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MULTIPLE PERIODIC SOLUTIONS FOR SOME DIFFERENCE EQUATIONS SUBJECTED TO ALLEE EFFECTS

SMITA PATI, SESHADEV PADHI, AND E. THANDAPANI

Department of Applied Mathematics, Birla Institute of Technology
Mesra, Ranchi-835215, India
E-mail: spatimath@yahoo.com
Department of Applied Mathematics, Birla Institute of Technology
Mesra, Ranchi-835215, India
E-mail: ses_2312@yahoo.co.in
Ramanujan Institute for Advanced Study in Mathematics, University of Madras,
Chennai-600 005, India
E-mail: ethandapani@yahoo.co.in

Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. This paper deals with the existence of multiple positive periodic solutions to a nonautonomous scalar difference equation subjected to Allee effects. Existence is established using Leggett-Williams multiple fixed point theorem. This result is employed to find the minimum number of positive periodic solutions admitted by a model representing dynamics of a renewable resource that is subjected to Allee effects in a seasonally varying environment.

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1. Introduction

In this paper, we use Leggett-Williams fixed point theorem to study the existence of multiple positive periodic solutions of a certain type of first order difference equation. This result is used to find the minimum number of positive periodic solutions admitted by some models representing dynamics of a renewable resource that is subjected to Allee effects in a seasonally varying environment.

There has been considerable contribution in recent years on the existence of periodic solutions of difference equations having periodic casual functions, see [12, 28, 33, 40, 41, 42, 43, 49, 50], and the references cited therein. Many authors [1, 20, 23, 31, 37] have argued that the discrete time models governed by difference equation are more appropriate than the continuous ones when the populations have non overlapping generations.

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Motivated by the above observation and by the work of Padhi, Srinivasu and Kiran Kumar [39], in this paper we investigate the existence of multiple periodic solutions of a first order nonlinear difference equation representing growth of a renewable resource that is subjected to Allee effects in a seasonally varying environment.

Let a, b be given integers and $a < b$. We denote discrete sets such as

$$Z[a,b] = \{a, a+1, \dots, b\}, \quad Z[a,b) = \{a, \dots, b-1\}, \quad Z[a,\infty) = \{a, a+1, \dots\},$$

etc. Let $T \in Z[1,\infty)$ be fixed.

The difference equation representing dynamics of a renewable resource $y(n)$, that is subjected to Allee effects is

$$\Delta y(n) = ay(n)(y(n) - b)(c - y(n)), \quad n \in Z(-\infty,\infty), \tag{1.1}$$

where $a > 0, 0 < b < c$ and the constants a, c and b represent respectively intrinsic growth rate, carrying capacity of the resource and the threshold value below which the growth rate of the resource is negative. It is well known that equation (1.1) admits two positive solutions given by $y_n = b$ and $y_n = c$, and one trivial solution as its equilibrium solution.

Since we are interested in the dynamics of a renewable resource in a seasonally varying environment we assume the coefficients a, b and c to be positive T-periodic functions of the same period, and study the existence of T-periodic solutions. Thus, we consider

$$\Delta y(n) = a(n)y(n)(y(n) - b(n))(c(n) - y(n)), \quad n \in Z(-\infty,\infty), \tag{1.2}$$

where the positive real sequences $c(n)$ and $b(n)$ stand for seasonal dependent carrying capacity and threshold function of the species respectively satisfying

$$0 < b(n) < c(n) \text{ and } 0 < a(n)b(n)c(n) < 1, \tag{1.3}$$

where $a(n)$ represents time dependent intrinsic growth rate of the resource. Clearly, we have the trivial solution ($y(n) \equiv 0$) to be a periodic solution of equation (1.2). Since the study deals with resource dynamics, we are interested in the existence of positive periodic solutions of the equation (1.2).

Equation (1.2) can be rewritten as

$$\Delta y(n) = -a(n)b(n)c(n)y(n) + a(n)(b(n) + c(n) - y(n))y^2(n). \tag{1.4}$$

Clearly (1.4) is a particular case of a general scalar difference equation of the form

$$\Delta y(n) = -A(n) + f(n, y(n)), \ n \in Z(-\infty,\infty), \tag{1.5}$$

where $A(n) : Z(-\infty,\infty) \to (0,1)$, $f : Z(-\infty,\infty) \times [0,\infty) \to (0,\infty)$ is continuous, and $A(n) = A(n+T)$, $f(n,u) = f(n+T,u)$.

Remark 1.1. We say a map $f : Z(-\infty, \infty) \times [0, \infty) \to [0, \infty)$ is continuous if it is continuous as a map of the topological space $Z(-\infty, \infty) \times [0, \infty)$ onto the topological space $[0.\infty)$. Throughout this paper the topology on $Z(-\infty, \infty)$ will be the discrete topology.

To conclude this section, we state a Leggett-Williams multiple fixed point theorem (see Theorem 3.5 in [32]) which will be needed in this paper.

Theorem 1.2. *Let $X = (X, \|\cdot\|)$ be a Banach space and let K be a cone in X. Suppose $E : \overline{K}_{c_3} \to K$ (here $\overline{K}_{c_3} = \{x \in K : \|x\| < c_3\}$) is completely continuous, and suppose there exists a concave nonnegative functional ψ with $\psi(x) \leq \|x\|$, $x \in K$ and numbers c_1 and c_2 with $0 < c_1 < c_2 < c_3$ satisfying the following conditions:*

(i) $\{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \emptyset$ *and* $\psi(Ex) > c_2$ *if* $x \in K(\psi, c_2, c_3) = \{x \in K : \psi(x) \geq c_2, \|x\| < c_3\}$;

(ii) $\|Ex\| < c_1$ *if* $x \in \overline{K}_{c_3}$;

(iii) $\psi(Ex) > \frac{c_2}{c_3}\|Ex\|$ *for each* $x \in \overline{K}_{c_3}$ *such that* $\|Ex\| > c_3$.

Then E has at least two fixed points in $\overline{K}_{c_3}$.

Now, for any positive bounded T-periodic sequence $p(n)$, we set

$$p_* = \min_{0 \leq n \leq T-1} p(n) \text{ and } p^* = \max_{0 \leq n \leq T-1} p(n).$$

2. Existence of Positive Periodic Solutions

In this section we establish the existence of positive periodic solutions to equation (1.5). Since $0 < A(n) < 1$, we can define

$$\delta = \left(\prod_{\theta=0}^{T-1} (1 - A(\theta)) \right)^{-1}. \tag{2.1}$$

Finding a T-periodic solutions of equation (1.5) is equivalent to finding a T-periodic solutions of the equation

$$y(n) = \sum_{s=n}^{n+T-1} G(n, s) f(s, y(s)), \tag{2.2}$$

where

$$G(n, s) = \frac{\prod_{\theta=s+1}^{n+T-1}(1 - A(\theta))}{1 - \prod_{\theta=0}^{T-1}(1 - A(\theta))}, \quad s \in [n, n + T - 1].$$

It is easy to see that, for $\theta \in [n, n + T - 1]$, we have

$$0 < \frac{1}{\delta - 1} \leq G(n, s) \leq \frac{\delta}{\delta - 1},$$

where δ is given as in (2.1). Let $X = \{y(n) : y(n) \in C(Z(-\infty,\infty),\mathbb{R}),\ y(n+T) = y(n)\}$ and define

$$\|y\| = \sup_{\theta \in Z[0,T-1]} \{y(\theta) : y \in X\}.$$

Then X with the norm $\|\cdot\|$ is a Banach space. Now solving (2.2) is equivalent to solving

$$y = Ey,$$

where E is defined by

$$(E(y))(n) = \sum_{s=n}^{n+T-1} G(n,s) f(s,y(s)) \tag{2.3}$$

for $y \in X$. Clearly, E is well defined. Let

$$K = \{y \in X : y(n) \geq 0\}.$$

Then it is not difficult to verify that K is a cone in X.

Theorem 2.1. *Suppose that there exists a positive constant c_3 such that $\sum_{s=0}^{T-1} f(s,y(s)) > 0$ holds when $0 < y(n) \leq c_3$ for all $s \in [0,T-1]$, and*

(H$_1$) $\sum_{s=0}^{T-1} f(s,y(s)) \leq \left(\frac{\delta-1}{\delta}\right) c_3$ *for* $\frac{c_3}{\delta} \leq y(s) \leq c_3$, $s \in [0,T-1]$

and

(H$_2$) $\lim_{\|y\|\to 0} \frac{1}{\|y\|} \sum_{s=0}^{T-1} f(s,y(s)) < \frac{\delta-1}{\delta}$

hold. Then equation (1.5) *has at least two positive T-periodic solutions in $\overline{K}_{c_3}$.*

Proof. Consider the Banach space X defined above, and the cone $K \subseteq X$. Let c_3 be the constant satisfying the conditions laid in the hypothesis. Define the operator $E : \overline{K}_{c_3} \to K$ as (2.3). We shall apply Leggett-Williams multiple fixed point theorem to the operator E to prove the existence of at least two positive periodic solutions for equation (1.5).

It can be easily verified that E is completely continuous and $E(\overline{K}_{c_3}) \subset K$. Now, let us consider a nonnegative concave continuous functional ψ defined on K as

$$\psi(y) = \min_{0 \leq n \leq T-1} y(n).$$

For $c_2 = \frac{c_3}{\delta}$ and $\phi_0 = \frac{1}{2}(c_2 + c_3)$, we have $c_2 < \phi_0 < c_3$ and the set

$$\{y \in K(\psi, c_2, c_3) : \psi(y) > c_2\} \neq \emptyset.$$

For $y(n) \in K(\psi, c_2, c_3)$, consider

$$\begin{aligned}\psi(Ey) &= \min_{0 \leq n \leq T-1} \sum_{s=n}^{n+T-1} G(n,s) f(s,y(s)) \\ &> \frac{1}{\delta-1} \sum_{s=0}^{T-1} f(s,y(s)) \geq \left(\frac{1}{\delta-1}\right)\left(\frac{\delta-1}{\delta}\right) c_3\end{aligned}$$

$$= \frac{c_3}{\delta}.$$

Hence the condition (i) of Theorem 1.2 is satisfied. Since $\lim_{\|y\|\to 0}\sum_{s=0}^{T-1} f(s,y(s)) < (\frac{\delta-1}{\delta})$ (from (H_2)), there exists a real $\xi,\ 0<\xi<c_2$ such that

$$\sum_{s=0}^{T-1} f(s,y(s)) < \left(\frac{\delta-1}{\delta}\right)\|y\| \quad \text{for } 0 \le \|y\| \le \xi.$$

Choose $c_1 = \xi$. Then, we have $0 < c_1 < c_2$ and for $0 \le y(n) \le c_1$, we have

$$\begin{aligned}\|Ey\| &= \sup_{0\le n\le T-1} \sum_{s=n}^{n+T-1} G(n,s) f(s,y(s)) \\ &< \left(\frac{\delta}{\delta-1}\right)\sum_{s=0}^{T-1} f(s,y(s)) \le \|y\| \le c_1.\end{aligned}$$

Hence condition (ii) of Theorem 1.2 is satisfied. Now consider

$$\begin{aligned}\psi(Ey) &= \min_{0\le n\le T-1} \sum_{s=n}^{T-1} G(n,s) f(s,y(s)) \\ &< \frac{1}{\delta-1}\sum_{s=0}^{T-1} f(s,y(s)).\end{aligned}$$

Let $0 < y(n) < c_3$ be such that $\|Ey\| > c_3$. For such a choice of $y(n)$, we have

$$\begin{aligned}c_3 < \|Ey\| &= \sup_{0\le n\le T-1} \sum_{s=n}^{n+T-1} G(n,s) f(s,y(s)) \\ &< \left(\frac{\delta}{\delta-1}\right)\sum_{s=0}^{T-1} f(s,y(s)) \\ &< \delta\psi(Ey).\end{aligned}$$

Therefore $\psi(Ey) > \frac{1}{\delta}\|Ey\|$, and this implies that $\psi(Ey) > \frac{c_2\|Ey\|}{c_3}$ for each y with $0 < y(n) < c_3$ satisfying $\|Ey\| > c_3$. Hence condition (iii) of Theorem 1.2 is satisfied. Therefore by Theorem 1.2, the operator (2.3) has at least two fixed point in $\overline{K}_{c_3}$. One may observe that the existence of a fixed point of E is equivalent to the existence of a positive periodic solution of equation (1.5). Hence under the hypothesis of theorem, equation (1.5) admits at least two positive T- periodic solutions. This completes the proof. □

Corollary 2.2. *Suppose that there exists a positive constant c_3 such that*
(H_1^*) $\quad \sum_{s=0}^{T-1} f(s,y(s)) > 0$ *for* $0 < y \le c_3$.
Furthermore, for the above choice of c_3, assume that

$$\sum_{s=0}^{T-1} f(s,y(s)) > \left(\frac{\delta}{\delta-1}\right) c_3 \quad \textit{for} \quad \frac{c_3}{\delta} \le y < c_3$$

and

$$\sum_{s=0}^{T-1} f(s,y(s)) = \left(\frac{\delta}{\delta-1}\right) y \quad \textit{for} \quad y = c_3,$$

and

(H$_2^*$) $\lim_{y\to 0} \frac{1}{y}\sum_{s=0}^{T-1} f(s,y(s)) < \frac{\delta-1}{\delta}$

hold. Then equation (1.5) *has at least two positive T-periodic solutions in* $\overline{K}_{c_3}$.

Proof. Assume that there exists a positive constant c_3 such that $\sum_{s=0}^{T-1} f(s,y(s)) > 0$ for $0 < y \le c_3$. Now, let $y(n) \in K$ be such that $0 < y(n) \le c_3$. From the above assumption it clearly follows that $\sum_{s=0}^{T-1} f(s,y(s)) > 0$ when $0 < y(s) \le c_3$ for all $s \in [0, T-1]$. Further, let us assume that

$$\sum_{s=0}^{T-1} f(s,y) = \left(\frac{\delta-1}{\delta}\right) y \quad \text{for } y = c_3$$

and

$$\sum_{s=0}^{T-1} f(s,y) > \left(\frac{\delta-1}{\delta}\right) c_3 \quad \text{for } \frac{c_3}{\delta} \le y < c_3.$$

This assumption implies that

$$\sum_{s=0}^{T-1} f(s,y(s)) \ge \left(\frac{\delta-1}{\delta}\right) c_3 \quad \text{for } \frac{c_3}{\delta} \le y(s) \le c_3, \ \ s \in [0, T-1]$$

and hence the condition (H_1^*) implies (H_1). Now assume that

$$\lim_{y\to 0} \sum_{s=0}^{T-1} \frac{f(s,y)}{y} < \left(\frac{\delta-1}{\delta}\right).$$

We have

$$\frac{1}{\|y\|}\sum_{s=0}^{T-1} f(s,y(s)) = \sum_{s=0}^{T-1} \frac{f(s,y(s))}{\|y\|} \le \sum_{s=0}^{T-1} \frac{f(s,y(s))}{y(s)}$$

for $s \in [0, T-1]$. Observe that $\|y\| \to 0$ if and only if $y(s)$ also tends to zero for all $s \in [0, T-1]$. Therefore, in view of (H$_2^*$) we have

$$\lim_{y\to 0} \frac{1}{\|y\|}\sum_{s=0}^{T-1} f(s,y(s)) \le \lim_{y(s)\to 0} \sum_{s=0}^{T-1} \frac{f(s,y(s))}{y(s)} < \left(\frac{\delta-1}{\delta}\right)$$

for all $s \in [0, T-1]$. Hence condition (H_2^*) implies (H_2). The proof is now complete. □

3. Application to Renewable Resource Dynamics-I

In this section, we shall apply the results developed in the previous section to investigate the existence of positive T-periodic solutions for the difference equation (1.2) representing dynamics of a renewable resource that is subjected to Allee effects.

Allee effects refer to a reduction in individual fitness at low population density that can lead to extinction [2, 3, 4, 5, 9, 14, 15, 19, 21, 31, 36, 38, 46]. It is a phenomenon in Biology characterized by a positive interaction between population density and the per-capita population growth rate in small populations. A strong Allee effect, where a population exhibits "Critical size density", below which the population declines on average, and above which it may increase. It is strongly related to the extinction vulnerability of populations. Any ecological mechanism that can lead to a positive relationship between a component of individual fitness and either the number or density of conspecifics can be termed a mechanism of the Allee effect [29, 45], or depensation [13, 18, 34], or negative competition effect [47]. A few mechanisms generating Allee effects in species dynamics have been suggested in the literature [5, 15]. There are several real world examples exhibiting the presence of Allee effects [8, 16, 25, 27]. Hence, system analysis in the presence of Allee effects has gained importance in real world problems in various fields such as population management [5], interacting species [7], biological invasions [11], marine systems [22], conservation biology [24], pest control, biological control [26], sustainable harvesting [35], and meta population dynamics [51]. A critical review of single species models subject to Allee effects can be found in [6].

Studying the consequences of Allee effects on a renewable resource under the influence of seasonal variations is a vital problem with real world applications. Recently, Padhi et al. [39] applied the Leggett-Williams multiple fixed point theorem to obtain sufficient conditions for the existence of at least two positive periodic solutions of a differential equation governing the dynamics of a renewable resource subject to Allee effects in a seasonally varying environment. The results obtained in [39] give estimates on the number of periodic solutions admitted by the model.

Describing species dynamics using periodic differential equations enables us to study the influence of seasonal variations on the species of interest. Periodicity and almost periodicity play important roles in problems associated with real world applications. In trying to analyze the consequences of such periodic or almost periodic variations in the environment, it is reasonable, as a first approximation, to consider the parameters involved to be periodic of the same period. Thus, a natural approach might then be to study the effects of periodic variations in the appropriate parameters of the model equations that have been used to describe the growth dynamics in constant environments such as in [10, 17, 39]. We note that an Allee effect refers to a decrease in a population growth rate at low population densities [4, 9, 19, 21, 36, 38, 46]. Classifications of the effects can be found in [5, 6].

Consider the transformation

$$y(n) = c(n)x(n).$$

The equation (1.2) is transformed to

(3.1)

$$\Delta x(n) = -\left(\frac{a(n)c^3(n)k(n)}{c(n+1)} + \frac{\Delta c(n)}{c(n+1)}\right)x(n) + \frac{a(n)c^3(n)}{c(n+1)}(1+k(n)-x(n))x^2(n),$$

where

$$k(n) = \frac{b(n)}{c(n)}.$$

Comparing (3.1) with (1.5) we have

$$A(n) = \frac{a(n)c^3(n)k(n) + \Delta c(n)}{c(n+1)}, \tag{3.2}$$

and

$$f(n,x) = \frac{a(n)c^3(n)}{c(n+1)}(1+k(n)-x)x^2. \tag{3.3}$$

Let us consider the Banach space X defined earlier. From (3.3), we have $f(n,0) = 0$, $f(n,x(n)) > 0$ for $0 < x(n) < 1 + k_m$ and $f(n,x(n)) < 0$ for $x(n) > 1 + k_M$, where $k_m = \min_{0\le n\le T-1} k(n)$ and $k_M = \max_{0\le n\le T-1} k(n)$. Hereafter we denote

$$M = \sum_{n=0}^{T-1} \frac{a(n)c^3(n)}{c(n+1)} \quad \text{and } N = \sum_{n=0}^{T-1} \frac{a(n)c^3(n)k(n)}{c(n+1)}.$$

Since (1.3) holds, then $0 < k(n) < 1$, and hence $M > N > 0$. From (3.3) we observe that $\lim_{x\to 0} \frac{1}{x} \sum_{n=0}^{T-1} f(n,x) = 0$ and hence (H_2^*) of Corollary 2.2 is satisfied by the equation (3.1). We have the following theorem.

Theorem 3.1. *If*

$$\frac{(M+N) + \sqrt{(M+N)^2 - 4M(\frac{\delta-1}{\delta})}}{2M} > \frac{\delta^2 - \frac{1}{\delta}}{M+N} \tag{3.4}$$

then equation (1.2) *has at least two positive T-periodic solutions.*

Proof. We shall use Corollary 2.2 to prove the theorem. From (3.2), it is easy to see that $1 - A(n) = \frac{c(n)}{c(n+1)}(1 - a(n)b(n)c(n)) > 0$. To complete the proof of theorem, it is enough to find the existence of a positive constant $c_3 > 0$ such that (H_1^*) holds.

Let us take

$$c_3 = \frac{(M+N) + \sqrt{(M+N)^2 - 4M(\frac{\delta-1}{\delta})}}{2M}$$

and define $c_2 = \frac{c_3}{\delta}$. Clearly $0 < c_2 < c_3$. It is easy to verify that $p = c_3$ is a solution of

$$-Mp^2 + (M+N)p - (\frac{\delta-1}{\delta}) = 0 \tag{3.5}$$

which is equivalent to

$$(1-p)p\sum_{s=0}^{T-1}\frac{a(s)c^3(s)}{c(s+1)} + p\sum_{s=0}^{T-1}\frac{a(s)c^3(s)k(s)}{c(s+1)} = \left(\frac{\delta-1}{\delta}\right).$$

The above equation can be written as $\sum_{s=0}^{T-1} f(s,p) = \left(\frac{\delta-1}{\delta}\right)p$, that is, $p = c_3$ satisfies

$$\sum_{s=0}^{T-1} f(s,c_3) = \left(\frac{\delta-1}{\delta}\right)c_3.$$

Next, we consider the inequality

$$\sum_{s=0}^{T-1} f(s,\frac{c_3}{\delta}) > \left(\frac{\delta-1}{\delta}\right)c_3. \tag{3.6}$$

Then we have

$$\sum_{s=0}^{T-1}\frac{a(s)c^3(s)}{c(s+1)}(1+k(s)-\frac{c_3}{\delta})\frac{c_3^2}{\delta^2} > \left(\frac{\delta-1}{\delta}\right)c_3.$$

The above inequality is equivalent to

$$-Mc_3^2 + (M+N)\delta c_3 - \delta^2(\delta-1) > 0.$$

Since $p = c_3$ is a solution of (3.5), the last inequality yields

$$c_3 > \frac{\delta^2 - \frac{1}{\delta}}{M+N}. \tag{3.7}$$

Therefore (3.6) will be satisfied if the root $p = c_3$ of (3.5) satisfies the inequality (3.7). Thus (H_1^*) will be satisfied if the parameters of the associated equation (3.1) satisfies (3.7) which is nothing but (3.4). This completes the proof. □

Remark 3.2. Note that Theorem 3.1 is verified only if M and N satisfy the inequality

$$(M+N)^2 - 4M(\frac{\delta-1}{\delta}) > 0.$$

Example 3.3. Consider the difference equation (1.2) with

$$a(n) = (1.2+(-1)^n)^2, \quad b(n) = \frac{(1.2+(-1)^n)}{12}, \quad c(n) = \frac{1}{(1.2+(-1)^n)}. \tag{3.8}$$

Then $a(n)$, $b(n)$ and $c(n)$ are 2-periodic functions. Now we have

$$k(n) = \frac{b(n)}{c(n)} = \frac{(1.2+(-1)^n)^2}{12} < 1$$

and

$$a(n)b(n)c(n) = \frac{(1.2+(-1)^n)^2}{12} < 1.$$

Further, we have

$$M = \frac{122}{11}, \quad N = \frac{11}{150}, \quad \text{and } \delta = 1.68.$$

Clearly $M > N > 0$, and

$$\frac{(M+N) + \sqrt{(M+N)^2 - 4M(\frac{\delta-1}{\delta})}}{2M} = 0.9689$$

and $\frac{\delta^2 - \frac{1}{\delta}}{M+N} = 0.1994948$. Therefore (3.4) is satisfied and hence (1.2) admits at least two positive periodic solutions with $a(n)$, $b(n)$ and $c(n)$ as given in (3.8).

Now, we provide a sufficient condition different from Eq. (3.4) for the existence of at least two positive T-periodic solution of (1.2). Since (1.2) can be rewritten as (1.4), we set

$$A_1(n) = a(n)b(n)c(n) \quad \text{and} \quad f(n, y(n)) = a(n)(b(n) + c(n) - y(n))y^2(n).$$

Further, (1.3) implies that $0 < A_1(n) < 1$. Set

$$\delta_1 = \left(\prod_{\theta=0}^{T-1}(1 - A_1(\theta))\right)^{-1} > 1$$

and

$$G(n,s) = \frac{\prod_{\theta=s+1}^{n+T-1}(1 - A_1(\theta))}{1 - \prod_{\theta=0}^{T-1}(1 - A_1(\theta))}, \quad \theta \in [n, n+T-1].$$

Lemma 3.4.

$$\sum_{s=n}^{n+T-1} G_1(n,s)A_1(s) = 1.$$

Proof. Let

$$\mu = \sum_{s=n}^{n+T-1} G_1(n,s)A_1(s) = \sum_{s=n}^{n+T-1} A_1(s)\frac{\prod_{\theta=s+1}^{n+T-1}(1 - A_1(\theta))}{1 - \prod_{\theta=0}^{T-1}(1 - A_1(\theta))}.$$

Setting $1 - A_1(n) = B(n)$, we can express μ as

$$\mu = \sum_{s=n}^{n+T-1}(1 - B(s))\frac{\prod_{\theta=s+1}^{n+T-1} B(\theta)}{1 - \prod_{\theta=0}^{T-1} B(\theta)}.$$

The proof of the lemma will be completed if we can show that

$$\sum_{s=n}^{n+T-1}(1 - B(s))\prod_{\theta=s+1}^{n+T-1} B(\theta) = 1 - \prod_{\theta=0}^{T-1} B(\theta) \tag{3.9}$$

holds. Indeed,

$$\sum_{s=n}^{n+T-1}(1 - B(s))\prod_{\theta=s+1}^{n+T-1} B(\theta) = \sum_{s=n}^{n+T-1}\left(\prod_{\theta=s+1}^{n+T-1} B(\theta) - \prod_{\theta=s}^{n+T-1} B(\theta)\right)$$

$$
\begin{aligned}
&= \left(\prod_{\theta=n+1}^{n+T-1} B(\theta) - \prod_{\theta=n}^{n+T-1} B(\theta)\right) \\
&\quad + \left(\prod_{\theta=n+2}^{n+T-1} B(\theta) - \prod_{\theta=n+1}^{n+T-1} B(\theta)\right) \\
&\quad + \left(\prod_{\theta=n+3}^{n+T-1} B(\theta) - \prod_{\theta=n+2}^{n+T-1} B(\theta)\right) + \cdots \\
&\quad + \left(\prod_{\theta=n+T}^{n+T-1} B(\theta) - \prod_{\theta=n+T-1}^{n+T-1} B(\theta)\right). \\
&= 1 - \prod_{\theta=n}^{n+T-1} B(\theta) = 1 - \prod_{\theta=0}^{T-1} B(\theta)
\end{aligned}
$$

implies that (3.9) holds. The proof is complete. □

Theorem 3.5. *Let the sequences $b(n), c(n)$ be bounded and*

$$(b_* + c_*)^2 > 4\delta_1^3 b^* c^* \tag{3.10}$$

hold. Then (1.2) *has at least two positive T-periodic solutions.*

Proof. We consider a Banach space X and a cone K on X as in Theorem 3.1. Choose

$$c_2 = \frac{(b_* + c_*) + \sqrt{(b_* + c_*)^2 - 4\delta_1^3 b^* c^*}}{2\delta_1^2} \quad \text{and} \quad c_3 = \delta_1 c_2.$$

Then $0 < c_2 < c_3$ and $\frac{c_2+c_3}{2} \in \{y(n) \in K(\psi, c_2, c_3);\ \psi(y(n)) > c_2\}$ is not empty. Further, for $y \in K(\psi, c_2, c_3)$ we have, using Lemma 3.4

$$
\begin{aligned}
\psi(A_1 y) &= \min_{n\in[0,T-1]} \sum_{s=n}^{n+T-1} G_1(n,s)a(s)[b(s)+c(s)-y(s)]y^2(s) \\
&\geq \frac{(b_* + c_*)c_2^2 - (\delta_1 c_2)^3}{b^* c^*} \sum_{s=n}^{n+T-1} G_1(n,s)A_1(s) \\
&= \frac{(b_* + c_*)c_2^2 - (\delta_1 c_2)^3}{b^* c^*} = c_2.
\end{aligned}
$$

Further,

$$\lim_{y\to 0} \frac{a(n)[b(n)+c(n)-y(n)]y^2(n)}{a(n)b(n)c(n)y(n)} = 0$$

implies the existence of a constant $c_1 \in (0, c_2)$. Observe that for the above choice of c_3, we have that $f_1(n,y) > 0$ for $0 \leq y \leq c_3$. The Green's kernel $G_1(n,s)$ is bounded by

$$0 \leq \frac{1}{\delta_1 - 1} \leq G_1(n,s) \leq \frac{\delta_1}{\delta_1 - 1}, \quad s \in [n, n+T-1].$$

Then

$$\psi(A_1y) \geq \frac{1}{\delta_1 - 1}\sum_{s=n}^{n+T-1} a(s)[b(s)+c(s)-y(s)]y^2(s)$$

implies that

$$\begin{aligned} c_3 &< \|A_1y\| \\ &\leq \frac{\delta_1}{\delta_1-1}\sum_{s=n}^{n+T-1} a(s)[b(s)+c(s)-y(s)]y^2(s) \\ &\leq \delta_1\psi(A_1y). \end{aligned}$$

Consequently,

$$\psi(A_1y) \geq \frac{1}{\delta_1}\|A_1y\| = \frac{c_2}{c_3}\|A_1y\|$$

holds. Hence, by Theorem 1.2, Eq. (1.2) has at least two positive T-periodic solutions. The theorem is proved. ☐

Consider Eq. (1.2) with $a(n), b(n)$ and $c(n)$ as given in (3.8). Clearly, $A_1(n) = \frac{(1.2+(-1)^n)^2}{12}$ and $T = 2$ implies that $\delta_1 = 1.035067$, $\delta_1^3 = 1.108935$. It is easy to verify that

$$(b_* + c_*)^2 = 0.385904499 < 4.060095 = 4\delta_1^3 b^* c^*$$

holds. Consequently, Theorem 3.5 cannot be applied to this example.

Example 3.6. We consider Eq. (1.2) with

$$a(n) = \frac{1}{10}\left(0.999999 + \frac{1}{1.000000}(-1)^n\right),$$

$$b(n) = \frac{1}{10}\left(0.9999999 + \frac{1}{1.0000000}(-1)^n\right)$$

and

$$c(n) = \frac{1}{10}\left(1.9999999 + \frac{1}{1.0000000}(-1)^n\right).$$

Here $T = 2$, $a^* = \frac{1}{10}$, $b^* = \frac{1}{10}$, $c^* = \frac{2}{10}$, $a_* = \frac{0.999998}{10}$, $b_* = \frac{0.9999998}{10}$ and $c_* = \frac{1.9999998}{10}$. A simple calculation shows that $\delta_1 = 1.0004012$, $(b_* + c_*)^2 = 0.08999$ and $4\delta_1^3 b^* c^* = 0.08009$. This in turn implies that (3.10) holds. Hence, by Theorem 3.5, Eq. (1.2) with $a(n)$, $b(n)$ and $c(n)$ considered in this example, has at least two positive T-periodic solutions.

On the other hand, $\delta = 1.0041$, $M = 0.0079$ and $N = 0.0039$ implies that

$$\frac{(M+N) + \sqrt{(M+N)^2 - 4M(\frac{\delta-1}{\delta})}}{2M} = 0.94936708 \text{ and } \frac{\delta^2 - \frac{1}{\delta}}{M+N} = 1.0423786$$

hold. Thus, the condition (3.4) fails to hold and hence Theorem 3.1 cannot be applied to (1.2) with the above considered $a(n)$, $b(n)$ and $c(n)$.

4. **Application to Renewable Resource Dynamics-II**

This section deals with the existence of at least two positive T-periodic solutions of the equations

$$\Delta y(n) = y(n)\left[a(n) - \frac{b(n)y(n-\tau)}{1+c(n)y(n-\tau)}\right], \tag{4.1}$$

$$\Delta y(n) = y(n)\left[a(n) - \frac{b(n)y(n-\tau)}{1+c(n)y(n-\tau)}\right] - qEy(n) \tag{4.2}$$

and

$$\Delta y(n) = y(n)\left[a(n) - \frac{b(n)y(n)}{1+c(n)y(n)}\right] - qEy(n), \tag{4.3}$$

where $a(n)$, $b(n)$ and $c(n)$ are positive T-periodic sequences and $\tau > 0$ is a real number.

Equations (4.1)–(4.3) are discrete analogue of the Michaelis Menton models of the forms

$$y'(t) = y(n)\left[a(t) - \frac{b(t)y(t-\tau)}{1+c(t)y(t-\tau)}\right], \tag{4.4}$$

$$y'(t) = y(t)\left[a(t) - \frac{b(t)y(t-\tau)}{1+c(t)y(t-\tau)}\right] - qEy(t), \tag{4.5}$$

and

$$y'(t) = y(t)\left[a(t) - \frac{b(t)y(t)}{1+c(t)y(t)}\right] - qEy(t) \tag{4.6}$$

respectively, where $a(t)$, $b(t)$ and $c(t)$ are positive T-periodic real valued functions and $\tau > 0$, and $T > 0$ is a real number.

Equation (4.4) is a Generalized Michaelis Menton type single species growth model [30, 44] where as (4.5) is a Generalized Michaelis Menton model with harvesting and (4.6) is a Generalized Michaelis Menton model with harvesting but no delay. Equations (4.4)–(4.6) has been studied extensively in the literature, see for example [30, 37, 44] and the references cited there.

It seems that few results exist in the literature for the existence of at least one positive T-periodic solutions of (4.3). Let $c(n) \equiv c$ be a constant. The Zeng [48] used Krasnoselskii fixed point theorem to prove that, if

$$0 < qE < \frac{1-\sigma}{T} \quad \text{and} \quad \frac{b^*}{c} + qE > \frac{1-\sigma}{\sigma^2 T} \tag{4.7}$$

hold, then (4.3) has at least one positive T-periodic solution, where $\sigma = \prod_{k=0}^{T-1}(1 + a(k))^{-1}$.

It follows from (4.7), that $qE \neq 0$ and hence the result cannot be applied to Eq. (4.1) with no delay.

In this section, we have made an attempt to find sufficient conditions for the existence of at least two positive T-periodic solutions of (4.1)–(4.3).

Remark 4.1. We note that Eqs. (4.2) and (4.3) can be rewritten as

$$\Delta y(n) = -qEy(n) + \left[a(n) - \frac{b(n)y(n-\tau)}{1+c(n)y(n-\tau)}\right] y(n) \tag{4.8}$$

and

$$\Delta y(n) = -qEy(n) + \left[a(n) - \frac{b(n)y(n)}{1+c(n)y(n)}\right] y(n) \tag{4.9}$$

respectively.

Setting $f_1(n,y) = \left[a(n) - \frac{b(n)y(n)}{1+c(n)y(n)}\right] y(n)$ we observe that $f_1(n,0) = 0$, $f_1(n,y) > 0$ for some $y \in (0,\mu)$, $\mu \in R$ and $f_1(n,y) \to -\infty$ as $y \to \infty$ if $b_* > a^*c^*$. In a similar way, if we set $f_2(n,y) = \left[a(n) - \frac{b(n)y(n-\tau)}{1+c(n)y(n-\tau)}\right] y(n)$, then $f_2(n,0) = 0$ and $f_2(n,y) > 0$ for some $y \in (0,\mu_1)$, $\mu_1 \in R$ and $f_2(n,y) \to -\infty$ as $y \to \infty$. The above calculation shows that the models (4.8) and (4.9) exhibit Allee effect if $b_* > a^*c^*$.

Equation (4.1) is equivalent to

$$y(n+1) = (1+a(n))y(n) - \frac{b(n)y(n-\tau)y(n)}{1+c(n)y(n-\tau)}.$$

Assuming that $y(n)$ is a positive T-periodic sequence, we observe that (4.1) is equivalent to

$$y(n) = \sum_{s=n}^{n+T-1} G_2(n,s) \left[\frac{b(s)y(s-\tau)y(s)}{1+c(s)y(s-\tau)}\right],$$

where

$$G_2(n,s) = \frac{\prod_{\theta=s+1}^{n+T-1}(1+a(\theta))}{\prod_{\theta=0}^{T-1}(1+a(\theta)) - 1}, \quad s \in [n, n+T-1]$$

is the Green's kernel satisfying the property

$$0 < \alpha = \frac{\delta_2}{1-\delta_2} \le G_2(n,s) \le \frac{1}{1-\delta_2} = \beta \text{ and } \delta_2 = \left(\prod_{\theta=0}^{T-1}(1+a(\theta))\right)^{-1} < 1.$$

Lemma 4.2.

$$\sum_{s=n}^{n+T-1} G_2(n,s)a(s) = 1.$$

Proof. Let

$$\mu = \sum_{s=n}^{n+T-1} G_2(n,s)a(s) = \sum_{s=n}^{n+T-1} a(s) \frac{\prod_{\theta=s+1}^{n+T-1}(1+a(\theta))}{\prod_{\theta=0}^{T-1}(1+a(\theta)) - 1}.$$

Set $1 + a(n) = B(n)$. Then μ can be written as

$$\mu = \sum_{s=n}^{n+T-1} (B(s)-1) \frac{\prod_{\theta=s+1}^{n+T-1} B(\theta)}{\prod_{\theta=0}^{T-1} B(\theta) - 1}.$$

To complete the proof of the lemma, it is enough to show that

$$\sum_{s=n}^{n+T-1}(B(s)-1)\prod_{\theta=s+1}^{n+T-1}B(\theta)=\prod_{\theta=0}^{T-1}B(\theta)-1.$$

Clearly

$$\begin{aligned}\sum_{s=n}^{n+T-1}(B(s)-1)\prod_{\theta=s+1}^{n+T-1}B(\theta) &= \sum_{s=n}^{n+T-1}\left(\prod_{\theta=s}^{n+T-1}B(\theta)-\prod_{\theta=s+1}^{n+T-1}B(\theta)\right)\\ &= \left(\prod_{\theta=n}^{n+T-1}B(\theta)-\prod_{\theta=n+1}^{n+T-1}B(\theta)\right)\\ &\quad+\left(\prod_{\theta=n+1}^{n+T-1}B(\theta)-\prod_{\theta=n+2}^{n+T-1}B(\theta)\right)\\ &\quad+\left(\prod_{\theta=n+2}^{n+T-1}B(\theta)-\prod_{\theta=n+3}^{n+T-1}B(\theta)\right)+\cdots\\ &\quad+\left(\prod_{\theta=n+T-1}^{n+T-1}B(\theta)-\prod_{\theta=n+T}^{n+T-1}B(\theta)\right).\\ &= \prod_{\theta=n}^{n+T-1}B(\theta)-1=\prod_{\theta=0}^{T-1}B(\theta)-1\end{aligned}$$

holds. The proof is complete. □

Theorem 4.3. *Suppose that $a(n)$, $b(n)$ and $c(n)$ are bounded sequences and*

$$b_*>a^*c^*$$

holds. Then (4.1) *has at least two positive T-periodic solutions.*

Proof. Let X be the space of all positive T-periodic sequences under the norm

$$\|y\|=\max_{0\le n\le T-1}|y(n)|.$$

Then X forms a Banach space. On the space X, we define a cone K by

$$K=\{y\in X;\ y\ge\delta_2\|y\|,\ n\in[0,T-1]\}.$$

On the cone K, we define an operator A_2 by

$$(A_2y)(n)=\sum_{s=n}^{n+T-1}G_2(n,s)\frac{b(s)y(s-\tau)y(s)}{1+c(s)y(s-\tau)}.$$

We consider the nonnegative concave function ψ as in Theorem2.1.

Setting $f(n,y)=\frac{b(n)y(n-\tau)y(n)}{1+c(n)y(n-\tau)}$, we observe that $\limsup_{y\to\infty}\frac{f(n,y)}{a(n)y(n)}>\frac{b_*}{a^*c^*}$, which in turn implies that there exists a positive constant $c_2>0$ such that

$$\frac{b(n)y(n-\tau)y(n)}{1+c(n)y(n-\tau)}>\frac{b_*}{a^*c^*}c_2\ge c_2,$$

holds for $c_2 \le y \le c_3$ and $0 \le n \le T-1$, where $c_3 = \frac{c_2}{\delta_2}$. Hence, for $y \in K(\psi, c_2, c_3)$, we have

$$\begin{aligned}\psi(A_2 y) &= \min_{0\le n\le T-1} \sum_{s=n}^{n+T-1} G_2(n,s)\frac{b(s)y(s-\tau)y(s)}{1+c(s)y(s-\tau)} \\ &\ge c_2 \sum_{s=n}^{n+T-1} G_2(n,s)a(s) = c_2.\end{aligned}$$

To complete the proof of the theorem, it suffices to show, in view of the Leggett-Williams multiple fixed point theorem, Theorem 1.2, that the existence of a positive constant c_1, $c_1 \in (0, c_2)$ such that the condition (ii) of Theorem 1.2 holds. Since $\limsup_{y\to 0} \frac{f(n,y)}{a(n)y} = 0$, then there exists $0 < \epsilon < 1$ and $\delta_1 \in (0, c_2)$ such that $f(n,y) < \epsilon a(n) y$ for $0 < y < \delta_1$. Now choosing, $\delta_1 = c_1$ and using Lemma 4.2, we can prove the condition (ii) of Theorem 1.2. This completes the proof of the theorem. $\square$

Theorem 4.4. *Let $c(n) \equiv c > 0$ be a constant, $qET < 1 - \delta_2$ and*

$$\delta_2^2(1-\delta_2)\sum_{s=0}^{T-1} b(s) > c\left(\frac{1}{\delta_2}(1-\delta_2) - qET\right)$$

hold. Then (4.2) *has at least two positive T-periodic solutions.*

Proof. Consider the Banach space X and a cone K and a non negative concave functional ψ as in Theorem 4.3. We define an operator $A_3(y)$ on K by

$$(A_3 y)(n) = \sum_{s=n}^{n+T-1} G_2(n,s)\left[\frac{b(s)y(s)y(s-\tau)}{1+cy(s-\tau)} + qEy(s)\right]$$

Choose positive constants c_1, c_2 and c_3 be such that

$$c_1 \in \left(0, \frac{1-\delta_2 - qET}{\sum_{s=0}^{T-1} b(s)}\right], \; c_2 = \frac{1}{\delta_2^2}\frac{1}{\sum_{s=0}^{T-1} b(s)}\left(\frac{1-\delta_2}{\delta_2} - qET\right) \text{ and } c_3 = \frac{c_2}{\delta_2}.$$

Since $\delta_2 < 1$, then $0 < c_1 < c_2 < c_3$. It is easy to verify that $A_3(K) \subseteq K$ and is completely continuous. For $y \in \overline{K_{c_1}}$, we have

$$\begin{aligned}\|A_3 y\| &= \max_{0\le n\le T-1} \sum_{s=n}^{n+T-1} G_2(n,s)\left[\frac{b(s)y(s)y(s-\tau)}{1+cy(s-\tau)} + qEy(s)\right] \\ &\le \frac{1}{1-\delta_2}\sum_{s=0}^{T-1}[b(s)y(s)y(s-\tau) + qEy(s)] \\ &\le \frac{c_1}{1-\delta_2}\left[c_1\sum_{s=0}^{T-1} b(s) + qE\right] \\ &\le c_1.\end{aligned}$$

Hence $A_3 y \in \overline{K}_{c_1}$.

Next, for each $y \in K$ with $c_2 \le \psi(y)$ and $\|y\| \le c_3$, we have $c_2 \le \|y\| \le c_3$ and $\delta_2 c_2 \le y(n-\tau) \le \frac{c_2}{\delta_2}$, $n \in [0, T-1]$. Then for $y \in K(\psi, c_2, c_3)$, we have

$$\begin{aligned}
\psi(A_3 y) &\ge \frac{\delta_2}{1-\delta_2} \sum_{s=0}^{T-1} \left[\frac{b(s)y(s)y(s-\tau)}{1+cy(s-\tau)} + qEy(s) \right] \\
&\ge \frac{\delta_2}{1-\delta_2} \sum_{s=0}^{T-1} \left[\frac{b(s)c_2\delta_2 c_2}{1+c\frac{c_2}{\delta_2}} + qEc_2 \right] \\
&\ge \frac{c_2\delta_2}{1-\delta_2} \left[\frac{\delta_2^2 c_2}{\delta_2 + cc_2} \sum_{s=0}^{T-1} b(s) + qEc_2 \right] \\
&\ge c_2.
\end{aligned}$$

Now, consider

$$\begin{aligned}
\psi(A_3 y) &= \min_{0 \le n \le T-1} \sum_{s=n}^{n+T-1} G_2(n,s) \left[\frac{b(s)y(s)y(s-\tau)}{1+cy(s-\tau)} + qEy(s) \right] \\
&\ge \frac{\delta_2}{1-\delta_2} \sum_{s=0}^{T-1} \left[\frac{b(s)y(s)y(s-\tau)}{1+cy(s-\tau)} + qEy(s) \right].
\end{aligned}$$

Then for $c_3 < \|A_3 y\|$, we have

$$c_3 \le \|A_3 y\| \le \frac{1}{1-\delta_2} \sum_{s=0}^{T-1} \left[\frac{b(s)y(s)y(s-\tau)}{1+cy(s-\tau)} + qEy(s) \right]$$

The above two inequalities yield

$$\psi(A_3 y) \ge \delta_2 \|A_3 y\| = \frac{c_2}{c_3} \|A_3 y\|.$$

Hence by Theorem 1.2, Eq. (4.2) has at least two positive T-periodic solutions. The theorem is proved. □

Now, we consider the equation

$$\Delta y(n) = y(n) \left[a(n) - \frac{b(n)y(n)}{1+c(n)y(n)} \right]$$

whose growth law obeys Michaelis-Menton type growth equation. Moreover, we assume that the population is subject to harvesting. Then, under the catch-per-unit-effort hypothesis, the harvest population's growth equation can be expressed as (4.3), where $a(n)$, $b(n)$ and $c(n)$ are positive T-periodic real sequences, q and E are positive constants denoting the catch-ability-coefficient and the harvesting effort, respectively.

As pointed earlier, Zeng [48] assumed the condition (4.7) to obtain one positive T-periodic solution of (4.3). In the following theorem, we use the Leggett-Williams multiple fixed point theorem, Theorem 1.2 to obtain a new easily verifiable sufficient condition different from (4.7), for the existence of at least two positive T-periodic solutions of (4.3).

Theorem 4.5. *Let $a(n)$, $b(n)$ and $c(n)$ be bounded sequences. If*

$$a_* > qE \text{ and } b_* \geq c^*(a^* - qE)$$

hold, then (4.3) *has at least two positive T-periodic solutions.*

Proof. We can express Eq. (4.3) in the form

$$\Delta y(n) = \overline{A}(n)y(n) - \frac{b(n)y^2(n)}{1+c(n)y(n)},$$

where $\overline{A}(n) = a(n) - qE$. Clearly, $a_* > qE$ implies that $\overline{A}(n) > 0$, $n \in [0, T-1]$. We consider a Banach space X as in previous theorems and a cone K in X as in Theorem 4.3. On the cone K, we define an operator A_4 by

$$(A_4 y)(n) = \sum_{s=n}^{n+T-1} G_3(n,s)\left[\frac{b(s)y^2(s)}{1+c(s)y^2(s)}\right],$$

where

$$G_3(n,s) = \frac{\prod_{\theta=s+1}^{n+T-1}(1+\overline{A}(\theta))}{\prod_{\theta=0}^{T-1}(1+\overline{A}(\theta)) - 1}$$

is the Green's Kernel bounded by

$$\frac{\delta_3}{1-\delta_3} = \alpha \leq G_3(n,s) \leq \frac{1}{1-\delta_3} = \beta, \quad s \in [n, n+T-1]$$

and

$$\delta_3 = \prod_{\theta=0}^{T-1}(1+\overline{A}(\theta))^{-1} < 1.$$

It is easy to prove that $A_4(K) \subset K$ and A_4 is completely continuous on K. We consider a nonnegative continuous concave functional ψ as in Theorem 4.3.

Since

$$\lim_{y\to\infty} \frac{b(n)y(n)}{\overline{A}(n)(1+c(n)y(n))} = \lim_{y\to\infty} \frac{b(n)y(n)}{(a(n)-qE)(1+c(n)y(n))} > \frac{b_*}{c^*(a^*-qE)},$$

then there exists a positive constant $c_2 > \frac{a^*-qE}{b_*} > 0$ such that

$$\frac{b(n)y^2(n)}{(a(n)-qE)(1+c(n)y(n))} > \frac{b_*}{c^*(a^*-qE)} \geq c_2$$

holds. Set $c_3 = \frac{c_2}{\delta_3}$. Then $0 < c_2 < c_3$. For $y \in K(\psi, c_2, c_3)$, we have, using Lemma 4.2, that

$$\begin{aligned}\psi(A_4 y) &= \min_{n\in[0,T-1]} \sum_{s=n}^{n+T-1} G_3(n,s)\left[\frac{b(s)y^2(s)}{1+c(s)y(s)}\right] \\ &\geq c_2.\end{aligned}$$

Next, we choose a constant $c_1 \in (0, \frac{a_* - qE}{b^*})$. Then $0 < c_1 < c_2$. Further, for $y \in \overline{K}_{c_1}$, we have, again using Lemma 4.2, that

$$\begin{aligned}\|A_4 y\| &= \max_{n \in [0,T-1]} \sum_{s=n}^{n+T-1} G_3(n,s) \left[\frac{b(s) y^2(s)}{1 + c(s) y(s)} \right] \\ &< \frac{b^* c_1^2}{(a_* - qE)} \sum_{s=n}^{n+T-1} G_3(n,s) \overline{A}(s) \\ &= \frac{b^* c_1}{(a_* - qE)} . c_1 < c_1.\end{aligned}$$

Further,

$$c_3 < \|A_4 y\| \le \frac{1}{1 - \delta_3} \sum_{s=n}^{n+T-1} \frac{b(s) y^2(s)}{1 + c(s) y(s)}$$

implies that

$$\psi(A_4 y) \ge \delta_3 \|A_4 y\| = \frac{c_2}{c_3} \|A_4 y\|,$$

which in view of the Theorem 1.2, assures that (4.3) has at least two positive T-periodic solutions. Hence the theorem is proved. $\square$

Example 4.6. Consider Eq. (4.3) with $a(n) = 3 + (-1)^n$, $b(n) = 3 + (-1)^n$, $c(n) = \frac{3 + (-1)^n}{8}$ and $qE = 1$. Here $T = 2$, $a^* = 4$, $a_* = 2$, $b^* = 4$, $b_* = 2$, $c^* = \frac{1}{2}$ and $c_* = \frac{1}{4}$. Then

$$\sigma = \prod_{\theta=0}^{1} (1 + a(\theta)) = \frac{1}{15} < 1.$$

Observe that $a_* = 2 > 1 = qE$ and $c^*(a^* - qE) = \frac{1}{2}(4 - 1) = \frac{3}{2} < 2 = b_*$ implies, by Theorem 4.5 that (4.3) has at least two positive T-periodic solutions with the above choice of $a(n)$, $b(n)$ and $c(n)$. On the other hand,

$$\frac{1 - \sigma}{T} = \frac{1 - \frac{1}{15}}{2} = \frac{1}{2} . \frac{14}{15} = \frac{7}{15} < 1 = qE$$

implies that (4.7) fails to hold and hence the result due to Zeng [48] cannot be applied to this example.

Our final result give a sufficient condition for the nonexistence of positive T-periodic solution of (4.3).

Corollary 4.7. *Let $a(n)$ be a bounded sequence. If $a^* < qE < 1$, then (4.3) has no positive T-periodic solutions.*

Proof. We can rewrite Eq. (4.3) as (4.9). We consider the Banach space X as in previous theorem where as a cone K in X by

$$K = \{y;\, y \in X,\, y(n) \ge (1 - qE)^T \|y\|\}.$$

Further, we define an operator A_5 on X by

$$(A_5 y)(n) = \sum_{s=n}^{n+T-1} G_5(n,s)\left[a(s) - \frac{b(s)y(s)}{1+c(s)y(s)}\right] y(s),$$

where

$$G_5(n,s) = \frac{(1-qE)^{n+T-s-1}}{1-(1-qE)^T}$$

is the Green's Kernel satisfying the property $\sum_{s=n}^{n+T-1} G(n,s) = \frac{1}{qE}$.

Notice that the existence of a positive T-periodic solution of (4.3) is equivalent to the existence of a fixed point of A_5 in K. If possible, suppose that $y(n)$ is a positive T-periodic solution of (4.3). Then $y(n) = (A_5 y)(n)$ for all $y \in K$. This in turn, we have

$$\begin{aligned}
\|y\| = \|A_5 y\| &= \max_{0\le n\le T-1} \sum_{s=n}^{n+T-1} G_5(n,s)\left[a(s) - \frac{b(s)y(s)}{1+c(s)y(s)}\right] y(s) \\
&\le \max_{n\in[0,T-1]} \sum_{s=n}^{n+T-1} G_5(n,s) a^* \|y(n)\| \\
&\le \|y\| \max_{n\in[0,T-1]} \sum_{s=n}^{n+T-1} G_5(n,s) qE \\
&\le \|y\|,
\end{aligned}$$

a contradiction. Hence (4.3) has no eventually positive T-periodic solutions. This proves the corollary. □

5. **Remark**

In this article, we have examined the existence of at least two positive T-periodic solutions for a scalar difference equation which represents dynamics of a renewable resource that is subjected to Allee effects. It would be interesting to develop results that identify the exact number of positive periodic solutions admitted by the considered model and study their stability nature.

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