x$ is a fixed point of $T$. By the definition of $T$, we have

$$\begin{aligned} t^{2-\alpha}(Tu)(t) &= t^{2-\alpha}\int_0^1 G(t,s)f(s,u(s),D_{0+}^{\mu}u(s))ds \\ &= -t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),D_{0+}^{\mu}u(s))ds \\ &\quad +\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),D_{0+}^{\mu}u(s))ds \\ &\quad -\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),D_{0+}^{\mu}u(s))ds. \end{aligned}$$

It follows that

$$\begin{aligned} t^{\mu+2-\alpha}D_{0+}^{\mu}(Tu)(t) &= -t^{\mu+2-\alpha}\int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}f(s,u(s),D_{0+}^{\mu}u(s))ds \\ &\quad +\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\mu)}f(s,u(s),D_{0+}^{\mu}u(s))ds \\ &\quad -\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha-\mu)}f(s,u(s),D_{0+}^{\mu}u(s))ds. \end{aligned}$$

It is easy to see from Remark 2.2 that $Tu \in X$ and $(Tu)(t) \geq 0$ for all $t \in (0,1]$. Hence $T : P \to P$ is well defined.

We divide the remainder of the proof into two steps.

**Step 1.** $T$ is continuous.

Let $\{y_n\}_{n=0}^{\infty}$ be a sequence such that $y_n \to y_0$ in $X$ as $n \to \infty$. Then we have

$$\|y_n\| = \max\left\{\sup_{0<t\leq 1} t^{2-\alpha}|y_n(t)|, \sup_{t\in(0,1]} t^{\mu+2-\alpha}|D_{0+}^{\mu}y_n(t)|\right\} \leq r < \infty, \quad n = 0,1,\ldots$$

and

$$\lim_{n\to\infty}\sup_{t\in(0,1]} t^{2-\alpha}|y_n(t)-y_0(t)| = 0, \quad \lim_{n\to\infty}\sup_{t\in(0,1]} t^{\mu+2-\alpha}|D_{0+}^{\mu}y_n(t)-D_{0+}^{\mu}y_0(t)| = 0.$$

So there exist constants $M \geq 0$, $k > -1$ and $l \in (\mu-\alpha, 0]$ such that $\alpha+l+k \geq 0$ and

$$\begin{aligned} |f(t,y_n(t),D_{0+}^{\mu}y_n(t))| &= |f(t,t^{\alpha-2}t^{2-\alpha}y_n(t),t^{\alpha-\mu-2}t^{\mu+2-\alpha}D_{0+}^{\mu}y_n(t))| \\ &\leq Mt^k(1-t)^l, t \in (0,1). \end{aligned}$$

It follows that

$$\begin{aligned} &t^{2-\alpha}|(Ty_n)(t)-(Ty_0)(t)| \\ &= t^{2-\alpha}\left|\int_0^1 G(t,s)f(s,y_n(s),D_{0+}^{\mu}y_n(s))ds - \int_0^1 G(t,s)f(s,y_0(s),D_{0+}^{\mu}y_0(s))ds\right| \\ &\leq t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y_n(s),D_{0+}^{\mu}y_n(s))-f(s,y_0(s),D_{0+}^{\mu}y_0(s))|ds \end{aligned}$$

$$
\begin{aligned}
&+ \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y_n(s), D_{0^+}^{\mu} y_n(s)) - f(s, y_0(s), D_{0^+}^{\mu} y_0(s))| ds \\
&+ \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y_n(s), D_{0^+}^{\mu} y_n(s)) - f(s, y_0(s), D_{0^+}^{\mu} y_0(s))| ds \\
\le\, & 2Mt^{2-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds + 2M \frac{1}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds \\
&+ 2M \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds \\
\le\, & 2Mt^{2-\alpha} \int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^k ds + 2M \frac{1}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^k ds \\
&+ 2M \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(\eta-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^k ds \\
=\, & 2Mt^{2+l+k} \int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)} w^k dw + 2M \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha+l-1}}{\Gamma(\alpha)} s^k ds \\
&+ 2M \frac{\eta^{\alpha+l+k} t}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)} w^k dw \\
\le\, & 2M \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} + 2M \frac{\mathbf{B}(\alpha+l, k+1)}{1-\beta\eta^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \\
&+ 2M \frac{\eta^{\alpha+l+k}}{1-\beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
&t^{\mu+2-\alpha} |D_{0^+}^{\mu}(Ty_n)(t) - D_{0^+}^{\mu}(Ty_0)(t)| \\
\le\, & t^{\mu+2-\alpha} \int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} |f(s, y_n(s), D_{0^+}^{\mu} y_n(s)) - f(s, y_0(s), D_{0^+}^{\mu} y_0(s))| ds \\
&+ \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\mu)} |f(s, y_n(s), D_{0^+}^{\mu} y_n(s)) - f(s, y_0(s), D_{0^+}^{\mu} y_0(s))| ds \\
&+ \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha-\mu)} |f(s, y_n(s), D_{0^+}^{\mu} y_n(s)) - f(s, y_0(s), D_{0^+}^{\mu} y_0(s))| ds \\
\le\, & 2Mt^{\mu+2-\alpha} \int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds + 2M \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \\
&+ 2M \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \\
\le\, & 2Mt^{\mu+2-\alpha} \int_0^t \frac{(t-s)^{\alpha+l-\mu-1}}{\Gamma(\alpha-\mu)} s^k ds + 2M \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha+l-1}}{\Gamma(\alpha-\mu)} s^k ds \\
&+ 2M \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^\eta \frac{(\eta-s)^{\alpha+l-1}}{\Gamma(\alpha-\mu)} s^k ds \\
=\, & 2Mt^{2+l+k} \int_0^1 \frac{(1-w)^{\alpha+l-\mu-1}}{\Gamma(\alpha-\mu)} w^k dw + 2M \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha+l-1}}{\Gamma(\alpha-\mu)} s^k ds
\end{aligned}
$$

$$+2M\frac{t\eta^{\alpha+k+l}}{1-\beta\eta^{\alpha-1}}\int_0^1\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha-\mu)}w^kdw$$
$$\leq 2M\frac{\mathbf{B}(\alpha+l-\mu,k+1)}{\Gamma(\alpha-\mu)}+2M\frac{1\mathbf{B}(\alpha+l,k+1)}{1-\beta\eta^{\alpha-1}}\frac{1}{\Gamma(\alpha-\mu)}$$
$$+2M\frac{\eta^{\alpha+k+l}}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha-\mu)}<\infty.$$

From the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty}\sup_{t\in(0,1)}t^{2-\alpha}|(Ty_n)(t)-(Ty_0)(t)|=0,$$

and

$$\lim_{n\to\infty}\sup_{t\in(0,1)}t^{\mu+2-\alpha}[D^{\mu}_{0^+}(Ty_n)(t)-D^{\mu}_{0^+}(Ty_0)(t)]=0.$$

It follows that $T$ is continuous.

**Step 2.** We prove that $T$ is compact, i.e., for each nonempty open bounded subset $\Omega$ of $X$, prove that $T(\overline{\Omega})$ is relatively compact. We must prove that $T(\overline{\Omega})$ is uniformly bounded, equi-continuous on each subinterval $[a,b]\subseteq(0,1]$, and $T(\overline{\Omega})$ is equi-convergent as $t\to 0$.

Let $\Omega$ be a non-empty bounded open subset of $X$. We have

$$\|x\|=\max\left\{\sup_{t\in(0,1)}t^{2-\alpha}|x(t)|,\ \sup_{t\in(0,2)}t^{\mu+2-\alpha}|D^{\mu}_{0^+}x(t)|\right\}=r<+\infty \tag{2.4}$$

for all $x\in\overline{\Omega}$. So there exist constants $M\geq 0$, $k>-1$ and $l\in(\mu-\alpha,0]$ such that $\alpha+l+k\geq 0$ and

$$\begin{aligned}|f(t,x(t),D^{\mu}_{0^+}x(t))|&=|f(t,t^{\alpha-2}t^{2-\alpha}x(t),t^{\alpha-\mu-2}t^{\mu+2-\alpha}D^{\mu}_{0^+}x(t))|\\&\leq Mt^k(1-t)^l,t\in(0,1).\end{aligned}$$

It follows that

$$\begin{aligned}&t^{2-\alpha}|(Tx)(t)|\\&=t^{2-\alpha}\left|\int_0^1G(t,s)f(s,x(s),D^{\mu}_{0^+}x(s))ds\right|\\&\leq t^{2-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s),D^{\mu}_{0^+}x(s))|ds\\&\quad+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s),D^{\mu}_{0^+}x(s))|ds\\&\quad+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s),D^{\mu}_{0^+}x(s))|ds\\&\leq Mt^{2-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^lds+M\frac{1}{1-\beta\eta^{\alpha-1}}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^lds\\&\quad+M\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^lds\end{aligned}$$

$$
\begin{aligned}
&\le Mt^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^k ds + M\frac{1}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^k ds \\
&\quad + M\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^k ds \\
&= Mt^{2+l+k}\int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dw + M\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^k ds \\
&\quad + M\frac{\eta^{\alpha+l+k}t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dw \\
&\le M\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)} + M\frac{\mathbf{B}(\alpha+l,k+1)}{1-\beta\eta^{\alpha-1}}\frac{1}{\Gamma(\alpha)} \\
&\quad + M\frac{\eta^{\alpha+l+k}}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)} < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
&t^{\mu+2-\alpha}|D_{0^+}^{\mu}(Tx)(t)| \\
&\le t^{\mu+2-\alpha}\int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}|f(s,x(s),D_{0^+}^{\mu}x(s))|ds \\
&\quad + \frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\mu)}|f(s,x(s),D_{0^+}^{\mu}x(s))|ds \\
&\quad + \frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha-\mu)}|f(s,x(s),D_{0^+}^{\mu}x(s))|ds \\
&\le Mt^{\mu+2-\alpha}\int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}s^k(1-s)^l ds + M\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\mu)}s^k(1-s)^l ds \\
&\quad + M\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha-\mu)}s^k(1-s)^l ds \\
&\le Mt^{\mu+2-\alpha}\int_0^t \frac{(t-s)^{\alpha+l-\mu-1}}{\Gamma(\alpha-\mu)}s^k ds + M\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha+l-1}}{\Gamma(\alpha-\mu)}s^k ds \\
&\quad + M\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha+l-1}}{\Gamma(\alpha-\mu)}s^k ds \\
&= Mt^{2+l+k}\int_0^1 \frac{(1-w)^{\alpha+l-\mu-1}}{\Gamma(\alpha-\mu)}w^k dw + M\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha+l-1}}{\Gamma(\alpha-\mu)}s^k ds \\
&\quad + M\frac{t\eta^{\alpha+k+l}}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha-\mu)}w^k dw \\
&\le M\frac{\mathbf{B}(\alpha+l-\mu,k+1)}{\Gamma(\alpha-\mu)} + M\frac{\mathbf{B}(\alpha+l,k+1)}{1-\beta\eta^{\alpha-1}}\frac{1}{\Gamma(\alpha-\mu)} \\
&\quad + M\frac{\eta^{\alpha+k+l}}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha-\mu)} < \infty.
\end{aligned}
$$

It is easy to see that $T(\overline{\Omega})$ is uniformly bounded.

Let $t_1, t_2 \in [a, b] \subseteq (0, 1]$ with $t_1 < t_2$ and $u \in \overline{\Omega}$. We have

$$
\begin{aligned}
&|t_1^{2-\alpha}(Tu)(t_1) - t_2^{2-\alpha}(Tu)(t_2)| \\
&\leq \left| t_1^{2-\alpha} \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), D_{0^+}^{\mu} u(s)) ds \right. \\
&\quad \left. - t_2^{2-\alpha} \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), D_{0^+}^{\mu} u(s)) ds \right| \\
&\quad + \frac{|t_1 - t_2|}{1 - \beta\eta^{\alpha-1}} \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), D_{0^+}^{\mu} u(s))| ds \\
&\quad + \frac{|t_1 - t_2|}{1 - \beta\eta^{\alpha-1}} \int_0^{\eta} \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), D_{0^+}^{\mu} u(s))| ds \\
&\leq |t_1^{2-\alpha} - t_2^{2-\alpha}| \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), D_{0^+}^{\mu} u(s))| ds \\
&\quad + t_1^{2-\alpha} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), D_{0^+}^{\mu} u(s))| ds \\
&\quad + t_1^{2-\alpha} \int_0^{t_1} \frac{|(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}|}{\Gamma(\alpha)} |f(s, u(s), D_{0^+}^{\mu} u(s))| ds \\
&\quad + M \frac{|t_1 - t_2|}{1 - \beta\eta^{\alpha-1}} \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1 - s)^l ds + M \frac{|t_1 - t_2|}{1 - \beta\eta^{\alpha-1}} \int_0^{\eta} \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1 - s)^l ds \\
&\leq M|t_1^{2-\alpha} - t_2^{2-\alpha}| \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1 - s)^l ds \\
&\quad + M t_1^{2-\alpha} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1 - s)^l ds \\
&\quad + M t_1^{2-\alpha} \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1 - s)^l ds \\
&\quad + M \frac{|t_1 - t_2|}{1 - \beta\eta^{\alpha-1}} \int_0^1 \frac{(1 - s)^{\alpha+l-1}}{\Gamma(\alpha)} s^k ds + M \frac{|t_1 - t_2|}{1 - \beta\eta^{\alpha-1}} \int_0^{\eta} \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} s^k (\eta - s)^l ds \\
&\leq M|t_1^{2-\alpha} - t_2^{2-\alpha}| \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} s^k (t_2 - s)^l ds \\
&\quad + M t_1^{2-\alpha} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} s^k (t_2 - s)^l ds \\
&\quad + M t_1^{2-\alpha} \left[ t_2^{\alpha+l+k} \int_0^{\frac{t_1}{t_2}} \frac{(1 - w)^{\alpha+l-1}}{\Gamma(\alpha)} w^k dw - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1} (t_2 - s)^l}{\Gamma(\alpha)} s^k ds \right] \\
&\quad + M \frac{|t_1 - t_2|}{1 - \beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha + l, k + 1)}{\Gamma(\alpha)} + M\eta^{\alpha+l+k} \frac{|t_1 - t_2|}{1 - \beta\eta^{\alpha-1}} \int_0^1 \frac{(1 - w)^{\alpha+l-1}}{\Gamma(\alpha)} w^k dw \\
&\leq M|t_1^{2-\alpha} - t_2^{2-\alpha}| b^{\alpha+l+k} \frac{\mathbf{B}(\alpha + l, k + 1)}{\Gamma(\alpha)} + M b^{2+l+k} t_2^{\alpha+l+k} \int_{\frac{t_1}{t_2}}^1 \frac{(1 - w)^{\alpha+l-1}}{\Gamma(\alpha)} w^k dw
\end{aligned}
$$

$$
\begin{aligned}
&+M\left[t_2^{\alpha+l+k}\int_0^{\frac{t_1}{t_2}}\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dw - t_1^{\alpha+l+k}\int_0^1\frac{(1-w)^{\alpha-1}\left(\frac{t_2}{t_1}-w\right)^l}{\Gamma(\alpha)}w^k dw\right]\\
&+M\frac{|t_1-t_2|}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}+M\eta^{\alpha+l+k}\frac{|t_1-t_2|}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\\
\le{}& M|t_1^{2-\alpha}-t_2^{2-\alpha}|b^{\alpha+l+k}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}+Mb^{2+l+k}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dw\\
&+M|t_2^{\alpha+l+k}-t_1^{\alpha+l+k}|\int_0^{\frac{t_1}{t_2}}\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dw\\
&+Mt_1^{\alpha+l+k}\int_0^{\frac{t_1}{t_2}}\left(\frac{(1-w)^l}{\Gamma(\alpha)}-\frac{\left(\frac{t_2}{t_1}-w\right)^l}{\Gamma(\alpha)}\right)(1-w)^{\alpha-1}w^k dw\\
&+Mt_1^{\alpha+l+k}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha-1}\left(\frac{t_2}{t_1}-w\right)^l}{\Gamma(\alpha)}w^k dw\\
&+M\frac{|t_1-t_2|}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}+M\eta^{\alpha+l+k}\frac{|t_1-t_2|}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\\
\le{}& M|t_1^{2-\alpha}-t_2^{2-\alpha}|b^{\alpha+l+k}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}+Mf^{2+l+k}b^{\alpha+l+k}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dw\\
&+M|t_2^{\alpha+l+k}-t_1^{\alpha+l+k}|\int_0^{\frac{t_1}{t_2}}\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dw\\
&+Mb^{\alpha+l+k}\int_0^{\frac{t_1}{t_2}}\left(\frac{(1-w)^l}{\Gamma(\alpha)}-\frac{\left(\frac{t_2}{t_1}-w\right)^l}{\Gamma(\alpha)}\right)(1-w)^{\alpha-1}w^k dw\\
&+Mb^{\alpha+l+k}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dw\\
&+M\frac{|t_1-t_2|}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}+M\eta^{\alpha+l+k}\frac{|t_1-t_2|}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\\
\to{}& 0 \text{ uniformly as } t_1\to t_2.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&|t_1^{\mu+2-\alpha}D_{0^+}^{\mu}(Ty)(t_1)-t_2^{\mu+2-\alpha}D_{0^+}^{\mu}(Ty)(t_2)|\\
&\le\left|t_2^{\mu+2-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}f(s,u(s),D_{0^+}^{\mu}u(s))ds\right.\\
&\qquad\left.-t_1^{\mu+2-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}f(s,u(s),D_{0^+}^{\mu}u(s))ds\right|\\
&\qquad+\frac{|t_1-t_2|}{1-\beta\eta^{\alpha-1}}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\mu)}|f(s,u(s),D_{0^+}^{\mu}u(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|t_1 - t_2|}{1-\beta\eta^{\alpha-1}} \int_0^{\eta} \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha-\mu)} |f(s, u(s), D_{0^+}^{\mu} u(s))| ds \\
&\le M \left| t_2^{\mu+2-\alpha} - t_1^{\mu+2-\alpha} \right| \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \\
&+ M t_2^{\mu+2-\alpha} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \\
&+ M t_2^{\mu+2-\alpha} \int_0^{t_1} \frac{|(t_2 - s)^{\alpha-\mu-1} - (t_1 - s)^{\alpha-\mu-1}|}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \\
&+ M \frac{|t_1 - t_2|}{1-\beta\eta^{\alpha-1}} \int_0^{1} \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \\
&+ M \frac{|t_1 - t_2|}{1-\beta\eta^{\alpha-1}} \int_0^{\eta} \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \\
&\le M \left| t_2^{\mu+2-\alpha} - t_1^{\mu+2-\alpha} \right| \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} s^k (t_1-s)^l ds \\
&+ M t_2^{\mu+2-\alpha} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} s^k (t_2-s)^l ds \\
&+ M t_2^{\mu+2-\alpha} \int_0^{t_1} \frac{|(t_2 - s)^{\alpha-\mu-1} - (t_1 - s)^{\alpha-\mu-1}|}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \\
&+ M \frac{|t_1 - t_2|}{1-\beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} + M\eta^{\alpha+l+k} \frac{|t_1 - t_2|}{1-\beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \\
&\le M \left| t_2^{\mu+2-\alpha} - t_1^{\mu+2-\alpha} \right| t_1^{\alpha+l+k-\mu} \frac{\mathbf{B}(\alpha+l-\mu, k+1)}{\Gamma(\alpha-\mu)} \\
&+ M t_2^{\mu+2-\alpha} \int_{\frac{t_1}{t_2}}^{1} \frac{(1 - w)^{\alpha+l-\mu-1}}{\Gamma(\alpha-\mu)} w^k dw \\
&+ M t_1^{\mu+2-\alpha} \int_0^{t_1} \frac{|(t_2 - s)^{\alpha-\mu-1} - (t_1 - s)^{\alpha-\mu-1}|}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \\
&+ M \frac{|t_1 - t_2|}{1-\beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} + M\eta^{\alpha+l+k} \frac{|t_1 - t_2|}{1-\beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} \\
&\le M \left| t_2^{\mu+2-\alpha} - t_1^{\mu+2-\alpha} \right| \max\left\{ a^{\alpha+l+k-\mu}, b^{\alpha+l+k-\mu} \right\} \frac{\mathbf{B}(\alpha+l-\mu, k+1)}{\Gamma(\alpha-\mu)} \\
&+ M \max\left\{ a^{\mu+2-\alpha}, b^{\mu+2-\alpha} \right\} \int_{\frac{t_1}{t_2}}^{1} \frac{(1 - w)^{\alpha+l-\mu-1}}{\Gamma(\alpha-\mu)} w^k dw \\
&+ M t_1^{\mu+2-\alpha} \int_0^{t_1} \frac{|(t_2 - s)^{\alpha-\mu-1} - (t_1 - s)^{\alpha-\mu-1}|}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \\
&+ M \frac{|t_1 - t_2|}{1-\beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} + M\eta^{\alpha+l+k} \frac{|t_1 - t_2|}{1-\beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)}.
\end{aligned}
$$

If $\alpha - \mu - 1 \ge 0$, then

$$
\int_0^{t_1} \frac{|(t_2 - s)^{\alpha-\mu-1} - (t_1 - s)^{\alpha-\mu-1}|}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds
$$

$$\le \int_0^{t_1} \frac{(t_2-s)^{\alpha-\mu-1}-(t_1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} s^k (t_2-s)^l ds$$

$$= t_2^{\alpha+k+l-\mu} \int_0^{\frac{t_1}{t_2}} \frac{(1-w)^{\alpha+l-\mu-1}}{\Gamma(\alpha-\mu)} w^k dw - t_1^{\alpha+k+l-\mu} \int_0^1 \frac{(1-w)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} w^k \left(\frac{t_2}{t_1}-w\right)^l dw$$

$$= [t_2^{\alpha+k+l-\mu} - t_1^{\alpha+k+l-\mu}] \int_0^{\frac{t_1}{t_2}} \frac{(1-w)^{\alpha+l-\mu-1}}{\Gamma(\alpha-\mu)} w^k dw$$

$$+ t_1^{\alpha+k+l-\mu} \int_0^{\frac{t_1}{t_2}} \left( \frac{(1-w)^l}{\Gamma(\alpha-\mu)} - \frac{\left(\frac{t_2}{t_1}-w\right)^l}{\Gamma(\alpha-\mu)} \right) (1-w)^{\alpha-\mu-1} w^k dw$$

$$+ \max\left\{e^{\alpha+k+l-\mu}, f^{\alpha+k+l-\mu}\right\} \int_{\frac{t_1}{t_2}}^1 \frac{(1-w)^{\alpha+l-\mu-1}}{\Gamma(\alpha-\mu)} w^k dw$$

$\to 0$ uniformly as $t_1 \to t_2$.

If $\alpha-\mu-1<0$, similarly we can show that

$$\int_0^{t_1} \frac{|(t_2-s)^{\alpha-\mu-1}-(t_1-s)^{\alpha-\mu-1}|}{\Gamma(\alpha-\mu)} s^k (1-s)^l ds \to 0 \text{ uniformly as } t_1 \to t_2.$$

Hence

$$|t_1^{\mu+2-\alpha} D_{0^+}^{\mu} (Ty)(t_1) - t_2^{\mu+2-\alpha} D_{0^+}^{\mu} (Ty)(t_2)| \to 0 \text{ uniformly as } t_1 \to t_2.$$

Therefore, $T(\overline{\Omega})$ is equicontinuous on $[a,b] \subseteq (0,1]$.

Finally, we prove that $T(\overline{\Omega})$ is equi-convergent as $t \to 0$. In fact, for $u \in \overline{\Omega}$, we have

$$t^{2-\alpha}|(Tu)(t)| \le t^{2-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s),D_{0^+}^{\mu}u(s))| ds$$

$$+ \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s),D_{0^+}^{\mu}u(s))| ds$$

$$+ \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s),D_{0^+}^{\mu}u(s))| ds$$

$$\le M t^{2-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds + M \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds$$

$$+ M \frac{t}{1-\beta\eta^{\alpha-1}} \int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^l ds$$

$$\le M t^{2+k+l} \frac{\mathbf{B}(\alpha+k+l,k+1)}{\Gamma(\alpha)} + M \frac{t}{1-\beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha+k+l,k+1)}{\Gamma(\alpha)}$$

$$+ M \frac{t}{1-\beta\eta^{\alpha-1}} \eta^{\alpha+k+l} \frac{\mathbf{B}(\alpha+k+l,k+1)}{\Gamma(\alpha)}$$

$\to 0$ uniformly as $t \to 0$.

Similarly we get

$$
\begin{aligned}
t^{\mu+2-\alpha}|D_{0^+}^{\mu}(Tu)(t)| &\le t^{\mu+2-\alpha}\int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)}|f(s,u(s),D_{0^+}^{\mu}u(s))|ds\\
&+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\mu)}|f(s,u(s),D_{0^+}^{\mu}u(s))|ds\\
&+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha-\mu)}|f(s,u(s),D_{0^+}^{\mu}u(s))|ds\\
&\le Mt^{2+k+l}\frac{\mathbf{B}(\alpha+l-\mu,k+1)}{\Gamma(\alpha-\mu)}+M\frac{t}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+k+l,k+1)}{\Gamma(\alpha-\mu)}\\
&+M\frac{t}{1-\beta\eta^{\alpha-1}}\eta^{\alpha+k+l}\frac{\mathbf{B}(\alpha+k+l,k+1)}{\Gamma(\alpha-\mu)}\\
&\to 0 \text{ uniformly as } t\to 0.
\end{aligned}
$$

The Arzela-Ascoli theorem implies that $T(\overline{\Omega})$ is relatively compact. Thus, the operator $T: P\to P$ is completely continuous. The proof is completed. □

## 3. MAIN RESULTS

In this section, we prove the main results. It is supposed that $1>\beta\eta^{\alpha-1}$ and $\beta\ge 0$, $f$ satisfies (H2). For $M_i\ge 0$, $k_i>-1$, $l_i\in(\mu-\alpha,0]$ $(i=1,2)$ such that $\alpha+l_i+k_i\ge 0$, denote

$$
\begin{aligned}
\Pi &= \max\left\{\frac{1}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\sum_{i=1}^{2}M_i\mathbf{B}(\alpha+k_i+l_i,k_i+1)(2-\beta\eta^{\alpha-1}+\eta^{\alpha+l_i+k_i}),\right.\\
&\left.\frac{1}{\Gamma(\alpha-\mu)(1-\beta\eta^{\alpha-1})}\sum_{i=1}^{2}M_i\mathbf{B}(\alpha+l_i-\mu,k_i+1)(2-\beta\eta^{\alpha-1}+\eta^{\alpha+k_i+l_i})\right\}.
\end{aligned}
$$

**Theorem 3.1.** *Suppose that*

**(H3)** *there exist $M_i\ge 0$, $k_i>-1$, $l_i\in(\mu-\alpha,0]$ $(i=1,2)$ such that $\alpha+k_i+l_i\ge 0$ $(i=1,2)$ and*

$$
\begin{aligned}
&|f(t.t^{\alpha-2}u_1,t^{\alpha-\mu-2}v_1)-f(t.t^{\alpha-2}u_2,t^{\alpha-\mu-2}v_2)|\\
&\le M_1t^{k_1}(1-t)^{l_1}|u_1-u_2|+M_2t^{k_2}(1-t)^{l_2}|v_1-v_2|.
\end{aligned}
\tag{3.1}
$$

*holds for all $t\in(0,1),u_1,u_2,v_1,v_2\in R$. Then BVP(1) has a unique positive solution if*

$$\Pi<1. \tag{3.2}$$

*Proof.* We shall prove that under the assumptions (3.1) and (3.2), $T$ is a contraction operator. Indeed, by the definition of $T$ for $x_1,x_2\in X$, one has

$$
\begin{aligned}
t^{2-\alpha}(Tx_i)(t) &= -t^{2-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_i(s),D_{0^+}^{\mu}x_i(s))ds\\
&+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_i(s),D_{0^+}^{\mu}x_i(s))ds\\
&-\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x_i(s),D_{0^+}^{\mu}x_i(s))ds
\end{aligned}
$$

and

$$\begin{aligned}
&t^{\mu+2-\alpha}D_{0^+}^{\mu}(Tx_i)(t) = -t^{\mu+2-\alpha}\int_0^t \tfrac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} f(s,x_i(s),D_{0^+}^{\mu}x_i(s))ds\\
&+\tfrac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \tfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\mu)} f(s,x_i(s),D_{0^+}^{\mu}x_i(s))ds\\
&-\tfrac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \tfrac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha-\mu)} f(s,x_i(s),D_{0^+}^{\mu}x_i(s))ds.
\end{aligned}$$

Suppose that $\max\{\|x_1\|,\|x_2\|\} = r$. Then there exist $M_i > 0$, $k_i > -1$, $l_i \in (\mu-\alpha]$, $(i=1,2)$ such that $\alpha + k_i + l_i \geq 0$ $(i=1,2)$ and (3.1) holds for all $t \in (0,1), u_1,u_2,v_1,v_2 \in R$. It is easy to see that

$$\begin{aligned}
&|f(s,x_1(s),D_{0^+}^{\mu}x_1(s)) - f(s,x_2(s),D_{0^+}^{\mu}x_2(s))|\\
&= |f(s,s^{\alpha-2}s^{2-\alpha}x_1(s),s^{\alpha-\mu-2}s^{\mu+2-\alpha}D_{0^+}^{\mu}x_1(s))x_2(s))|\\
&\leq M_1 s^{k_1}(1-s)^{l_1}|s^{2-\alpha}x_1(s) - s^{2-\alpha}x_2(s)|\\
&\quad +M_2 s^{k_2}(1-s)^{l_2}|s^{\mu+2-\alpha}D_{0^+}^{\mu}x_1(s) - s^{\mu+2-\alpha}D_{0^+}^{\mu}x_2(s)|\\
&\leq [M_1 s^{k_1}(1-s)^{l_1} + M_2 s^{k_2}(1-s)^{l_2}]\|x_1-x_2\|.
\end{aligned}$$

So we have the estimate

$$\begin{aligned}
&t^{2-\alpha}|(Tx_1)(t)-(Tx_2)(t)|\\
&\leq t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x_1(s),D_{0^+}^{\mu}x_1(s)) - f(s,x_2(s),D_{0^+}^{\mu}x_2(s))|ds\\
&\quad+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x_1(s),D_{0^+}^{\mu}x_1(s)) - f(s,x_2(s),D_{0^+}^{\mu}x_2(s))|ds\\
&\quad+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x_1(s),D_{0^+}^{\mu}x_1(s)) - f(s,x_2(s),D_{0^+}^{\mu}x_2(s))|ds\\
&\leq t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_1 s^{k_1}(1-s)^{l_1} + M_2 s^{k_2}(1-s)^{l_2}]\|x_1-x_2\|ds\\
&\quad+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}[M_1 s^{k_1}(1-s)^{l_1} + M_2 s^{k_2}(1-s)^{l_2}]\|x_1-x_2\|ds\\
&\quad+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}[M_1 s^{k_1}(1-s)^{l_1} + M_2 s^{k_2}(1-s)^{l_2}]\|x_1-x_2\|ds\\
&\leq \|x_1-x_2\|\sum_{i=1}^2\left[M_i t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_i}(1-s)^{l_i}ds\right.\\
&\quad+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} M_i s^{k_i}(1-s)^{l_i}ds\\
&\quad\left.+\frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} M_i s^{k_i}(1-s)^{l_i}ds\right]\\
&\leq \|x_1-x_2\|\sum_{i=1}^2\left[M_i t^{2+k_i+l_i}\frac{\mathbf{B}(\alpha+k_i+l_i,k_i+1)}{\Gamma(\alpha)}\right.\\
&\quad\left.+M_i\frac{t}{1-\beta\eta^{\alpha-1}}\eta^{\alpha+k_i+l_i}\frac{\mathbf{B}(\alpha+k_i+l_i,k_i+1)}{\Gamma(\alpha)}\right]
\end{aligned}$$

$$\leq \|x_1 - x_2\| \sum_{i=1}^{2} \Bigg[ M_i \frac{\mathbf{B}(\alpha + k_i + l_i, k_i + 1)}{\Gamma(\alpha)} + M_i \frac{1}{1 - \beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha + k_i + l_i, k_i + 1)}{\Gamma(\alpha)}$$
$$+ M_i \frac{1}{1 - \beta\eta^{\alpha-1}} \eta^{\alpha+k_i+l_i} \frac{\mathbf{B}(\alpha + k_i + l_i, k_i + 1)}{\Gamma(\alpha)} \Bigg] \leq \Pi \|x_1 - x_2\|$$

and similarly we get

$$\begin{aligned} &t^{2+\mu-\alpha}|D^{\mu}_{0^+}(Tx_1)(t) - D^{\mu}_{0^+}(Tx_2)(t)| \\ &\leq \|x_1 - x_2\| \textstyle\sum_{i=1}^{2} \Big[ M_i \frac{\mathbf{B}(\alpha+l_i-\mu,k_i+1)}{\Gamma(\alpha-\mu)} + M_i \frac{1}{1-\beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha+k_i+l_i,k_i+1)}{\Gamma(\alpha-\mu)} \\ &\quad + M_i \frac{1}{1-\beta\eta^{\alpha-1}} \eta^{\alpha+k_i+l_i} \frac{\mathbf{B}(\alpha+k_i+l_i,k_i+1)}{\Gamma(\alpha-\mu)} \Big] \leq \Pi \|x_1 - x_2\|. \end{aligned}$$

Then

$$\|Tx_1 - Tx_2\| \leq \Pi \|x_1 - x_2\|.$$

Hence the contraction map principle implies that BVP(1) has a unique solution $x \in X$. By $x(t) = (Tx)(t) = \int_0^1 G(t,s) f(s, x(s), D^{\mu}_{0^+} x(s)) ds$ and Remark 2.2, it is easy to show that $x$ is a positive solution of BVP(1.1). The proof is completed. □

**Theorem 3.2.** *Suppose that*

**(H4)** *there exist* $M_i > 0$, $k_i > -1$, $l_i \in (\mu - \alpha, 0]$ $(i = 1, 2, 3)$ *such that* $\alpha + k_i + l_i \geq 0$ $(i = 1, 2, 3)$ *and*

$$|f(t.t^{\alpha-2}x, t^{\alpha-\mu-2}y)| \leq M_3 t^{k_3}(1-t)^{l_3} + M_2 t^{k_2}(1-t)^{l_2}|x| + M_1 t^{k_1}(1-t)^{l_1}|y|. \tag{3.3}$$

*holds for all* $t \in (0,1), x, y \in R$. *Then* $BVP(1.1)$ *has at least positive solution if* $(10)$ *holds.*

*Proof.* To apply Lemma 2.3, we should define an open bounded subset $\Omega$ of $X$ centered at zero such that assumptions in Lemma 2.3 hold.

Let $\Omega_1 = \{x \in X : \ x = \lambda Tx$ for some $\lambda \in (0,1)\}$. We prove that $\Omega_1$ is bounded. For $x \in \Omega_1$, we get $x = \lambda Tx$. It follows that

$$\begin{aligned} t^{2-\alpha}x(t) = t^{2-\alpha}\lambda(Tx)(t) &= -\lambda t^{2-\alpha} \textstyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), D^{\mu}_{0^+}x(s)) ds \\ &+ \lambda \tfrac{t}{1-\beta\eta^{\alpha-1}} \textstyle\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), D^{\mu}_{0^+}x(s)) ds \\ &- \lambda \tfrac{t}{1-\beta\eta^{\alpha-1}} \textstyle\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), D^{\mu}_{0^+}x(s)) ds \end{aligned}$$

and

$$\begin{aligned} t^{\mu+2-\alpha}D^{\mu}_{0^+}x(t) = t^{\mu+2-\alpha}\lambda D^{\mu}_{0^+}(Tx)(t) &= -\lambda t^{\mu+2-\alpha} \textstyle\int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} f(s, x(s), D^{\mu}_{0^+}x(s)) ds \\ &+ \lambda \tfrac{t}{1-\beta\eta^{\alpha-1}} \textstyle\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\mu)} f(s, x(s), D^{\mu}_{0^+}x(s)) ds \\ &- \lambda \tfrac{t}{1-\beta\eta^{\alpha-1}} \textstyle\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha-\mu)} f(s, x(s), D^{\mu}_{0^+}x(s)) ds. \end{aligned}$$

By (3.3) there exist $M_i \geq 0$, $k_i > -1$, $l_i \in (\mu - \alpha]$, $(i = 1, 2, 3)$ such that $\alpha + k_i + l_i \geq 0$ $(i = 1, 2, 3)$ such that

$$|f(t.t^{\alpha-2}u, t^{\alpha-\mu-2}v)| \leq M_3 t^{k_3}(1-t)^{l_3} + M_2 t^{k_2}(1-t)^{l_2}|u| + M_1 t^{k_1}(1-t)^{l_1}|v|$$

holds for all $t \in (0,1), u, v \in R$. It is easy to see that

$$\begin{aligned}&|f(s,x(s),D^{\mu}_{0^+}x(s))| = |f(s, s^{\alpha-2}s^{2-\alpha}x(s), s^{\alpha-\mu-2}s^{\mu+2-\alpha}D^{\mu}_{0^+}x(s))|\\ &\le M_3s^{k_3}(1-s)^{l_3} + M_2s^{k_2}(1-s)^{l_2}|s^{2-\alpha}|x(s)| + M_1s^{k_1}(1-s)^{l_1}s^{\mu+2-\alpha}|D^{\mu}_{0^+}x_1(s)|\\ &\le M_3s^{k_3}(1-s)^{l_3} + M_2s^{k_2}(1-s)^{l_2}\|x\| + M_1s^{k_1}(1-s)^{l_1}s^{\mu+2-\alpha}\|x\|.\end{aligned}$$

So we have the estimate

$$\begin{aligned}&t^{2-\alpha}|x(t)| \le t^{2-\alpha}|(Tx)(t)| \le t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s),D^{\mu}_{0^+}x(s))|ds\\ &\quad + \frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s),D^{\mu}_{0^+}x(s))|ds\\ &\quad + \frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s),D^{\mu}_{0^+}x(s))|ds\\ &\le t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[M_3s^{k_3}(1-s)^{l_3} + M_1s^{k_1}(1-s)^{l_1}\|x\|\\ &\quad + M_2s^{k_2}(1-s)^{l_2}\|x\|]ds\\ &\quad + \frac{t}{1-\beta\eta^{\alpha-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}[M_3s^{k_3}(1-s)^{l_3} + M_1s^{k_1}(1-s)^{l_1}\|x\|\\ &\quad + M_2s^{k_2}(1-s)^{l_2}\|x\|]ds\\ &\quad + \frac{t}{1-\beta\eta^{\alpha-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}[M_3s^{k_3}(1-s)^{l_3} + M_1s^{k_1}(1-s)^{l_1}\|x\|\\ &\quad + M_2s^{k_2}(1-s)^{l_2}\|x\|]ds\\ &\le \|x\|\sum_{i=1}^{2}\left[M_i\frac{\mathbf{B}(\alpha+k_i+l_i,k_i+1)}{\Gamma(\alpha)} + M_i\frac{1}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+k_i+l_i,k_i+1)}{\Gamma(\alpha)}\right.\\ &\quad \left.+M_i\frac{1}{1-\beta\eta^{\alpha-1}}\eta^{\alpha+k_i+l_i}\frac{\mathbf{B}(\alpha+k_i+l_i,k_i+1)}{\Gamma(\alpha)}\right]\\ &\quad + M_3\frac{\mathbf{B}(\alpha+k_3+l_3,k_3+1)}{\Gamma(\alpha)} + M_3\frac{1}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+k_3+l_3,k_3+1)}{\Gamma(\alpha)}\\ &\quad + M_3\frac{1}{1-\beta\eta^{\alpha-1}}\eta^{\alpha+k_3+l_i}\frac{\mathbf{B}(\alpha+k_3+l_3,k_3+1)}{\Gamma(\alpha)}\end{aligned}$$

and similarly we get

$$\begin{aligned}&t^{\mu+2-\alpha}|D^{\mu}_{0^+}x(t)| \le t^{2+\mu-\alpha}|D^{\mu}_{0^+}(Tx)(t)| \le \|x\|\textstyle\sum_{i=1}^2\left[M_i\frac{\mathbf{B}(\alpha+l_i-\mu,k_i+1)}{\Gamma(\alpha-\mu)}\right.\\ &\quad \left.+M_i\frac{1}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+k_i+l_i,k_i+1)}{\Gamma(\alpha-\mu)} + M_i\frac{1}{1-\beta\eta^{\alpha-1}}\eta^{\alpha+k_i+l_i}\frac{\mathbf{B}(\alpha+k_i+l_i,k_i+1)}{\Gamma(\alpha-\mu)}\right]\\ &\quad +M_3\frac{\mathbf{B}(\alpha+l_3-\mu,k_3+1)}{\Gamma(\alpha-\mu)} + M_3\frac{1}{1-\beta\eta^{\alpha-1}}\frac{\mathbf{B}(\alpha+k_3+l_3,k_3+1)}{\Gamma(\alpha-\mu)}\\ &\quad +M_3\frac{1}{1-\beta\eta^{\alpha-1}}\eta^{\alpha+k_3+l_3}\frac{\mathbf{B}(\alpha+k_3+l_3,k_3+1)}{\Gamma(\alpha-\mu)}.\end{aligned}$$

Then

$$\|x\| \le \Pi\|x\| + M_3\frac{\mathbf{B}(\alpha+l_3-\mu,k_3+1)}{\Gamma(\alpha-\mu)}$$

$$+M_3 \frac{1}{1-\beta\eta^{\alpha-1}} \frac{\mathbf{B}(\alpha+k_3+l_3, k_3+1)}{\Gamma(\alpha-\mu)}$$
$$+M_3 \frac{1}{1-\beta\eta^{\alpha-1}} \eta^{\alpha+k_3+l_3} \frac{\mathbf{B}(\alpha+k_3+l_3, k_3+1)}{\Gamma(\alpha-\mu)}.$$

It follows from $\Pi < 1$ that there exists a constant $M > 0$ such that $\|x\| \leq M$. It follows that $\Omega_1$ is bounded.

To apply Lemma 2.3, let $\Omega$ be a non-empty open bounded subset of $X$ such that $\Omega \supset \overline{\Omega_1}$ centered at zero.

It is easy to see from Lemma 2.5 that $T$ is a completely continuous operator. One can see that

$$x \neq \lambda Tx \text{ for all } x \in \partial\Omega \text{ and } \lambda \in (0,1).$$

Thus, from Lemma 2.3, $x = Tx$ has at least one solution $x \in \overline{\Omega}$. So $x$ is a solution of BVP(1.1). It is easy to how that $x$ is a positive solution of BVP(1). The proof of Theorem 3.2 is complete. □

**Remark 3.1.** In (H4), $f$ may be a non-Caratheodory function. For example, $\alpha = \frac{7}{4}$, $\mu = \frac{1}{2}$, and $f(t,x,y) = t^{-\frac{1}{2}}(1-t)^{-1}(1 + t^{\frac{1}{4}}x + t^{\frac{3}{4}}y)$ is not a Caratheodory function. But $f$ satisfies (H4).

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