Dynamic Systems and Applications 23 (2014) 235-244

PERIODIC AND ASYMPTOTICALLY PERIODIC SOLUTIONS IN COUPLED NONLINEAR SYSTEMS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

MUHAMMAD N. ISLAM AND YOUSSEF N. RAFFOUL

Department of Mathematics, University of Dayton, Dayton, OH 45469-2316 USA

Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. The existence of periodic and asymptotically period solutions of a system of coupled nonlinear Volterra integro-differential equations with infinite delay has been studied in this paper. The fixed point theorem of Schauder has been employed as the primary mathematical tool of analysis.

AMS (MOS) Subject Classification. 34K20, 45J05, 45D05.

1. **Introduction**

Coupled differential or integrodifferential equations have been used in many areas of biological and environmental sciences. Lotka-Volterra models for competitive species are probably the most well-known examples of such coupled equations (cf. [3], [7], [9]). A particular case where the Lotka-Volterra model has successfully been used is the famous predator-prey problem for two competing species. Two interlocked or coupled equations are required to model such a problem. Coupled equations are also used in other fields to study various qualitative properties of solutions (cf. [4], [5], [11], [13], [14]). In [4], and [5] the authors studied the existence of asymptotically periodic solutions of linear systems of Volterra difference equations, and in [11], [13], and [14], the authors studied oscillation properties and asymptotic or limiting properties of solutions.

In this paper we study the existence of periodic and asymptotically period solutions of the following coupled nonlinear Volterra integro-differential equations with infinite delay

$$(1) \qquad \begin{cases} x'(t) = h(t)x(t) + \int_{-\infty}^{t} a(t,s)f(y(s))ds, \\ y'(t) = p(t)y(t) + \int_{-\infty}^{t} b(t,s)g(x(s))ds, \end{cases}$$

where the functions a, b, f, g, h, and p are assumed to be continuous in their arguments.

Received July 2013

For the readers interested in periodic, asymptotically periodic, and almost periodic solutions of Volterra equations, we refer to the partial list [1], [2], [3], [6], [8], [10], [12], and the references therein.

Although some of the studies mentioned above deal with the periodicity on systems of Volterra integral equations with infinite delay, our results are different with respect to assumptions and methods. For asymptotic periodicity we can hardly find any study on equation like the one we considered in this paper. In [3], the author considered a forced asymptotic periodicity on a predator-prey system, in which the equations, assumptions, and the methods are very different from ours.

In the analysis of our paper, we invert both equations in (1), transform them into integral equations, and then use Schauder's fixed point theorem. We show the existence of periodic solutions in Section 2, and the existence of asymptotic periodic solutions in Section 3.

We assume that there exists a least positive real number T, such that

$$\begin{gathered} a(t+T, s+T) = a(t,s), \quad b(t+T, s+T) = b(t,s) \\ p(t+T) = p(t) \quad \text{and} \quad h(t+T) = h(t), \end{gathered} \tag{2}$$

for all $t \in \mathbb{R}$.

To have well defined mappings we assume that

$$\int_0^T h(s)ds \neq 0, \quad \int_0^T p(s)ds \neq 0. \tag{3}$$

Define $P_T = \{(\varphi, \psi) : (\varphi, \psi)(t+T) = (\varphi, \psi)(t)\}$ where, both ϕ and ψ are real valued continuous functions on $\mathbb{R}$. Then P_T is a Banach space endowed with the maximum norm

$$\|(x,y)\| = \max\left\{ \max_{t\in[0,T]} |x(t)|, \max_{t\in[0,T]} |y(t)| \right\}.$$

Lemma 1.1. *Assume* (2) *and* (3). *Then a function* $(x(t), y(t))$ *is a* T*-periodic solution of equation* (1) *if and only if*

$$x(t) = \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \int_{-\infty}^u a(u,s) f(y(s))\, dsdu, \tag{4}$$

and

$$y(t) = \int_t^{t+T} \frac{e^{\int_u^{t+T} p(s)ds}}{1 - e^{\int_0^T p(s)ds}} \int_{-\infty}^u b(u,s) g(x(s))\, dsdu. \tag{5}$$

Proof. It is easy to verify that $(x(t), y(t))$ given by (4) and (5) are T-periodic. Let $(x(t), y(t))$ be a solution of (1). Multiply both sides of the first equation in (1) with

$e^{-\int_0^t h(s)ds}$, and then integrate from t to $t+T$ to obtain

$$x(t+T)e^{-\int_0^{t+T} h(s)ds} - x(t)e^{-\int_0^t h(s)ds} = \int_t^{t+T} \int_{-\infty}^u a(u,s)f(y(s))ds e^{-\int_0^u h(s)ds} du.$$

Now multiply both sides by $e^{\int_0^{t+T} h(s)ds}$ and then use the fact that $x(t+T) = x(t)$ and $e^{-\int_t^{t+T} h(s)ds} = e^{-\int_0^T h(s)ds}$, to arrive at Eq. (4). One can reverse these steps to obtain the first equation of (1) from Eq. (4). Therefore these two equations are equivalent. Similarly, the second equation of (1) and Eq. (5) are equivalent. This completes the proof. □

2. **Periodic Solutions**

Theorem 2.1 (Schauder's Fixed Point Theorem). *Let X be a Banach space, and K be a closed, bounded and convex subset of X. If $T : K \to K$ is completely continuous then T has a fixed point in K.*

A map is completely continuous if it is continuous and it maps bounded sets into relatively compact sets.

We assume there exist positive constants M_1, M_2, K_1 and K_2 such that

$$|f(y)| \le M_1, \tag{6}$$

$$|g(x)| \le M_2, \tag{7}$$

$$\int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \right| \int_{-\infty}^u |a(u,s)| ds\, du \le K_1, \tag{8}$$

and

$$\int_t^{t+T} \left| \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \right| \int_{-\infty}^u |b(u,s)| ds\, du \le K_2. \tag{9}$$

Let

$$M = \max\{M_1K_1, M_2K_2\}.$$

We define a subset $\Omega_{x,y}$ of P_T as follows: $\Omega_{x,y} = \{(x,y) : (x,y) \in P_T$ with $\|(x,y)\| \le M\}$. Then Ω_{xy} is bounded, closed and convex subset of P_T. Now, for $(x,y) \in \Omega_{xy}$ we can define an operator $E : \Omega_{xy} \to P_T$ by

$$E\,(x,y)\,(t) = \left(E_1(y)(t), E_2(x)(t)\right),$$

where

$$E_1(y)(t) = \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \int_{-\infty}^u a(u,s)f(y(s))ds\, du,$$

and

$$E_2(x)(t) = \int_t^{t+T} \frac{e^{\int_u^{t+T} p(s)ds}}{1 - e^{\int_0^T p(s)ds}} \int_{-\infty}^u b(u,s)g(x(s))ds\,du.$$

Theorem 2.2. *Suppose* (2), (3), (6), (7), (8) *and* (9) *hold. Then* (1) *has a T-periodic solution.*

Proof. It is clear from Lemma 1.1 that $E_1(y)(t+T) = E_1(y)(t)$ and $E_2(x)(t+T) = E_2(x)(t)$. Therefore, $E(x,y)(t+T) = E(x,y)(t)$. Moreover, if $(x,y) \in \Omega_{xy}$, then

$$\big|E_1(y)(t)\big| \le \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}}\Big| \int_{-\infty}^u |a(u,s)||f(y(s))|ds\,du \le M_1K_1.$$

Similarly,

$$\big|E_2(x)(t)\big| \le M_2K_2.$$

Thus, E maps Ω_{xy} into itself, i.e., $E(\Omega_{xy}) \subseteq \Omega_{xy}$. Now, we have to show that E is continuous. Let $\big\{(x^l, y^l)\big\}$ be a sequence in $\Omega_{x,y}$ such that,

$$\lim_{l\to\infty} \|(x^l, y^l) - (x,y)\| = 0.$$

Since $\Omega_{x,y}$ is closed, we have $(x,y) \in \Omega_{x,y}$. Then by the definition of E we have

$$\begin{aligned}\big\|E(x^l,y^l) - E(x,y)\big\| = \max\Big\{&\max_{t\in[0,T]} \big|E_1(y^l)(t) - E_1(y)(t)\big|\,, \\ &\max_{t\in[0,T]} \big|E_2(x^l)(t) - (E_2(x)(t)\big|\Big\},\end{aligned}$$

in which

$$\begin{aligned}&\big|E_1(y^l)(t) - E_1(y)(t)\big| \\ &= \Big|\int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \int_{-\infty}^u a(u,s)f(y^l(s))ds\,du \\ &\quad - \int_t^{t+T} \frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}} \int_{-\infty}^u a(u,s)f(y(s))ds\,du\Big| \\ &\le \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h(s)ds}}{1 - e^{\int_0^T h(s)ds}}\Big| \left(\int_{-\infty}^u |a(u,s)||f(y^l(s)) - f(y(s))|ds\right) du\end{aligned}$$

and

$$\begin{aligned}&\big|E_2(x^l)(t) - E_2(x)(t)\big| \\ &= \Big|\int_t^{t+T} \frac{e^{\int_u^{t+T} p(s)ds}}{1 - e^{\int_0^T p(s)ds}} \int_{-\infty}^u b(u,s)g(x^l(s))ds\,du \\ &\quad - \int_t^{t+T} \frac{e^{\int_u^{t+T} p(s)ds}}{1 - e^{\int_0^T p(s)ds}} \int_{-\infty}^u b(u,s)g(x(s))ds\,du\Big| \\ &\le \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} p(s)ds}}{1 - e^{\int_0^T p(s)ds}}\Big| \left(\int_{-\infty}^u |b(u,s)||g(x^l(s)) - g(x(s))|ds\right) du.\end{aligned}$$

The continuity of f and g along with the Lebesgue dominated convergence theorem implies that

$$\lim_{l\to\infty} \max_{t\in[0,T]} |E_1(y^l)(t) - E_1(y)(t)| = 0,$$

and

$$\lim_{l\to\infty} \max_{t\in[0,T]} |E_2(x^l)(t) - E_2(x)(t)| = 0.$$

Thus,

$$\lim_{l\to\infty} \|E(x^l, y^l) - E(x,y)\| = 0.$$

This shows that E is a continuous map. To show that the map E is completely continuous, we will show that $E\Omega_{x,y}$ is relatively compact. We already know that $E(\Omega_{xy}) \subseteq \Omega_{xy}$, which means $E(\Omega_{xy})$ is uniformly bounded because Ω_{xy} is uniformly bounded. It is an easy exercise to show that for all $(x,y) \in \Omega_{xy}$, there exists a constant $L > 0$ such that $|\frac{d}{dt}E_1(y)(t)| \le L$, and $|\frac{d}{dt}E_2(x)(t)| \le L$. This means $|\frac{d}{dt}E(x,y)(t)| \le L$. Therefore the set $E(\Omega_{xy})$ is equicontinuous, and hence by Arzela-Ascoli's theorem, it is relatively compact.

By Schauder's fixed point theorem, we conclude that there exist $(x,y) \in \Omega_{x,y}$ such that $(x,y) = E(x,y)$. This (x,y) is a T-periodic solution of (1). □

In the next theorem we relax condition (7).

Theorem 2.3. *Suppose* (2), (3), (6), (8) *and* (9) *hold. In addition, we ask that g be nondecreasing and*

$$|g(x)| \le g(|x|). \tag{10}$$

Then (1) *has a T-periodic solution.*

Proof. Set

$$M = \max\{M_1K_1, M_2\, g(M_1K_1)\}.$$

For $(x,y) \in \Omega_{x,y}$, we have by the previous theorem that

$$\Big|E_1(y)(t)\Big| \le M_1K_1.$$

Moreover,

$$\begin{aligned}
\big|E_2(x)(t)\big| &\le \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} p(s)ds}}{1-e^{\int_0^T p(s)ds}}\Big| \int_{-\infty}^u |b(u,s)||g(x(s))|ds\,du \\
&\le \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} p(s)ds}}{1-e^{\int_0^T p(s)ds}}\Big| \int_{-\infty}^u |b(u,s)|g(|E_1(y(s))|)|ds\,du \\
&\le M_2\, g(M_1K_1).
\end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 2.2. □

In the next theorem we relax condition (6).

Theorem 2.4. *Suppose* (2), (3), (7), (8) *and* (9) *hold. In addition, we ask that f be nondecreasing and*

$$|f(y)| \leq f(|y|). \tag{11}$$

Then (1) *has a T-periodic solution.*

Proof. Set

$$M = \max\{M_1\ f(M_2K_2), M_2K_2\}.$$

Then we have

$$\big|E_2(x)(t)\big| \leq M_2K_2,$$

For $(x,y) \in \Omega_{x,y}$.

$$\begin{aligned}\big|E_1(y)(t)\big| &\leq \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}}\Big| \int_{-\infty}^u |b(u,s)||f(y(s))|ds\,du \\ &\leq \int_t^{t+T} \Big|\frac{e^{\int_u^{t+T} h(s)ds}}{1-e^{\int_0^T h(s)ds}}\Big| \int_{-\infty}^u |a(u,s)|f(|E_2(x(s))|)|ds\,du \\ &\leq M_1 f(M_2K_2).\end{aligned}$$

The rest of the proof follows along the lines of the proof of Theorem 2.2. □

3. **Asymptotically Periodic Solutions**

Definition 3.1. A function $x(t)$ is called asymptotically T-periodic if there exist two functions $x_1(t)$ and $x_2(t)$ such that $x_1(t)$ is T-periodic, $\lim_{t\to\infty} x_2(t) = 0$ and $x(t) = x_1(t) + x_2(t)$ for all t.

In this section we do not assume the periodicity condition on the functions $a(t,s)$ and $b(t,s)$. We only assume $h(t)$ and $p(t)$ are T-periodic, and

$$\int_0^T h(s)ds = 0 \text{ and } \int_0^T p(s)ds = 0. \tag{12}$$

Since h and p are T-periodic, there are constants m_k, M_k^*, $k = 1,2$, such that

$$m_1 \leq e^{\int_0^t h(s)ds} \leq M_1^*, \text{ and } m_2 \leq e^{\int_0^t p(s)ds} \leq M_2^*. \tag{13}$$

Also, we assume that there are positive numbers A and B such that

$$\int_0^\infty \int_{-\infty}^u |a(u,s)|ds\,du \leq A, \text{ and } \int_0^\infty \int_{-\infty}^u |b(u,s)|ds\,du \leq B \tag{14}$$

In addition, we suppose that

$$\lim_{t\to\infty} \int_t^\infty \int_{-\infty}^u |a(u,s)|ds\,du = \lim_{t\to\infty} \int_t^\infty \int_{-\infty}^u |b(u,s)|ds\,du = 0. \tag{15}$$

Theorem 3.2. *Suppose that* (6), (7), (12), (13), (14), *and* (15) *hold. Then system* (1) *has an asymptotically T-periodic solution (x, y) satisfying*

$$x(t) := x_1(t) + x_2(t)$$
$$y(t) := y_1(t) + y_2(t)$$

where

$$x_1(t) = c_1 e^{\int_0^t h(s)ds}, \quad y_1(t) = c_2 e^{\int_0^t p(s)ds}, \quad t \in \mathbb{R}$$

for arbitrary constants c_1, c_2 and

$$x_2(t) = -\int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t h(l)dl}}{e^{\int_0^u h(l)dl}} a(u,s) f(y(s)) ds\, du,$$

$$y_2(t) = -\int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t p(l)dl}}{e^{\int_0^u p(l)dl}} b(u,s) g(x(s)) ds\, du.$$

Proof. Define $P_T^* = \{(\varphi, \psi) : \varphi = \varphi_1 + \varphi_2, \psi = \psi_1 + \psi_2, (\varphi_1, \psi_1)(t+T) = (\varphi_1, \psi_1)(t),$ and $(\varphi_2, \psi_2)(t) \to (0,0)$ as $t \to \infty\}$. Then P_T^* is a Banach space when endowed with the maximum norm

$$\|(x,y)\| = \max\left\{ \max_{t \in [0,T]} |x(t)|, \max_{t \in [0,T]} |y(t)| \right\}.$$

We define a subset $\Omega_{x,y}$ of P_T^* as follows. For a constant W^* to be defined later in the proof, let $\Omega_{x,y} = \{(x,y) : (x,y) \in P_T^* \text{ with } \|(x,y)\| \le W^*\}$. Then Ω_{xy} is bounded, closed and convex subset of P_T^*. Now, for $(x,y) \in \Omega_{xy}$ we can define an operator $F : \Omega_{xy} \to P_T^*$ by

$$F(x,y)(t) = (F_1(y)(t), F_2(x)(t)),$$

where

$$F_1(y)(t) = c_1 e^{\int_0^t h(s)ds} - \int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t h(l)dl}}{e^{\int_0^u h(l)dl}} a(u,s) f(y(s)) ds\, du,$$

and

$$F_2(x)(t) = c_2 e^{\int_0^t p(s)ds} - \int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t p(l)dl}}{e^{\int_0^u p(l)dl}} b(u,s) g(x(s)) ds\, du.$$

We will show that the mapping F has a fixed point in Ω_{xy}. First, we demonstrate that $F\Omega_{x,y} \subseteq \Omega_{x,y}$. If $(x,y) \in \Omega_{x,y}$, then

$$\begin{aligned} \left| F_1(y)(t) - c_1 e^{\int_0^t h(s)ds} \right| &\le M_1^* m_1^{-1} M_1 \int_t^\infty \int_{-\infty}^u |a(u,s)|\, ds\, du \\ &\le M_1^* m_1^{-1} M_1 \int_0^\infty \int_{-\infty}^u |a(u,s)|\, ds\, du \\ &= M_1^* m_1^{-1} M_1 A, \end{aligned} \tag{16}$$

and

$$\left|F_2(x)(t) - c_2 e^{\int_0^t p(s)ds}\right| \le M_2^* m_2^{-1} M_2 \int_t^\infty \int_{-\infty}^u |b(u,s)|\, ds\, du$$
$$\le M_2^* m_2^{-1} M_2 \int_0^\infty \int_{-\infty}^u |b(u,s)|\, ds\, du$$
$$= M_2^* m_2^{-1} M_2 B. \tag{17}$$

This implies that

$$|F_1(y)(t)| \le M_1^* m_1^{-1} M_1 A + c_1 M_1$$

and

$$|F_2(x)(t)| \le M_2^* m_2^{-1} M_2 B + c_2 M_2.$$

If we set

$$W^* = \max\{M_1^* m_1^{-1} M_1 A + c_1 M_1, M_2^* m_2^{-1} M_2 B + c_2 M_2\}$$

then we have $F\Omega_{x,y} \subseteq \Omega_{x,y}$ as desired.

The work to show that F is completely continuous is similar to the corresponding work in Theorem 2.2, and hence we omit it here. Therefore, by Schauder's fixed point theorem, there exists a fixed point $(x, y) \in \Omega_{xy}$ such that $F(x, y)(t) = (F_1(y)(t), F_2(x)(t)) = (x(t), y(t))$. Now we show that this fixed point is a solution of (1). Let

$$x(t) = c_1 e^{\int_0^t h(s)ds} - \int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t h(l)dl}}{e^{\int_0^u h(l)dl}} a(u,s) f(y(s))ds\, du.$$

Then a differentiation with respect to t gives

$$\begin{aligned}
x'(t) &= c_1 h(t) e^{\int_0^t h(s)ds} + \int_{-\infty}^t \frac{e^{\int_0^t h(l)dl}}{e^{\int_0^t h(l)dl}} a(t,s) f(y(s))ds \\
&\quad - h(t) \int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t h(l)dl}}{e^{\int_0^u h(l)dl}} a(u,s) f(y(s))ds\, du \\
&= h(t) \left[c_1 e^{\int_0^t h(s)ds} - \int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t h(l)dl}}{e^{\int_0^u h(l)dl}} a(u,s) f(y(s))ds\, du \right] \\
&\quad + \int_{-\infty}^t a(t,s) f(y(s))ds \\
&= h(t)x(t) + \int_{-\infty}^t a(t,s) f(y(s))ds.
\end{aligned}$$

Thus $x(t)$ satisfies the first equation of (1). In a similar fashion we can easily show that if

$$y(t) = c_2 e^{\int_0^t p(s)ds} - \int_t^\infty \int_{-\infty}^u \frac{e^{\int_0^t p(l)dl}}{e^{\int_0^u p(l)dl}} b(u,s) g(x(s))ds\, du,$$

then it is a solution to the second equation of (1).

We still need to show that x_1 and y_1 are T-periodic, and

$$\lim_{t\to\infty} x_2(t) = \lim_{t\to\infty} y_2(t) = 0.$$

From (12), one can see

$$\begin{aligned} x_1(t+T) &= c_1 e^{\int_0^{t+T} h(s)ds} = c_1 e^{\int_0^t h(s)ds + \int_t^{t+T} h(s)ds} \\ &= c_1 e^{\int_0^t h(s)ds} e^{\int_t^{t+T} h(s)ds} = c_1 e^{\int_0^t h(s)ds} = x_1(t). \end{aligned}$$

Similarly, $y_1(t)$ is T-periodic.

Finally, by (6), (13) and (15),

$$\lim_{t\to\infty} |x_2(t)| \le M_1^* m_1^{-1} M_1 \lim_{t\to\infty} \int_t^\infty \int_{-\infty}^u |a(u,s)| ds\, du = 0.$$

Hence,

$$\lim_{t\to\infty} x_2(t) = 0.$$

Similarly,

$$\lim_{t\to\infty} y_2(t) = 0.$$

This concludes the proof. □

Example 3.3. Let $h(t) = p(t) = \cos(t)$, $a(t,s) = b(t,s) = e^{-2t+s}$. Also assume that $f(y) = \sin(y)$ and $g(x) = \cos(2x)$. Then all conditions of Theorem 3.2 are satisfied and hence the system

$$\begin{cases} x'(t) = \cos(t)x(t) + \int_{-\infty}^t e^{-2t+s} \sin(y(s))ds, \\ y'(t) = \cos(t)y(t) + \int_{-\infty}^t e^{-2t+s} \cos(2x(s))ds \end{cases}$$

has asymptotically 2π- periodic solution.

REFERENCES

[1] T. Burton, P. W. Eloe and M. N. Islam, Periodic solutions of linear integrodifferential equations, *Math. Nachr.*, 147, (1990), 175–184.

[2] T. Burton and T. Furumochi, Periodic and asymptotically periodic solutions of Volterra integral equations, *Funkcialaj Ekvacioj*, 39 (1996), 87–107.

[3] J. Cushing, Forced Asymptotically Periodic Solutions of Predator-Prey Systems With or Without Hereditary Effects, *SIAM J. APPL. MATH.*, Vol. 30. (1976), 665–674.

[4] J. Diblik, E. Schmeidel and M. Ruzickova, Asymptotically periodic solutions of Volterra system of difference equations, *Computers and Mathematics with Applications*, 59, (2010), 2854–2867.

[5] J. Diblik, E. Schmeidel and M. Ruzickova, Existence of asymptotically periodic solutions of system of Volterra difference equations, *Journal of Difference Equations and Applications*, Vol. 15., (2009), 1165–1177.

[6] T. Furumochi, Asymptotically periodic solutions of Volterra difference equations, *Vietnam J. Math.*, 30 (2002), 537–550.

[7] K. Gopalsamy, Nonoscillation in time delayed Lotka-Volterra competition systems, *Oscillations, bifurcations and chaos: CMS Conf. Proc.*, 8 (Toronto, Ont., 1986), 443–446.

[8] G. Gripenberg, S. Londen and O. Staffans, *Volterra Integral and Functional Equations*, Cambridge U. Press, Cambridge, 1990.

[9] X. He and K. Gopalsamy, Dynamics of Lotka-Volterra mutualism in changing environments, *Nonlinear world I* (1994), 173–185.

[10] Y. Hino, S. Murakami and T. Yoshizawa, Existence of almost periodic solutions of some functional differential equations with infinite delay in a Banach space, *Tohoku Math. J.*, 49 (1997), pp. 133–147

[11] H. Huo and W. Li, Oscillation criteria for certain two-dimensional differential systems, *Int. J. Appl. Math.*, 6(3)(2001), 253–261.

[12] M. Islam and Y. Raffoul, Bounded solutions and periodic solutions of almost linear Volterra equations, *CUBO - A Mathematical J., Special Issue: Qualitative Properties of Functional Equations*, 11 (2009), 115–124.

[13] W. Li and S. Cheng, Limiting behavior of nonoscillatory solutions of a pair of coupled nonlinear differential equations, *Proc. Edingburgh Math. Soc.*, 43 (2000), 457–473.

[14] Y. Raffoul, Classification of positive solutions of nonlinear systems of Volterra integral equations, *Ann. Funct. Anal.*, 2 (2011), 34–41.