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EXISTENCE OF SOLUTIONS FOR NONLINEAR FRACTIONAL DIFFERENCE EQUATIONS WITH INITIAL CONDITIONS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In this paper, we use the well-known upper and lower solution method to prove the existence of solutions for a nonlinear nabla fractional difference equation with an initial condition on the finite discrete domain. We prove that the solution of the initial value problem stays between lower and upper solutions under some assumptions on the nonlinear term. We illustrate the main result of the paper with two examples where one can see the necessity of the well-order in the lower and upper solutions (i.e. a lower solution is less than or equal to an upper solution) to obtain a solution between the upper and lower solutions. As an application, we use the quasilinearization method to construct two sequences of upper and lower solutions which converge to the solution of the initial value problem.

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1. INTRODUCTION

The method of upper and lower solutions has been successfully applied to many problems (initial value problems, boundary value problems, inclusions, etc.) of differential equations since 1931 when G. Scorza Dragoni [6] first introduced the method for the Dirichlet problem of ordinary differential equations. This method has also been applied to many problems in difference equations and more recently to problems in dynamic equations on time scales. We refer the reader to [7, 5, 1]. The upper and lower solution method provides two important characteristics of the solutions: Existence and location.

We are motivated by the work of Cabada and et. al [4] and we employ the lower and upper solution method to the following nonlinear nabla fractional difference equation with an initial condition

$$\begin{cases} \nabla_{t_0}^{\alpha} y(t) = f(t, y(t)), \quad t = t_0 + 1, t_0 + 2, \dots, t_0 + n = t_n \\ y(t_0) = y_0, \end{cases} \tag{1.1}$$

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where $0 < \alpha \leq 1$, and $f : [t_0+1, t_n] \cap \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. Our results in this paper generalize the results of the above mentioned paper for the non-integer order case.

In Section 2, we have a preliminary work to prove the main result of the paper. We give definitions of the nabla fractional sum and difference operators along with some lemmas such as the power rule for the discrete fractional operators. We then define the upper and lower solutions for the initial value problem (IVP) (1.1). In Section 3, we give necessary conditions on f to have the upper and lower solutions in the well order (i.e. a lower solution is less than or equal to an upper solution). Under the assumption that the lower and upper solutions for the IVP (1.1) exist and the continuity assumption on the nonlinear term f, we prove the existence of the solution of the IVP (1.1) between a lower solution and an upper solution. We illustrate the main result of the paper with two examples where one can see the necessity of the well-order in the lower and upper solutions to obtain a solution between the upper and the lower solutions. In Section 4, we use the quasilinearization method to construct two sequences of upper and lower solutions which converge to the solution of the IVP (1.1).

2. PRELIMINARIES

Let a be any real number and α be any positive real number such that $0 < n-1 \leq \alpha < n$ where n is an integer.

The α-th order fractional sum of f is defined by

$$\nabla_a^{-\alpha} f(t) = \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(s), \tag{2.1}$$

where $t \in \mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\rho(t) = t-1$ is backward jump operator of the time scale calculus.

The α-th order fractional difference (a Riemann-Liouville fractional difference) of f is defined by

$$\nabla_a^{\alpha} f(t) = \nabla^n \nabla_a^{-(n-\alpha)} f(t) = \nabla^n \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} f(s),$$

where $f : \mathbb{N}_a \to \mathbb{R}$.

The proofs of the following two lemmas can be found in [2] and [3], respectively.

Lemma 2.1 (Power Rule). *Let $\alpha > 0$ and μ be two real numbers so that $\frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}$ is defined. Then the following holds*

$$\nabla_a^{-\alpha}(t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a+1)^{\overline{\alpha+\mu}},$$

for every $t \in \mathbb{N}_a$.

Lemma 2.2. *For any $\alpha > 0$, the following equality holds:*

$$\nabla_{a+1}^{-\alpha}\nabla f(t) = \nabla\nabla_a^{-\alpha} f(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a),$$

where f is defined on $\mathbb{N}_a$.

Definition 2.3. Let the functions v, w be defined on $[t_0, t_n]$, where $[t_0, t_n] = \{t_0, t_0 + 1, t_0 + 2, \ldots, t_0 + n = t_n\}$. The function v is said to be a lower solution of the IVP (1.1) if

$$\begin{cases} \nabla_{t_0}^{\alpha} v(t) \leq f(t, v(t)), \\ v(t_0) \leq y_0. \end{cases} \tag{2.2}$$

The function w is said to be an upper solution of the IVP (1.1) if

$$\begin{cases} \nabla_{t_0}^{\alpha} w(t) \geq f(t, w(t)), \\ w(t_0) \geq y_0. \end{cases} \tag{2.3}$$

A lower solution $v(t)$ and an upper solution $w(t)$ are called well ordered if $v(t) \leq w(t)$, for all $t \in [t_0, t_n]$. The notation $[v, w]$ represents a set of functions which are greater than or equal to v and less than or equal to w for each t.

Lemma 2.4. *Let $f(t)$ and $g(t)$ be real valued functions defined on $[t_0, t_n]$. If $f(t) \leq g(t)$ on $[t_0, t_n]$, then for each $t \in [t_0, t_n]$*

$$\nabla_{t_0}^{-\alpha} f(t) \leq \nabla_{t_0}^{-\alpha} g(t),$$

where $0 < \alpha \leq 1$.

Proof. We first observe that for each $t \in [t_0, t_n]$ the coefficients of f and g in the sums are positive:

$$\frac{(t-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)} = \frac{\Gamma(t-t_0+\alpha)}{\Gamma(t-t_0+1)\Gamma(\alpha)} > 0, \quad \text{when } s = t_0.$$

$$\frac{(t-t_0)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)} = \frac{\Gamma(t-t_0+\alpha-1)}{\Gamma(t-t_0)\Gamma(\alpha)} > 0, \quad \text{when } s = t_0 + 1.$$

$$\vdots$$

$$\frac{(t-t+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)} = \frac{\Gamma(\alpha)}{\Gamma(1)\Gamma(\alpha)} = 1 > 0, \quad \text{when } s = t.$$

Then we have the following inequalities:

$$\frac{(t-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)} f(t_0) \leq \frac{(t-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)} g(t_0),$$

$$\frac{(t-t_0)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)} f(t_0+1) \leq \frac{(t-t_0)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)} g(t_0+1),$$

$$\vdots$$

$$\frac{(t-t+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}f(t) \leq \frac{(t-t+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}g(t).$$

If we sum each side of the above inequalities from t_0 to t side by side, then we obtain

$$\sum_{s=t_0}^{t}\frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(s) \leq \sum_{s=t_0}^{t}\frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)}g(s).$$

Therefore,

$$\nabla_{t_0}^{-\alpha}f(t) \leq \nabla_{t_0}^{-\alpha}g(t),$$

where $t \in [t_0, t_n]$. □

3. LOWER SOLUTIONS AND UPPER SOLUTIONS IN ORDER

This section is devoted to prove the main result of this paper which is existence of solutions for the IVP (1.1) where we assume the existence of a pair of ordered lower and upper solutions.

Theorem 3.1. *Suppose $f(t,y)$ is continuous in y for each t on $[t_0, t_n]$ and is differentiable such that $0 < \frac{\partial f}{\partial y} < 1$ for each t on $[t_0, t_n]$. Assume $v(t)$ and $w(t)$ are a lower and an upper solution for the IVP* (1.1)*, respectively. Then $v(t) \leq w(t)$, for $t_0 \leq t \leq t_n$.*

Proof. We define $m(t) = v(t) - w(t)$. We prove that $m(t) \leq 0$ on $[t_0, t_n]$ by using mathematical induction.

Initial Step. If $t = t_0$, we have $m(t_0) = v(t_0) - w(t_0) \leq y_0 - y_0 = 0$.

Inductive Step. Assume that $m(t) \leq 0$ for all $t_0 \leq t \leq k$. We prove that the inequality is true for $t = k+1$.

First we note that by use of Lemma 2.4, the following inequalities can be obtained.

$$v(k+1) \leq \nabla_{t_0+1}^{-\alpha}f(k+1, v(k+1)) + \frac{(k+1-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}\nabla_{k_0}^{-(1-\alpha)}v(k)|_{k=t_0},$$

and

$$w(t) \geq \nabla_{t_0+1}^{-\alpha}f(k+1, w(k+1)) - \frac{(k+1-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}\nabla_{t_0}^{-(1-\alpha)}w(k)|_{k=t_0}.$$

Hence we have

$$\begin{aligned} m(k+1) &= v(k+1) - w(k+1) \\ &\leq \nabla_{t_0+1}^{-\alpha}f(k+1, v(k+1)) + \frac{(k+1-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}\nabla_{k_0}^{-(1-\alpha)}v(k)|_{k=t_0} \\ &\quad - \nabla_{t_0+1}^{-\alpha}f(k+1, w(k+1)) - \frac{(k+1-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}\nabla_{t_0}^{-(1-\alpha)}w(k)|_{k=t_0} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=t_0+1}^{k+1} \frac{(k+1-\rho(s))^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}(f(s,v(s))-f(s,w(s))) \\
&\quad + \frac{(k+1-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}(v(t_0)-w(t_0)) \\
&\leq \sum_{s=t_0+1}^{k+1} \frac{(k+1-\rho(s))^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}(f(s,v(s))-f(s,w(s))) \\
&= \sum_{s=t_0+1}^{k} \frac{(k+1-\rho(s))^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}(f(s,v(s))-f(s,w(s))) \\
&\quad + \frac{(k+1-(k+1-1))^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}(f(k+1,v(k+1))-f(k+1,w(k+1))),
\end{aligned}
$$

where we used $\frac{(k+1-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)} > 0$ and $v(t_0)-w(t_0) \leq 0$.

By using Mean Value Theorem, there exist ξ_s and μ_{k+1} such that

$$f(s,v(s))-f(s,w(s)) = f_y(s,\xi_s)(v(s)-w(s)),$$

where ξ_s between $v(s)$ and $w(s)$.

$$f(k+1,v(k+1))-f(k+1,w(k+1)) = f_y(k+1,\mu_{k+1})(v(k+1)-w(k+1)),$$

where μ_{k+1} between $v(k+1)$ and $w(k+1)$. Hence we obtain the following

$$
\begin{aligned}
m(k+1) &\leq \sum_{s=t_0+1}^{k} \frac{(k+1-\rho(s))^{\overline{(\alpha-1)}}}{\Gamma(\alpha)} f_y(s,\xi_s)(v(s)-w(s)) \\
&\quad + f_y(k+1,\mu_{k+1})(v(k+1)-w(k+1)) \\
&\leq f_y(k+1,\mu_{k+1})(v(k+1)-w(k+1)),
\end{aligned}
$$

since $\sum_{s=t_0+1}^{k} \frac{(k+1-\rho(s))^{\overline{(\alpha-1)}}}{\Gamma(\alpha)} f_y(s,\xi_s)(v(s)-w(s)) < 0$. Therefore, we have

$$m(k+1) = v(k+1)-w(k+1) \leq f_y(k+1,\mu_{k+1})(v(k+1)-w(k+1)).$$

This implies that

$$(1-f_y(k+1,\mu_{k+1}))(v(k+1)-w(k+1)) \leq 0.$$

Since $0 < f_y(k+1,\mu_{k+1}) < 1$, we obtain $v(k+1) \leq w(k+1)$. $\square$

Next we show that there exists a solution between the ordered lower and upper solutions.

Theorem 3.2. *Suppose there exist lower and upper solutions* $v = \{v_0, v_1, \ldots, v_n\}$ *and* $w = \{w_0, w_1, \ldots, w_n\}$*, respectively, of the IVP* (1.1) *such that* $v \leq w$*. Assume also that* $f(\cdot, y_i)$ *is a continuous function in* $[v_i, w_i]$ *for all* $i \in [0,n]$*. Then the IVP* (1.1) *has at least one solution* λ *such that* $v(t) \leq \lambda(t) \leq w(t)$ *for each* $t \in [t_0, t_n]$.

Proof. Consider the following modified problem:

$$\left\{\begin{array}{l}\nabla_{t_0}^{\alpha}\lambda(t)=f(t,p(t,\lambda(t))),0<\alpha<1\\ \nabla_{t_0}^{-(1-\alpha)}\lambda(t)|_{t=t_0}=\lambda(t_0)=y_0.\end{array}\right. \tag{3.1}$$

where $p(t_i,r)=\max\{v_i,\min\{r,w_i\}\}$, for all $i\in\{0,\dots,n\}$ and $r\in\mathbb{R}$.

One can easily see that the IVP (3.1) is equivalent to the following sum equation (see [2])

$$\lambda(t)=\nabla_{t_0+1}^{-\alpha}f(t,p(t,\lambda(t)))+\frac{(t-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}\lambda(t_0).$$

Define K=$\{y:y(t)$ is defined on $[t_0,t_n]$ and $\|y\|\leq k$ for some $k\in\mathbb{R}^+\}$.

For $y\in K$, define Ty by

$$Ty(t)=\nabla_{t_0+1}^{-\alpha}f(t,p(t,y(t)))+\frac{(t-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}y(t_0).$$

By the definition of p, we have $f(t,p(t,r))$ for $t_0\leq t\leq t_n$

$$f(t,p(t,r))=\begin{cases}f(t,w_i), & r\geq w_i\\ f(t,r), & v_i\leq r\leq w_i\\ f(t,v_i), & r\leq v_i.\end{cases}$$

Since f is continuous as a function of p for each t on $[t_0,t_n]$, f is bounded. Hence the operator T is continuous and bounded. By the Brouwer fixed point theorem, there exists a solution for the modified problem (3.1).

Next, we use another approach to define the operator T for the IVP (3.1).

Now x is a solution of the above problem if and only if x= col$(x_0,x_1,\dots,x_n)$ is a solution of the matrix equation

$$Ax=F(x) \tag{3.2}$$

where $A=(a_{ij})$ is defined by

$$A=\begin{bmatrix}\alpha & 1 & 0 & 0 & \cdots & 0\\ \frac{3^{\overline{-\alpha}}-2^{\overline{-\alpha}}}{\Gamma(1-\alpha)} & -\alpha & 1 & 0 & \cdots & 0\\ \frac{4^{\overline{-\alpha}}-3^{\overline{-\alpha}}}{\Gamma(1-\alpha)} & \frac{3^{\overline{-\alpha}}-2^{\overline{-\alpha}}}{\Gamma(1-\alpha)} & -\alpha & 1 & \cdots & 0\\ \vdots & & & \vdots & & \\ \frac{(n+2)^{\overline{-\alpha}}-(n+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} & & \cdots & & & -\alpha\end{bmatrix}$$

and $F(x)$ is the transpose of the vector

$$(f(t_0,p(t_0,x(t_0))),\dots,f(t_n,p(t_n,x(t_n))),-\sum_{i=1}^{n}f(t_i,p(t_i,x(t_i)))-\lambda(t_0)).$$

To show the existence of A^{-1}, we need to prove that $det(A) \neq 0$. One can prove this by mathematical induction.

Then we rewrite (3.2) as the fixed-point equation $x = A^{-1}F(x) \equiv Tx$. Obviously, T is a continuous map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$. By definition of p there exist $L > 0$ such that $\|Tx\| \leq L$, where $\|x\| = \max\{|x(t_i)|, i = 0, \ldots, n\}$. Thus, the Brouwer fixed point theorem implies the existence of a fixed point of the operator T, and in consequence, there exists a solution of the IVP (3.1).

We now show that $v(t) \leq \lambda(t) \leq w(t)$ for each $t \in [t_0, t_n]$. Assume to the contrary that there exist some $t \in [t_1, t_n]$ such that $\lambda(t) < v(t)$. Let $j_0 = \min\{j \in [t_1, t_n] : v(j_0) > \lambda(j_0)\}$. Then obviously, $v(j_0 - 1) \leq \lambda(j_0 - 1)$, in consequence, we have

$$\nabla^\alpha \lambda(j_0) = f(j_0, p(j_0, \lambda(j_0))) = f(j_0, v(j_0)) \geq \nabla^\alpha v(j_0),$$

where $p(j_0, \lambda(j_0)) = \max\{v(j_0), \min\{\lambda(j_0), w(j_0)\}\} = v(j_0)$.

First we employ the definition of the nabla fractional difference operator and we obtain

$$\nabla\nabla^{-(1-\alpha)}\lambda(t)|_{t=j_0} \geq \nabla\nabla^{-(1-\alpha)}v(t)|_{t=j_0}$$

$$\nabla \sum_{s=0}^{t} \frac{(t-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\lambda(s)|_{t=j_0} \geq \nabla \sum_{s=0}^{t} \frac{(t-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)}v(s)|_{t=j_0}.$$

By using the power rule (Lemma 2.1) we obtain

$$\sum_{s=0}^{t} \nabla \frac{(t-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)}\lambda(s)|_{t=j_0} \geq \sum_{s=0}^{t} \nabla \frac{(t-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)}v(s)|_{t=j_0}$$

$$\sum_{s=0}^{t} \frac{(-\alpha)(t-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(1-\alpha)}\lambda(s)|_{t=j_0} \geq \sum_{s=0}^{t} \frac{(-\alpha)(t-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(1-\alpha)}v(s)|_{t=j_0}$$

$$\sum_{s=0}^{t} \frac{(t-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}\lambda(s)|_{t=j_0} \geq \sum_{s=0}^{t} \frac{(t-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}v(s)|_{t=j_0}$$

$$\sum_{s=0}^{j_0} \frac{(j_0-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}\lambda(s) \geq \sum_{s=0}^{j_0} \frac{(j_0-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}v(s).$$

Then we have

$$\sum_{s=0}^{j_0-1} \frac{(j_0-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}\lambda(s) + \frac{(j_0-(j_0-1))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}\lambda(j_0)$$
$$\geq \sum_{s=0}^{j_0-1} \frac{(j_0-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}v(s) + \frac{(j_0-(j_0-1))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}v(j_0),$$

which can be written in the form

$$\frac{(j_0-(j_0-1))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}\lambda(j_0) - \frac{(j_0-(j_0-1))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}v(j_0)$$

$$\geq \sum_{s=0}^{j_0-1} \frac{(j_0-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}(v(s)-\lambda(s)).$$

Hence we have

$$0 > \lambda(j_0) - v(j_0) \geq \sum_{s=0}^{j_0-1} \frac{(j_0-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}(v(s)-\lambda(s)) \geq 0.$$

Finally, we attain the contradiction that $0 > \lambda(j_0) - v(j_0) \geq 0$. Similarly, we can prove that $\lambda(t) \leq w(t)$ for each $t \in [t_0, t_n]$. □

Next we illustrate our results with two examples. In the first example we show the existence of a solution of the initial value problem between lower and upper solutions, which are well ordered. In the second example, we show that if a lower solution and an upper solution are not well-ordered, then we cannot assure the existence of a solution of the initial value problem lying between them.

Example 3.3. The sequences $v = \{1, -1, -1\}$, and $w = \{1, 1, 2\}$ are, respectively, a lower and an upper solution of the initial value problem

$$\nabla_0^\alpha x(t) = \frac{2}{x^2(t)+1} - \alpha, \quad t = \{1, 2\},$$

$$x(0) = 1.$$

This initial value problem can be written as

$$x(t) = \sum_{s=1}^{t} \frac{(t-\rho(s))^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}\left(\frac{2}{x^2(s)+1} - \alpha\right) + \frac{(t-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}.$$

For $t = 1$, we have $x(1) = 1$. For $t = 2$, we have $x^3(2) - (\frac{\alpha}{2} - \frac{\alpha^2}{2})x^2(2) + x(2) + \frac{\alpha^2}{2} - \frac{\alpha}{2} - 2 = 0$. Let's choose $\alpha = \frac{1}{2}$. Then $x(2)$ satisfies the following equation

$$\frac{1}{8}x^2 + x - \frac{17}{8} = 0.$$

The only real zero of the equation is 1.0635064. Therefore, the solution is $\{1,\ 1,\ 1.0635064\}$. And it stays between v and w.

Example 3.4. The sequences $v = \{1, 1, 0\}$, and $w = \{1, 0, 0\}$ are a lower and an upper solution of the initial value problem respectively,

$$\nabla_0^\alpha x(t) = x^2(t) - \alpha, \quad t = \{1, 2\},$$

$$x(0) = 1.$$

This initial value problem can be written as

$$x(t) = \sum_{s=1}^{t} \frac{(t-\rho(s))^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}\left(\frac{x^2(t)}{2} - \alpha\right) + \frac{(t-t_0+1)^{\overline{(\alpha-1)}}}{\Gamma(\alpha)}.$$

For $t = 1$, we obtain $x(1) = 0$ or $x(1) = 1$. For $t = 2$, we obtain $x(2) = \frac{1 \pm \sqrt{2\alpha^2+2\alpha+1}}{2}$ or $x(2) = \frac{1 \pm \sqrt{2\alpha^2-2\alpha+1}}{2}$. Clearly, $x(2)$ doesn't lie between v and w for any $0 < \alpha \leq 1$.

4. AN APPLICATION: THE QUASILINEARIZATION METHOD

In this section, as an application of Theorem 3.2 we employ monotone iterative method called the quasilinearization method to study the existence of solutions for the nabla fractional difference equation with an initial condition.

Theorem 4.1. *In addition to the hypothesis of Theorem 3.2, we assume that there exists a constant $m > 0$ such that*

$$f(t, y) + my \leq f(t, x) + mx, \quad \text{for } v(t) \leq y \leq x \leq w(t), \quad t \in [t_0, t_n].$$

Then there exist two monotone and convergent sequences in $\mathbb{R}^{N+1}$, v_n and w_n such that $v = v_0 \leq v_n \leq w_n \leq w_0 = w$, where $v_0(t)$ is a lower solution and $w_0(t)$ is an upper solution for the initial value problem (1.1).

Proof. We first show that v_0 and w_0 are a lower solution and an upper solution for the following modified problem

$$\nabla_{t_0}^{\alpha} \lambda(t) = f(t, v_0(t)) + m(v_0(t) - \lambda(t))$$

$$\lambda(t_0) = y_0.$$

v_0 is a lower solution of the initial value problem (1.1). This implies that

$$\nabla_{t_0}^{\alpha} v_0(t) \leq f(t, v_0(t)),$$

$$\nabla_{t_0}^{\alpha} v_0(t) \leq f(t, v_0(t)) + m(v_0(t) - v_0(t)).$$

In consequence, $v_0(t)$ is a lower solution of the equation

$$\nabla_{t_0}^{\alpha} \lambda(t) = f(t, v_0(t)) + m(v_0(t) - \lambda(t)).$$

Since $v_0(t) \leq w_0(t)$, we have

$$f(t, v_0(t)) + m(v_0(t)) \leq f(t, w_0(t)) + m(w_0(t)),$$

$$f(t, v_0(t)) + m(v_0(t)) - m(w_0(t)) \leq f(t, w_0(t)).$$

w_0 is an upper solution of the initial value problem (1.1). This implies that

$$f(t, v_0(t)) + m(v_0(t)) - m(w_0(t)) \leq \nabla_{t_0}^{\alpha} w_0(t).$$

Hence, $w_0(t)$ is an upper solution of the equation

$$\nabla_{t_0}^{\alpha} \lambda(t) = f(t, v_0(t)) + m(v_0(t) - \lambda(t)).$$

By using Theorem 3.2, there exists a solution v_1 of the modified problem

$$\nabla_{t_0}^{\alpha} \lambda(t) = f(t, v_0(t)) + m(v_0(t) - \lambda(t))$$

$$\lambda(t_0) = y_0.$$

such that $v_0 \leq v_1 \leq w_0$.

Next, we want to show that v_0 and w_0 are a lower solution and an upper solution for the following modified problem

$$\begin{cases} \nabla_{t_0}^{\alpha}\gamma(t) = f(t, w_0(t)) + m(w_0(t) - \gamma(t)), \\ \gamma(t_0) = y_0. \end{cases} \tag{4.1}$$

Since w_0 is an upper solution of $\nabla_{t_0}^{\alpha} y(t) = f(t, y(t))$, we have

$$\nabla_{t_0}^{\alpha} w_0(t) \geq f(t, w_0(t)),$$

$$\nabla_{t_0}^{\alpha} w_0(t) \geq f(t, w_0(t)) + m(w_0(t) - w_0(t)).$$

Hence $w_0(t)$ is an upper solution of $\nabla_{t_0}^{\alpha}\gamma(t) = f(t, w_0(t)) + m(w_0(t) - \gamma(t))$.

We also have $v_0(t) \leq w_0(t)$, then it follows that

$$f(t, v_0(t)) + m(v_0(t)) \leq f(t, w_0(t)) + m(w_0(t)),$$

$$f(t, v_0(t)) \leq f(t, w_0(t)) + m(w_0(t)) - m(v_0(t)).$$

Since $v_0(t)$ is a lower solution of $\nabla_{t_0}^{\alpha} y(t) = f(t, y(t))$, we obtain

$$\nabla_{t_0}^{\alpha} v_0(t) \leq f(t, v_0(t)),$$

$$\nabla_{t_0}^{\alpha} v_0(t) \leq f(t, v_0(t)) + m(w_0(t)) - m(v_0(t)).$$

Therefore, $v_0(t)$ is a lower solution of $\nabla_{t_0}^{\alpha}\gamma(t) = f(t, w_0(t)) + m(w_0(t) - \gamma(t))$. Similarly, by using Theorem 3.2, there exists a solution w_1 for the modified problem (4.1).

Next we claim that $v_1(t) \leq w_1(t)$. Assume to the contrary that there exists a smallest k such that $v_1(k) > w_1(k)$. Obviously, $v_1(k-1) \leq w_1(k-1)$.

Define $\beta(k) = w_1(k) - v_1(k)$. We note that $\beta(k) < 0$. Applying $\nabla_{t_0}^{\alpha}$ to each side, we obtain

$$\nabla_{t_0}^{\alpha}\beta(k) = \nabla_{t_0}^{\alpha} w_1(k) - \nabla_{t_0}^{\alpha} v_1(k),$$

$$\nabla_{t_0}^{\alpha}\beta(k) = f(k, w_0(k)) + m(w_0(k) - w_1(k)) - f(k, v_0(k)) - m(v_0(k) - v_1(k)),$$

$$\nabla_{t_0}^{\alpha}\beta(k) = f(k, w_0(k)) + m w_0(k) - f(k, v_0(k)) - m v_0(k) + m(v_1(k) - w_1(k)) > 0.$$

Therefore,

$$\nabla_{t_0}^{\alpha}\beta(k) = \nabla_{t_0}^{\alpha} w_1(k) - \nabla_{t_0}^{\alpha} v_1(k) > 0.$$

After using the definition of the fractional difference operator, we have

$$\nabla_{t_0}^{\alpha} w_1(k) > \nabla_{t_0}^{\alpha} v_1(k)$$

$$\nabla_{t_0}^{\alpha} w_1(t)|_{t=k} > \nabla_{t_0}^{\alpha} v_1(t)|_{t=k}$$

$$\nabla\nabla_{t_0}^{-(1-\alpha)} w(t)|_{t=k} > \nabla\nabla_{t_0}^{-(1-\alpha)} v_1(t)|_{t=k}$$

$$\nabla \sum_{s=t_0}^{t} \frac{(t-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)} w_1(t)|_{t=k} > \nabla \sum_{s=t_0}^{t} \frac{(t-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)} v_1(t)|_{t=k}$$

$$\sum_{s=t_0}^{t} \nabla \frac{(t-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)} w_1(t)|_{t=k} > \sum_{s=t_0}^{t} \nabla \frac{(t-\rho(s))^{\overline{-\alpha}}}{\Gamma(1-\alpha)} v_1(t)|_{t=k}$$

$$\sum_{s=t_0}^{t} \frac{(-\alpha)(t-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(1-\alpha)} w_1(t)|_{t=k} > \sum_{s=0}^{t} \frac{(-\alpha)(t-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(1-\alpha)} v_1(t)|_{t=k}$$

$$\sum_{s=t_0}^{t} \frac{(t-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} w_1(t)|_{t=k} > \sum_{s=t_0}^{t} \frac{(t-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} v_1(t)|_{t=k}.$$

Hence we have

$$0 > w_1(k) - v_1(k) > \sum_{s=t_0}^{k-1} \frac{(k-1-\rho(s))^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} (v_1(s) - w_1(s)) \geq 0.$$

We obtain a contradiction. Therefore, $v_1(k) \leq w_1(k)$. By doing the same process repeatedly, we have

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_1 \leq w_0.$$

Now we need to show that v_n and w_n are convergent to the solution of the initial value problem.

v_{n+1} solves the following problem

$$\nabla_{t_0}^{\alpha} \lambda(t) = f(t, v_n(t)) + m(v_n(t) - \lambda(t)).$$

This implies that

$$v_{n+1} = \nabla_{t_0+1}^{-\alpha}(f(t, v_n(t) + m(v_n(t) - v_{n+1})) + \frac{(t-t_0+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_{t_0}^{-(1-\alpha)} v_{n+1}|_{t=t_0}.$$

Let's employ the limit each side, we have

$$\begin{aligned} \lim_{n\to\infty} v_{n+1}(t) = \nabla_{t_0+1}^{-\alpha}(f(t, \lim_{n\to\infty} v_n(t) + m(\lim_{n\to\infty} v_n(t) - \lim_{n\to\infty} v_{n+1}(t))) \\ + \frac{(t-t_0+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_{t_0}^{-(1-\alpha)} \lim_{n\to\infty} v_{n+1}(t)|_{t=t_0}. \end{aligned}$$

Since $v_n(t)$ is a monotone and convergent sequence bounded by w_0, we have

$$\lim_{n\to\infty} v_n(t) = \lim_{n\to\infty} v_{n+1}(t),$$

$$\lim_{n\to\infty} v_n(t) = \nabla_{t_0+1,}^{-\alpha}(f(t, \lim_{n\to\infty} v_n(t))) + \frac{(t-t_0+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \nabla_{t_0}^{-(1-\alpha)} \lim_{n\to\infty} v_n(t)|_{t=t_0}.$$

Hence $\lim_{n\to\infty} v_n(t)$ is a solution of the IVP (1.1). Similarly, we show that $\lim_{n\to\infty} w_n(t)$ is also a solution of the IVP (1.1). One can see that $\lim_{n\to\infty} v_n(t) \leq \lim_{n\to\infty} w_n(t)$. This finishes the proof. $\square$

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