Dynamic Systems and Applications 23 (2014) 163-178

MONOTONE POSITIVE SOLUTION OF THIRD-ORDER BOUNDARY VALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS

JIAN-PING SUN, YA-ZHUO CHEN, AND XIAO-LI ZHANG

Department of Applied Mathematics, Lanzhou University of Technology,
Lanzhou, Gansu 730050, People's Republic of China

Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. This paper is concerned with the following third-order boundary value problem with integral boundary conditions

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, \ t \in [0, 1], \\ u(0) = 0, \ u'(0) = \int_0^1 g_1(t)u'(t)dt, u'(1) = \int_0^1 g_2(t)u'(t)dt, \end{cases}$$

where $f \in C([0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$ and $g_i \in C([0, 1], [0, +\infty))$, $i = 1, 2$. The existence of monotone positive solution to the above problem is obtained when f is superlinear or sublinear. The main tool used is the Guo-Krasnoselskii fixed point theorem.

AMS (MOS) Subject Classification. 34B10.

1. INTRODUCTION

Third-order differential equations arise from a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [8].

Recently, third-order two-point or multi-point boundary value problems (BVPs for short) have attracted a lot of attention, see [1, 4, 6, 7, 10, 11, 14, 15, 16, 17, 18, 19, 21, 22, 23, 25, 26, 27, 28] and the references therein. It is known that BVPs with integral boundary conditions cover multi-point BVPs as special cases. Although there are many excellent works on third-order two-point or multi-point BVPs, a little work has been done for third-order BVPs with integral boundary conditions [2, 3, 12, 13, 20, 24]. Especially, in 2010, Sun and Li [20] employed the Guo-Krasnoselskii fixed point theorem to study the existence and nonexistence of monotone positive solution for the following third-order BVP with integral boundary conditions

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, \ t \in [0, 1], \\ u(0) = 0, \ u'(0) = 0, \ u'(1) = \int_0^1 g(t)u'(t)dt. \end{cases} \tag{1.1}$$

Received May 2013

Among the boundary conditions in (1.1), only the slope of the tangent of the solution $u(t)$ at the point $(1, u(1))$ is related to the area under the curve of the product of $u'(t)$ and some function from $t = 0$ to $t = 1$. A natural question is that whether we can obtain similar results when the slope of the tangent of the solution $u(t)$ at the point $(0, u(0))$ is also related to the area under the curve of the product of $u'(t)$ and some function from $t = 0$ to $t = 1$. To answer this question, in this paper, we are concerned with the following third-order BVP with integral boundary conditions

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, \ t \in [0, 1], \\ u(0) = 0, \ u'(0) = \int_0^1 g_1(t)u'(t)dt, \ u'(1) = \int_0^1 g_2(t)u'(t)dt. \end{cases} \tag{1.2}$$

Throughout this paper, we always assume $f \in C([0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$, $g_i \in C([0, 1], [0, +\infty))$ $(i = 1, 2)$. Existence results of at least one monotone positive solution for the BVP (1.2) are established when$f$ is superlinear or sublinear. Here, a solution u of the BVP (1.2) is said to be monotone, if $u'(t) \geq 0$ for $t \in [0, 1]$. Our main tool is the following Guo-Krasnoselskii fixed point theorem [9].

Theorem 1.1. *Let E be a Banach space and K be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that either*

(1) $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$, or

(2) $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. PRELIMINARIES

For convenience, we denote

$$\mu = \left(1 - \int_0^1 g_1(r)dr\right) \int_0^1 rg_2(r)dr - \left(1 - \int_0^1 g_2(r)dr\right) \int_0^1 rg_1(r)dr + \int_0^1 g_1(r)dr.$$

Lemma 2.1. *Let $\mu \neq 1$. Then for any $\sigma \in C[0, 1]$, the BVP*

$$\begin{cases} -u'''(t) = \sigma(t), \ t \in [0, 1], \\ u(0) = 0, \ u'(0) = \int_0^1 g_1(t)u'(t)dt, \ u'(1) = \int_0^1 g_2(t)u'(t)dt \end{cases} \tag{2.1}$$

has a unique solution

$$\begin{aligned} u(t) = & \int_0^1 G_1(t, s)\sigma(s)ds + \frac{t^2}{2(1-\mu)} \int_0^1 \int_0^1 G_2(\tau, s) \\ & \times \left[\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau)\right] d\tau\sigma(s)ds \\ & + \frac{t}{1-\mu} \int_0^1 \int_0^1 G_2(\tau, s) \\ & \times \left[\int_0^1 rg_1(r)drg_2(\tau) + \left(1 - \int_0^1 rg_2(r)dr\right) g_1(\tau)\right] d\tau\sigma(s)ds, \ t \in [0, 1], \end{aligned} \tag{2.2}$$

where

$$G_1(t,s)=\frac{1}{2}\begin{cases}(2t-t^2-s)\,s, & 0\le s\le t\le 1,\\ (1-s)t^2, & 0\le t\le s\le 1\end{cases}$$

and

$$G_2\,(t,s)=\begin{cases}(1-t)\,s, & 0\le s\le t\le 1,\\ (1-s)\,t, & 0\le t\le s\le 1.\end{cases}$$

Proof. Let u be a solution of the BVP (2.1). Then, we may suppose that

$$u(t)=\int_0^1 G_1(t,s)\sigma(s)ds+At^2+Bt+C,\quad t\in[0,1].$$

By the boundary conditions in (2.1), we have

$$\begin{aligned}A=&\frac{1}{2(1-\mu)}\int_0^1\int_0^1 G_2(\tau,s)\\ &\times\left[\left(1-\int_0^1 g_1(r)dr\right)g_2(\tau)-\left(1-\int_0^1 g_2(r)dr\right)g_1(\tau)\right]d\tau\sigma(s)ds,\\ B=&\frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)\\ &\times\left[\int_0^1 rg_1(r)drg_2(\tau)+\left(1-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau\sigma(s)ds,\\ C=&\,0.\end{aligned}$$

Therefore, the BVP (2.1) has a unique solution

$$\begin{aligned}u(t)=&\int_0^1 G_1(t,s)\sigma(s)ds+\frac{t^2}{2(1-\mu)}\int_0^1\int_0^1 G_2(\tau,s)\\ &\times\left[\left(1-\int_0^1 g_1(r)dr\right)g_2(\tau)-\left(1-\int_0^1 g_2(r)dr\right)g_1(\tau)\right]d\tau\sigma(s)ds\\ &+\frac{t}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)\\ &\times\left[\int_0^1 rg_1(r)drg_2(\tau)+\left(1-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau\sigma(s)ds,\ t\in[0,1].\end{aligned}$$

□

Lemma 2.2 ([23])**.** *For any* $(t,s)\in[0,1]\times[0,1]$,

$$\frac{t^2}{2}(1-s)s\le G_1\,(t,s)\le\frac{1}{2}\,(1-s)\,s.$$

Lemma 2.3 ([5])**.** *For any* $(t,s)\in[0,1]\times[0,1]$,

$$0\le G_2(t,s)\le(1-s)\,s.$$

In the remainder of this paper, we always assume that $\alpha \in (0,1)$, $\beta = \frac{\alpha^2}{2}$ and the following two conditions are satisfied:

$(H1)$ $g_1(r) \le g_2(r),\ r \in [0,1]$;

$(H2)$ $\int_0^1 r g_2(r)dr + \int_0^1 g_1(r)dr \left(1 + \int_0^1 g_2(r)dr\right) < 1$.

Lemma 2.4. *If $\sigma \in C[0,1]$ and $\sigma(t) \ge 0$ for $t \in [0,1]$, then the unique solution u of the BVP (2.1) satisfies*
(1) $u(t) \ge 0,\ t \in [0,1]$;
(2) $u'(t) \ge 0$, $t \in [0,1]$ and $\min\limits_{t\in[\alpha,1]} u(t) \ge \beta \|u\|$, where $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$.

Proof. By $(H2)$, we can obtain that

$$\int_0^1 g_1(r)dr < 1, \tag{2.3}$$

$$\int_0^1 r g_2(r)dr < 1 \quad \text{and} \quad \mu < 1. \tag{2.4}$$

In view of (2.3) and $(H1)$, we have

$$\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau) \ge 0, \quad \tau \in [0,1]. \tag{2.5}$$

From (2.2), we get

$$\begin{aligned} u'(t) = {} & \int_0^1 G_2(t,s)\sigma(s)ds + \frac{t}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\ & \times \left[\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau)\right] d\tau\sigma(s)ds \\ & + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\ & \times \left[\int_0^1 r g_1(r)dr g_2(\tau) + \left(1 - \int_0^1 r g_2(r)dr\right) g_1(\tau)\right] d\tau\sigma(s)ds, t \in [0,1]. \end{aligned} \tag{2.6}$$

It follows from (2.2), (2.6), (2.4) and (2.5) that $u(t) \ge 0$ and $u'(t) \ge 0$ for $t \in [0,1]$.

On the one hand, in view of (2.2) and Lemma 2.2, we have

$$\begin{aligned} \|u\|_\infty \le {} & \int_0^1 (1-s)s\sigma(s)ds + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\ & \times \left[\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau)\right] d\tau\sigma(s)ds \\ & + \frac{2}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\ & \times \left[\int_0^1 r g_1(r)dr g_2(\tau) + \left(1 - \int_0^1 r g_2(r)dr\right) g_1(\tau)\right] d\tau\sigma(s)ds. \end{aligned} \tag{2.7}$$

On the other hand, by (2.6) and Lemma 2.3, we have

$$\begin{aligned}
\|u'\|_\infty \leq & \int_0^1 (1-s)s\sigma(s)ds + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\
& \times \left[\left(1-\int_0^1 g_1(r)dr\right)g_2(\tau) - \left(1-\int_0^1 g_2(r)dr\right)g_1(\tau)\right] d\tau\sigma(s)ds \\
& + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\
& \times \left[\int_0^1 rg_1(r)drg_2(\tau) + \left(1-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right] d\tau\sigma(s)ds. \qquad (2.8)
\end{aligned}$$

It follows from (2.7) and (2.8) that

$$\begin{aligned}
\|u\| \leq & \int_0^1 (1-s)s\sigma(s)ds + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\
& \times \left[\left(1-\int_0^1 g_1(r)dr\right)g_2(\tau) - \left(1-\int_0^1 g_2(r)dr\right)g_1(\tau)\right] d\tau\sigma(s)ds \\
& + \frac{2}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\
& \times \left[\int_0^1 rg_1(r)drg_2(\tau) + \left(1-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right] d\tau\sigma(s)ds,
\end{aligned}$$

which together with Lemma 2.2 implies that

$$\begin{aligned}
\min_{t\in[\alpha,1]} u(t) \geq & \min_{t\in[\alpha,1]} \frac{t^2}{2}\left\{\int_0^1 (1-s)s\sigma(s)ds + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)\right. \\
& \times \left[\left(1-\int_0^1 g_1(r)dr\right)g_2(\tau) - \left(1-\int_0^1 g_2(r)dr\right)g_1(\tau)\right] d\tau\sigma(s)ds \\
& + \frac{2}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\
& \left.\times \left[\int_0^1 rg_1(r)drg_2(\tau) + \left(1-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right] d\tau\sigma(s)ds\right\} \\
& \geq \beta\|u\|.
\end{aligned}$$

□

Let $E = C^1[0,1]$ be equipped with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$. Then E is a Banach space. If we denote

$$K = \left\{u \in E: \ u(t) \geq 0,\ u'(t) \geq 0,\ t \in [0,1] \text{ and } \min_{t\in[\alpha,1]} u(t) \geq \beta\|u\|\right\},$$

then it is easy to see that K is a cone in E. Now, we define an operator T on K by

$$\begin{aligned}
(Tu)(t) = & \int_0^1 G_1(t,s)f(s,u(s),u'(s))ds + \frac{t^2}{2(1-\mu)}\int_0^1\int_0^1 G_2(\tau,s) \\
& \times \left[\left(1-\int_0^1 g_1(r)dr\right)g_2(\tau) - \left(1-\int_0^1 g_2(r)dr\right)g_1(\tau)\right] d\tau f(s,u(s),u'(s))ds
\end{aligned}$$

$$+\frac{t}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)$$
$$\times\left[\int_0^1 rg_1(r)drg_2(\tau)+\left(1-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau f(s,u(s),u'(s))ds,\ t\in[0,1].$$

Obviously, if u is a fixed point of T, then u is a monotone nonnegative solution of the BVP (1.2).

Lemma 2.5. *$T:K\to K$ is completely continuous.*

Proof. First, by Lemma 2.4, we know that $T\,(K)\subset K$.

Next, we assume that $D\subset K$ is a bounded set. Then there exists a constant $M_1>0$ such that $\|u\|\le M_1$ for any $u\in D$. Now, we will prove that $T(D)$ is relatively compact in K. Suppose that $\{y_k\}_{k=1}^{\infty}\subset T(D)$. Then there exist $\{x_k\}_{k=1}^{\infty}\subset D$ such that $Tx_k=y_k$. Let

$$M_2=\sup\left\{f(t,x,y):(t,x,y)\in[0,1]\times[0,M_1]\times[0,M_1]\right\},$$

$$\begin{aligned}M_3=&\frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)\left[\left(1-\int_0^1 g_1(r)dr+2\int_0^1 rg_1(r)dr\right)g_2(\tau)\right.\\&\left.+\left(1+\int_0^1 g_2(r)dr-2\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau ds\end{aligned}$$

and

$$\begin{aligned}M_4=&\frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)\left[\left(1-\int_0^1 g_1(r)dr+\int_0^1 rg_1(r)dr\right)g_2(\tau)\right.\\&\left.+\left(\int_0^1 g_2(r)dr-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau ds.\end{aligned}$$

Then for any k, by Lemma 2.2, we have

$$\begin{aligned}|y_k(t)|&=|(Tx_k)(t)|\\&=\left|\int_0^1 G_1(t,s)f(s,x_k(s),x_k'(s))ds+\frac{t^2}{2(1-\mu)}\int_0^1\int_0^1 G_2(\tau,s)\right.\\&\times\left[\left(1-\int_0^1 g_1(r)dr\right)g_2(\tau)-\left(1-\int_0^1 g_2(r)dr\right)g_1(\tau)\right]d\tau f(s,x_k(s),x_k'(s))ds\\&\quad+\frac{t}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)\\&\left.\times\left[\int_0^1 rg_1(r)drg_2(\tau)+\left(1-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau f(s,x_k(s),x_k'(s))ds\right|\\&\le\frac{M_2}{2}\left(\frac{1}{6}+M_3\right),\ t\in[0,1],\end{aligned}$$

which implies that $\{y_k\}_{k=1}^{\infty}$ is uniformly bounded. At the same time, for any k, in view of Lemma 2.3, we have

$$\begin{aligned}
|y_k'(t)| &= |(Tx_k)'(t)| \\
&= \bigg| \int_0^1 G_2(t,s) f(s, x_k(s), x_k'(s)) ds + \frac{t}{1-\mu} \int_0^1 \int_0^1 G_2(\tau, s) \\
&\times \left[\left(1 - \int_0^1 g_1(r) dr \right) g_2(\tau) - \left(1 - \int_0^1 g_2(r) dr \right) g_1(\tau) \right] d\tau f(s, x_k(s), x_k'(s)) ds \\
&\quad + \frac{1}{1-\mu} \int_0^1 \int_0^1 G_2(\tau, s) \\
&\times \left[\int_0^1 r g_1(r) dr g_2(\tau) + \left(1 - \int_0^1 r g_2(r) dr \right) g_1(\tau) \right] d\tau f(s, x_k(s), x_k'(s)) ds \bigg| \\
&\le M_2 \left(\frac{1}{6} + M_4 \right), \quad t \in [0,1],
\end{aligned}$$

which shows that $\{y_k'\}_{k=1}^{\infty}$ is also uniformly bounded. This indicates that $\{y_k\}_{k=1}^{\infty}$ is equicontinuous. It follows from Arzela-Ascoli theorem that $\{y_k\}_{k=1}^{\infty}$ has a convergent subsequence in $C[0,1]$. Without loss of generality, we may assume that $\{y_k\}_{k=1}^{\infty}$ converges in $C[0,1]$. By the uniform continuity of $G_2(t,s)$, we know that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta_1$, we have

$$|G_2(t_1, s) - G_2(t_2, s)| < \frac{\varepsilon}{2(M_2+1)}, \quad s \in [0,1].$$

Let

$$\begin{aligned}
M_5 = &\frac{1}{1-\mu} \int_0^1 \int_0^1 G_2(\tau, s) \\
&\times \left[\left(1 - \int_0^1 g_1(r) dr \right) g_2(\tau) - \left(1 - \int_0^1 g_2(r) dr \right) g_1(\tau) \right] d\tau ds
\end{aligned}$$

and $\delta = \min\left\{\delta_1, \frac{\varepsilon}{2(M_2 M_5 + 1)}\right\}$. Then for any k and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}
&|y_k'(t_1) - y_k'(t_2)| = |(Tx_k)'(t_1) - (Tx_k)'(t_2)| \\
&\le \int_0^1 |(G_2(t_1, s) - G_2(t_2, s)| f(s, x_k(s), x_k'(s)) ds \\
&\quad + \frac{|t_1 - t_2|}{1-\mu} \int_0^1 \int_0^1 G_2(\tau, s) \\
&\times \left[\left(1 - \int_0^1 g_1(r) dr \right) g_2(\tau) - \left(1 - \int_0^1 g_2(r) dr \right) g_1(\tau) \right] d\tau f(s, x_k(s), x_k'(s)) ds \\
&\le M_2 \int_0^1 |G_2(t_1, s) - G_2(t_2, s)| ds + M_2 M_5 |t_1 - t_2| \\
&< \varepsilon,
\end{aligned}$$

which implies that $\{y_k'\}_{k=1}^{\infty}$ is equicontinuous. Again, by Arzela-Ascoli theorem, we know that $\{y_k'\}_{k=1}^{\infty}$ has a convergent subsequence in $C[0,1]$. Therefore, $\{y_k\}_{k=1}^{\infty}$ has a convergent subsequence in $C^1[0,1]$. Thus, we have shown that T is a compact operator.

Finally, we prove that T is continuous. Suppose that $u_m, u \in K$ and $\|u_m - u\| \to 0$ $(m \to \infty)$. Then there exists $M_6 > 0$ such that for any m, $\|u_m\| \le M_6$. Let

$$M_7 = \sup\{f(t,x,y) : (t,x,y) \in [0,1] \times [0,M_6] \times [0,M_6]\}.$$

Then for any m and $t \in [0,1]$, in view of Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
&\left\{G_1(t,s) + \frac{t^2}{2(1-\mu)} \int_0^1 G_2(\tau,s)\right. \\
&\times \left[\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau)\right] d\tau + \frac{t}{1-\mu} \int_0^1 G_2(\tau,s) \\
&\times \left.\left[\int_0^1 rg_1(r)dr g_2(\tau) + \left(1 - \int_0^1 rg_2(r)dr\right) g_1(\tau)\right] d\tau\right\} f(s, u_m(s), u_m'(s)) \\
&\le \frac{M_7}{2}\left\{1 + \frac{1}{1-\mu}\int_0^1 \left[\left(1 - \int_0^1 g_1(r)dr + 2\int_0^1 rg_1(r)dr\right) g_2(\tau)\right.\right. \\
&\quad + \left.\left.\left(1 + \int_0^1 g_2(r)dr - 2\int_0^1 rg_2(r)dr\right) g_1(\tau)\right] d\tau\right\} (1-s)s, \ \ s \in [0,1]
\end{aligned}$$

and

$$\begin{aligned}
&\left\{G_2(t,s) + \frac{t}{1-\mu} \int_0^1 G_2(\tau,s)\right. \\
&\times \left[\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau)\right] d\tau + \frac{1}{1-\mu} \int_0^1 G_2(\tau,s) \\
&\times \left.\left[\int_0^1 rg_1(r)dr g_2(\tau) + \left(1 - \int_0^1 rg_2(r)dr\right) g_1(\tau)\right] d\tau\right\} f(s, u_m(s), u_m'(s)) \\
&\le M_7\left\{1 + \frac{1}{1-\mu}\int_0^1 \left[\left(1 - \int_0^1 g_1(r)dr + \int_0^1 rg_1(r)dr\right) g_2(\tau)\right.\right. \\
&\quad + \left.\left.\left(\int_0^1 g_2(r)dr - \int_0^1 rg_2(r)dr\right) g_1(\tau)\right] d\tau\right\} (1-s)s, \quad s \in [0,1].
\end{aligned}$$

By applying Lebesgue Dominated Convergence theorem, we get

$$\begin{aligned}
&\lim_{m\to\infty} (Tu_m)(t) = \lim_{m\to\infty} \left\{\int_0^1 G_1(t,s) f(s, u_m(s), u_m'(s)) ds + \frac{t^2}{2(1-\mu)} \int_0^1 \int_0^1 G_2(\tau,s)\right. \\
&\times \left[\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau)\right] d\tau f(s, u_m(s), u_m'(s)) ds \\
&\quad + \frac{t}{1-\mu} \int_0^1 \int_0^1 G_2(\tau,s) \\
&\times \left.\left[\int_0^1 rg_1(r)dr g_2(\tau) + \left(1 - \int_0^1 rg_2(r)dr\right) g_1(\tau)\right] d\tau f(s, u_m(s), u_m'(s)) ds\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 G_1(t,s) f(s,u(s),u'(s))ds + \frac{t^2}{2(1-\mu)} \int_0^1 \int_0^1 G_2(\tau,s) \\
&\times \left[\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau)\right] d\tau f(s,u(s),u'(s))ds \\
&\quad + \frac{t}{1-\mu} \int_0^1 \int_0^1 G_2(\tau,s) \\
&\times \left[\int_0^1 r g_1(r) dr g_2(\tau) + \left(1 - \int_0^1 r g_2(r)dr\right) g_1(\tau)\right] d\tau f(s,u(s),u'(s))ds \\
&= (Tu)(t), \ \ t \in [0,1]
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{m\to\infty} (Tu_m)'(t) = \lim_{m\to\infty} \left\{ \int_0^1 G_2(t,s) f(s,u_m(s),u_m'(s))ds + \frac{t}{1-\mu} \int_0^1 \int_0^1 G_2(\tau,s) \right.\\
&\times \left[\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau)\right] d\tau f(s,u_m(s),u_m'(s))ds \\
&\quad + \frac{1}{1-\mu} \int_0^1 \int_0^1 G_2(\tau,s) \\
&\left.\times \left[\int_0^1 r g_1(r) dr g_2(\tau) + \left(1 - \int_0^1 r g_2(r)dr\right) g_1(\tau)\right] d\tau f(s,u_m(s),u_m'(s))ds \right\} \\
&= \int_0^1 G_2(t,s) f(s,u(s),u'(s))ds + \frac{t}{1-\mu} \int_0^1 \int_0^1 G_2(\tau,s) \\
&\times \left[\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau)\right] d\tau f(s,u(s),u'(s))ds \\
&\quad + \frac{1}{1-\mu} \int_0^1 \int_0^1 G_2(\tau,s) \\
&\times \left[\int_0^1 r g_1(r) dr g_2(\tau) + \left(1 - \int_0^1 r g_2(r)dr\right) g_1(\tau)\right] d\tau f(s,u(s),u'(s))ds \\
&= (Tu)'(t), \ \ t \in [0,1],
\end{aligned}
$$

which shows that T is continuous. Therefore, $T : K \to K$ is completely continuous. □

3. MAIN RESULTS

Define

$$
f^0 = \limsup_{x+y\to 0^+} \max_{t\in[0,1]} \frac{f(t,x,y)}{x+y}, \quad f_0 = \liminf_{x+y\to 0^+} \min_{t\in[\alpha,1]} \frac{f(t,x,y)}{x+y},
$$

$$
f^\infty = \limsup_{x+y\to +\infty} \max_{t\in[0,1]} \frac{f(t,x,y)}{x+y}, \quad f_\infty = \liminf_{x+y\to +\infty} \min_{t\in[\alpha,1]} \frac{f(t,x,y)}{x+y}.
$$

Theorem 3.1. *The BVP* (1.2) *has at least one monotone positive solution in the case* *(i)* $f^0 = 0$ *and* $f_\infty = +\infty$ *(superlinear);or*

(ii) $f_0 = +\infty$ *and* $f^\infty = 0$ *(sublinear).*

Proof. First, we consider the superlinear case: $f^0 = 0$ and $f_\infty = +\infty$.

Now, since $f^0 = 0$, we may choose $\rho_1 > 0$ so that

$$f(t,x,y) \leq \varepsilon_1 (x+y), \quad t \in [0,1], \quad (x+y) \in [0,\rho_1], \tag{3.1}$$

where $\varepsilon_1 > 0$ satisfies

$$2\varepsilon_1 \left(\frac{1}{6} + M_4\right) \leq 1. \tag{3.2}$$

Let $\Omega_1 = \left\{u \in E : \|u\| < \frac{\rho_1}{2}\right\}$. Then for any $u \in K \cap \partial\Omega_1$, in view of (3.1) and (3.2), we have

$$\begin{aligned}
(Tu)'(t) = & \int_0^1 G_2(t,s) f(s,u(s),u'(s))ds + \frac{t}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\
& \times \left[\left(1 - \int_0^1 g_1(r)dr\right) g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right) g_1(\tau)\right] d\tau \\
& \times f(s,u(s),u'(s))ds + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\
& \times \left[\int_0^1 r g_1(r)dr g_2(\tau) + \left(1 - \int_0^1 r g_2(r)dr\right) g_1(\tau)\right] d\tau f(s,u(s),u'(s))ds \\
& \leq \varepsilon_1 \int_0^1 (1-s)s\left(u(s)+u'(s)\right)ds \\
& \quad + \frac{\varepsilon_1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)\left[\left(1 - \int_0^1 g_1(r)dr + \int_0^1 r g_1(r)dr\right) g_2(\tau)\right. \\
& \quad \left. + \left(\int_0^1 g_2(r)dr - \int_0^1 r g_2(r)dr\right) g_1(\tau)\right] d\tau \left(u(s)+u'(s)\right)ds \\
& \leq 2\varepsilon_1\left(\frac{1}{6} + M_4\right)\|u\| \\
& \leq \|u\|, \quad t \in [0,1]. \qquad (3.3)
\end{aligned}$$

By integrating the above inequality on $[0,t]$, we get

$$(Tu)(t) \leq \|u\|, \quad t \in [0,1],$$

which together with (3.3) implies that

$$\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_1. \tag{3.4}$$

On the other hand, since $f_\infty = +\infty$, there exists $\rho_2 > \rho_1$ such that

$$f(t,x,y) \geq \varepsilon_2 (x+y), \quad t \in [\alpha,1], \quad (x+y) \in [\rho_2,+\infty), \tag{3.5}$$

where $\varepsilon_2 > 0$ satisfies

$$\begin{aligned}
&\frac{\varepsilon_2\beta}{2}\int_\alpha^1 (1-s)sds \\
&\quad + \frac{\varepsilon_2\beta}{2(1-\mu)}\int_\alpha^1\int_0^1 G_2(\tau,s)\left[\left(1-\int_0^1 g_1(r)dr + 2\int_0^1 rg_1(r)dr\right)g_2(\tau)\right. \\
&\quad \left. + \left(1+\int_0^1 g_2(r)dr - 2\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau ds \geq 1. \qquad (3.6)
\end{aligned}$$

Let $\Omega_2 = \left\{u \in E : \|u\| < \frac{\rho_2}{\beta}\right\}$. Then for any $u \in K \cap \partial\Omega_2$, in view of (3.5) and (3.6), we have

$$\begin{aligned}
(Tu)(1) &= \int_0^1 G_1(1,s)f(s,u(s),u'(s))ds + \frac{1}{2(1-\mu)}\int_0^1\int_0^1 G_2(\tau,s) \\
&\quad \times \left[\left(1-\int_0^1 g_1(r)dr\right)g_2(\tau) - \left(1-\int_0^1 g_2(r)dr\right)g_1(\tau)\right]d\tau f(s,u(s),u'(s))ds \\
&\quad + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\
&\quad \times \left[\int_0^1 rg_1(r)drg_2(\tau) + \left(1-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau f(s,u(s),u'(s))ds \\
&\geq \int_\alpha^1 G_1(1,s)f(s,u(s),u'(s))ds + \frac{1}{2(1-\mu)}\int_\alpha^1\int_0^1 G_2(\tau,s) \\
&\quad \times \left[\left(1-\int_0^1 g_1(r)dr\right)g_2(\tau) - \left(1-\int_0^1 g_2(r)dr\right)g_1(\tau)\right]d\tau f(s,u(s),u'(s))ds \\
&\quad + \frac{1}{1-\mu}\int_\alpha^1\int_0^1 G_2(\tau,s) \\
&\quad \times \left[\int_0^1 rg_1(r)drg_2(\tau) + \left(1-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau f(s,u(s),u'(s))ds \\
&\geq \frac{\varepsilon_2}{2}\int_\alpha^1 (1-s)s\left(u(s)+u'(s)\right)ds \\
&\quad + \frac{\varepsilon_2}{2(1-\mu)}\int_\alpha^1\int_0^1 G_2(\tau,s)\left[\left(1-\int_0^1 g_1(r)dr + 2\int_0^1 rg_1(r)dr\right)g_2(\tau)\right. \\
&\quad \left. + \left(1+\int_0^1 g_2(r)dr - 2\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau\left(u(s)+u'(s)\right)ds \\
&\geq \frac{\varepsilon_2\beta}{2}\int_\alpha^1 (1-s)sds\,\|u\| \\
&\quad + \frac{\varepsilon_2\beta}{2(1-\mu)}\int_\alpha^1\int_0^1 G_2(\tau,s)\left[\left(1-\int_0^1 g_1(r)dr + 2\int_0^1 rg_1(r)dr\right)g_2(\tau)\right. \\
&\quad \left. + \left(1+\int_0^1 g_2(r)dr - 2\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau ds\,\|u\| \\
&\geq \|u\|,
\end{aligned}$$

which implies that

$$\|Tu\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2. \tag{3.7}$$

Therefore, it follows from (3.4), (3.7) and Theorem 1.1 that the operator T has one fixed point $u \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which is a monotone positive solution of the BVP (1.2).

Next, we consider the sublinear case: $f_0 = +\infty$ and $f^\infty = 0$.

Since $f_0 = +\infty$, we may choose $\rho_3 > 0$ so that

$$f(t, x, y) \geq \varepsilon_3(x + y), \quad t \in [\alpha, 1], \quad (x + y) \in [0, \rho_3], \tag{3.8}$$

where $\varepsilon_3 > 0$ satisfies

$$\begin{aligned} &\frac{\varepsilon_3\beta}{2}\int_\alpha^1 (1-s)s\,ds \\ &\quad + \frac{\varepsilon_3\beta}{2(1-\mu)}\int_\alpha^1\int_0^1 G_2(\tau, s)\left[\left(1 - \int_0^1 g_1(r)dr + 2\int_0^1 rg_1(r)dr\right)g_2(\tau)\right. \\ &\quad \left. + \left(1 + \int_0^1 g_2(r)dr - 2\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau ds \geq 1. \end{aligned} \tag{3.9}$$

Let $\Omega_3 = \left\{u \in E : \|u\| < \frac{\rho_3}{2}\right\}$. Then for any $u \in K \cap \partial\Omega_3$, in view of (3.8) and (3.9), we have

$$\begin{aligned} (Tu)(1) &= \int_0^1 G_1(1, s)f(s, u(s), u'(s))ds + \frac{1}{2(1-\mu)}\int_0^1\int_0^1 G_2(\tau, s) \\ &\quad \times \left[\left(1 - \int_0^1 g_1(r)dr\right)g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right)g_1(\tau)\right]d\tau f(s, u(s), u'(s))ds \\ &\quad + \frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau, s) \\ &\quad \times \left[\int_0^1 rg_1(r)drg_2(\tau) + \left(1 - \int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau f(s, u(s), u'(s))ds \\ &\geq \int_\alpha^1 G_1(1, s)f(s, u(s), u'(s))ds + \frac{1}{2(1-\mu)}\int_\alpha^1\int_0^1 G_2(\tau, s) \\ &\quad \times \left[\left(1 - \int_0^1 g_1(r)dr\right)g_2(\tau) - \left(1 - \int_0^1 g_2(r)dr\right)g_1(\tau)\right]d\tau f(s, u(s), u'(s))ds \\ &\quad + \frac{1}{1-\mu}\int_\alpha^1\int_0^1 G_2(\tau, s) \\ &\quad \times \left[\int_0^1 rg_1(r)drg_2(\tau) + \left(1 - \int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau f(s, u(s), u'(s))ds \\ &\geq \frac{\varepsilon_3}{2}\int_\alpha^1 (1-s)s\left(u(s) + u'(s)\right)ds \\ &\quad + \frac{\varepsilon_3}{2(1-\mu)}\int_\alpha^1\int_0^1 G_2(\tau, s)\left[\left(1 - \int_0^1 g_1(r)dr + 2\int_0^1 rg_1(r)dr\right)g_2(\tau)\right. \\ &\quad \left. + \left(1 + \int_0^1 g_2(r)dr - 2\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau\left(u(s) + u'(s)\right)ds \end{aligned}$$

$$\begin{aligned}
&\geq \frac{\varepsilon_3\beta}{2}\int_\alpha^1 (1-s)sds\,\|u\| \\
&\quad+\frac{\varepsilon_3\beta}{2(1-\mu)}\int_\alpha^1\int_0^1 G_2(\tau,s)\left[\left(1-\int_0^1 g_1(r)dr+2\int_0^1 rg_1(r)dr\right)g_2(\tau)\right. \\
&\quad\left.+\left(1+\int_0^1 g_2(r)dr-2\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau ds\,\|u\| \\
&\geq \|u\|,
\end{aligned}$$

which implies that

$$\|Tu\| \geq \|u\|, \quad u\in K\cap\partial\Omega_3. \tag{3.10}$$

Now, since $f^\infty=0$, there exists $\widehat{\rho_4}>0$ such that

$$f(t,x,y)\leq \varepsilon_4(x+y), \quad t\in[0,1], \quad (x+y)\in[\widehat{\rho_4},+\infty), \tag{3.11}$$

where $\varepsilon_4>0$ satisfies

$$4\varepsilon_4\left(\frac{1}{6}+M_4\right)\leq 1. \tag{3.12}$$

Let $M^*=\max\{f(t,x,y):(t,x,y)\in[0,1]\times[0,\widehat{\rho_4}]\times[0,\widehat{\rho_4}]\}$. Then by (3.11), we have

$$f(t,x,y)\leq\varepsilon_4(x+y)+M^*, \quad (t,x,y)\in[0,1]\times[0,+\infty)\times[0,+\infty). \tag{3.13}$$

Choose

$$\rho_4\geq\max\left\{\rho_3,\frac{M^*}{2\varepsilon_4}\right\}. \tag{3.14}$$

Let $\Omega_4=\{u\in E:\|u\|<\rho_4\}$. Then for any $u\in K\cap\partial\Omega_4$, in view of (3.12), (3.13) and (3.14), we have

$$\begin{aligned}
(Tu)'(t)&=\int_0^1 G_2(t,s)f(s,u(s),u'(s))ds+\frac{t}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\
&\quad\times\left[\left(1-\int_0^1 g_1(r)dr\right)g_2(\tau)-\left(1-\int_0^1 g_2(r)dr\right)g_1(\tau)\right]d\tau \\
&\quad\times f(s,u(s),u'(s))ds+\frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s) \\
&\quad\times\left[\int_0^1 rg_1(r)drg_2(\tau)+\left(1-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau f(s,u(s),u'(s))ds \\
&\leq\int_0^1(1-s)s\left[\varepsilon_4(u(s)+u'(s))+M^*\right]ds \\
&\quad+\frac{1}{1-\mu}\int_0^1\int_0^1 G_2(\tau,s)\left[\left(1-\int_0^1 g_1(r)dr+\int_0^1 rg_1(r)dr\right)g_2(\tau)\right. \\
&\quad\left.+\left(\int_0^1 g_2(r)dr-\int_0^1 rg_2(r)dr\right)g_1(\tau)\right]d\tau\left[\varepsilon_4(u(s)+u'(s))+M^*\right]ds
\end{aligned}$$

$$
\begin{aligned}
&\leq (2\varepsilon_4 \|u\| + M^*) \left\{ \int_0^1 (1-s)sds \right. \\
&\quad + \frac{1}{1-\mu} \int_0^1 \int_0^1 G_2(\tau, s) \left[\left(1 - \int_0^1 g_1(r)dr + \int_0^1 rg_1(r)dr \right) g_2(\tau) \right. \\
&\quad \left. \left. + \left(\int_0^1 g_2(r)dr - \int_0^1 rg_2(r)dr \right) g_1(\tau) \right] d\tau ds \right\} \\
&= (2\varepsilon_4 \|u\| + M^*) \left(\frac{1}{6} + M_4 \right) \\
&\leq \|u\|, \ t \in [0,1]. \qquad (3.15)
\end{aligned}
$$

By integrating the above inequality on $[0, t]$, we get

$$(Tu)(t) \leq \|u\|, \quad t \in [0,1],$$

which together with (3.15) implies that

$$\|Tu\| \leq \|u\|, \quad u \in K \cap \partial\Omega_4. \qquad (3.16)$$

Therefore, it follows from (3.10), (3.16) and Theorem 1.1 that the operator T has one fixed point $u \in K \cap (\overline{\Omega}_4 \backslash \Omega_3)$, which is a monotone positive solution of the BVP (1.2). □

REFERENCES

[1] D. R. Anderson, Green's function for a third-order generalized right focal problem, *J. Math. Anal. Appl.*, 288:1–14, 2003.

[2] D. R. Anderson, C. C. Tisdell, Third-order nonlocal problems with sign-changing nonlinearity on time scales, *Electron. J. Differential Equations*, 19:1–12, 2007.

[3] A. Boucherif, S. M. Bouguima, N. Al-Malki, Z. Benbouziane, Third order differential equations with integral boundary conditions, *Nonlinear Anal.*, 71:e1736–e1743, 2009.

[4] Z. Du, W. Ge, X. Lin, Existence of solutions for a class of third-order nonlinear boundary value problems, *J. Math. Anal. Appl.*, 294:104–112, 2004.

[5] L. H. Erbe, H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.*, 120:743–748, 1994.

[6] Y. Feng, Solution and positive solution of a semilinear third-order equation, *J. Appl. Math. Comput.*, 29:153–161, 2009.

[7] Y. Feng, S. Liu, Solvability of a third-order two-point boundary value problem, *Appl. Math. Lett.*, 18:1034–1040, 2005.

[8] M. Gregus, *Third Order Linear Differential Equations*, Reidel, Dordrecht, 1987.

[9] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[10] L. -J. Guo, J. -P. Sun, Y. -H. Zhao, Existence of positive solution for nonlinear third-order three-point boundary value problem, *Nonlinear Anal.*, 68:3151–3158, 2008.

[11] J. R. Graef, L. Kong, Positive solutions for third order semipositone boundary value problems, *Appl. Math. Lett.*, 22:1154–1160, 2009.

[12] J. R. Graef, J. R. L. Webb, Third order boundary value problems with nonlocal boundary conditions, *Nonlinear Anal.*, 71:1542–1551, 2009.

[13] J. R. Graef, B. Yang, Positive solutions of a third order nonlocal boundary value problem, *Discrete Contin. Dyn. Syst. Ser. S*, 1:89–97, 2008.

[14] J. Henderson, C. C. Tisdell, Five-point boundary-value problems for third-order differential equations by solution matching, *Math. Comput. Modelling*, 42:133–137, 2005.

[15] B. Hopkins, N. Kosmatov, Third-order boundary value problems with sign-changing solutions, *Nonlinear Anal.*, 67:126–137, 2007.

[16] S. Li, Positive solutions of nonlinear singular third-order two-point boundary value problem, *J. Math. Anal. Appl.*, 323:413–425, 2006.

[17] Z. Liu, L. Debnath, S. M. Kang, Existence of monotone positive solutions to a third-order two-point generalized right focal boundary value problem, *Comput. Math. Appl.*, 55:356–367, 2008.

[18] Z. Liu, J. S. Ume, S. M. Kang, Positive solutions of a singular nonlinear third order two-point boundary value problem, *J. Math. Anal. Appl.*, 326:589–601, 2007.

[19] R. Ma, Multiplicity results for a third order boundary value problem at resonance, *Nonlinear Anal.*, 32:493–499, 1998.

[20] J. -P. Sun, H. -B. Li, Monotone positive solution of nonlinear third-order BVP with integral boundary conditions, *Bound. Value Probl.*, 2010:1–11, 2010.

[21] J. -P. Sun, Q. -Y. Ren, Y. -H. Zhao, The upper and lower solution method for nonlinear third-order three-point boundary value problem, *Electron. J. Qual. Theory Differ. Equ.*, 26:1–8, 2010.

[22] Y. Sun, Positive solutions of singular third-order three-point boundary value problem, *J. Math. Anal. Appl.*, 306:589–603, 2005.

[23] Y. Sun, Positive solutions for third-order three-point nonhomogeneous boundary value problems, *Appl. Math. Lett.*, 22:45–51, 2009.

[24] Y. Wang, W. Ge, Existence of solutions for a third order differential equation with integral boundary conditions, *Comput. Math. Appl.*, 53:144–154, 2007.

[25] B. Yang, Positive solutions of a third-order three-point boundary-value problem, *Electron. J. Differential Equations*, 99:1–10, 2008.

[26] Q. Yao, Positive solutions of singular third-order three-point boundary value problems, *J. Math. Anal. Appl.*, 354:207–212, 2009.

[27] Q. Yao, Successive iteration of positive solution for a discontinuous third-order boundary value problem, *Comput. Math. Appl.*, 53:741–749, 2007.

[28] Q. Yao, Y. Feng, The existence of solution for a third-order two-point boundary value problem, *Appl. Math. Lett.*, 15:227–232, 2002.