Dynamic Systems and Applications 23 (2014) 431-446

POSITIVE SOLUTIONS OF NONLOCAL CONTINUOUS SECOND ORDER BVP'S

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. A boundary value problem on the half-line to a class of second order differential equations is considered. In particular, the existence of solutions which start at the origin, are positive on the real half-line and tend to a nonzero constant as t tends to infinity, is studied. The solvability of this BVP is accomplished by a new approach which combines, in a suitable way, two separated problems on $[0, 1]$ and $[1, \infty)$ and uses some continuity arguments.

AMS (MOS) Subject Classification. Primary 34B10, Secondary 34B15.

1. INTRODUCTION

In this paper we study bounded solutions to a class of second order differential equations. We concentrate on solutions starting at the origin, having a nonzero limit as $t \to \infty$, and are positive for any $t > 0$.

To introduce the investigated problem, consider the Emden-Fowler differential equation

$$x'' + p(t)|x|^\gamma \text{sgn } x = 0 \quad (t \geq 0), \tag{EF}$$

where $\gamma > 0$, $\gamma \neq 1$, p is a nonnegative continuous function for $t \geq 0$ and $p \not\equiv 0$. Equation (EF) has solutions which approach a nonzero constant if and only if

$$\int_0^\infty \left(\int_t^\infty p(r)dr \right) dt < \infty,$$

see, e.g., [21, Theorems 17.1, 17.2]. The question is whether an eventually positive (bounded) solution of (EF) can have zeros on $[0, \infty)$. The answer can be negative, because, under certain assumptions, every solution of (EF) with a zero is oscillatory,

The first author is supported by the grant GAP 201/11/0768 of the Czech Grant Agency.
Received June 2013

see e.g. [1, Theorem 4, claim *(ii)*] or [18, Theorem 18.4 and its proof]. For instance, any solution with a zero of the equation

$$x'' + \frac{1}{4(t+1)^{(\beta+3)/2}}|x|^{\gamma}\operatorname{sgn} x = 0 \quad \gamma > 1,\ t \geq 0,$$

is oscillatory. Another example can be found in [22, pages 102–103].

When $p(t) \equiv 0$ for large t, for simplicity say for $t \geq 1$, then this fact does not occur. Indeed, in this case (EF) has bounded solutions x such that

$$x(0) = 0, \quad x(t) > 0 \text{ for } t > 0, \quad x(t) \equiv x(1) \text{ for } t \geq 1.$$

This follows from the existence of positive solutions x of (EF) in $[0,1]$, satisfying the boundary condition $x(0) = x'(1) = 0$, see, e.g., [22] or [17, Corollary 5.2, Corollary 5.5].

The interesting problem is how equation (EF), with $p(t) \equiv 0$ for large t, can be perturbed so as to admit solutions which start from the origin, are globally positive for $t > 0$ and have a nonzero limit as $t \to \infty$. This question leads to the following boundary value problem [BVP] on the half-line.

Consider the class of second order nonlinear differential equations

$$\big(a(t)\Phi(x')\big)' + f(t,x) = \lambda b(t) F(x) \quad (t \geq 0), \tag{1.1}$$

as a perturbation of equation

$$\big(a(t)\Phi(x')\big)' + f(t,x) = 0 \quad (t \geq 0).$$

We assume that $\Phi(u) = |u|^{\alpha}\text{sgn } u$, $0 < \alpha \leq 1$, a is a positive continuous function on $[0,\infty)$, f is a continuous function on $[0,\infty) \times [0,\infty)$ such that $f(t,u) \geq 0$ on $[0,1] \times [0,\infty)$, $f(t,u) \equiv 0$ on $[1,\infty) \times [0,\infty)$, and there exists $[T_1,T_2] \subseteq [0,1]$ such that $\min_{t\in[T_1,T_2]} f(t,u) > 0$ for any $u > 0$.

We also assume that $\lambda > 0$ is a real parameter, b is a continuous function on $[0,\infty)$ such that $b(t) \equiv 0$ on $[0,1]$, and F is a positive continuous function on $[0,\infty)$ and locally Lipschitz on $(0,\infty)$.

Observe that the function b is allowed to change its sign. In recent years, the existence of positive solutions to second order BVPs has been extensively studied. We refer to [3, 11, 15, 19] and references therein for BVPs with sign-changing coefficients, to [10, 16] for recent interesting contributions on BVPs with a positive parameter and to [8, 9] for the case with a general Φ.

The aim of our paper is to prove the existence of solutions x for (1.1) satisfying the nonlocal conditions

$$x(0) = 0, \quad x'(1) \leq 0, \quad x(t) > 0 \text{ for } t > 0, \quad 0 < \lim_{t\to\infty} x(t) < \infty, \tag{1.2}$$

under the following assumptions

$$\int_1^\infty \frac{1}{a^{1/\alpha}(t)}\,dt = \infty, \quad \int_1^\infty |b(t)|dt < \infty, \tag{1.3}$$

$$J = \int_1^\infty \frac{1}{a^{1/\alpha}(t)} \left(\int_t^\infty |b(r)|dr \right)^{1/\alpha} dt < \infty. \tag{1.4}$$

Due to the lack of useful lower bounds for solutions of (1.1)–(1.2), the solvability of this BVP is accomplished by a new approach, which combines two separated problems on $[0,1]$ and $[1,\infty)$, respectively, and uses some continuity arguments for solutions of both problems. This idea is suggested by [14], in which a similar method is employed for studying certain BVPs on compact intervals. Here, this approach is adapted in a suitable way for considering asymptotic boundary conditions. From this point of view, this method extends a similar one in [20], which deals with the existence of solutions x satisfying $x(0) = \lim_{t\to\infty} x(t) = 0, x(t) > 0$ on $(0,\infty)$.

Roughly speaking, the solvability of (1.1)–(1.2) will be obtained by pasting solutions of the first BVP, which satisfy suitable terminal conditions at $t = 1$, with solutions of the second BVP which satisfy suitable initial conditions at $t = 1$. In particular, these auxiliary BVPs are:

$$\begin{cases} (a(t)\Phi(x'))' + f(t,x) = 0, & t \in [0,1] \\ x(0) = 0, \ \gamma x(1) + \delta x'(1) = 0, \end{cases} \tag{1.5}$$

where $\gamma\delta = 0$, $\gamma + \delta > 0$, and

$$\begin{cases} (a(t)\Phi(x'))' = \lambda b(t)F(x), & t \in [1,\infty) \\ x(1) = c, \quad x(t) > 0, \quad 0 < \lim_{t\to\infty} x(t) < \infty, \end{cases} \tag{1.6}$$

where $\lambda > 0$.

Our results for both auxiliary BVPs (1.5), (1.6) are interesting by themselves, independently of the problem investigated here. The one for (1.5) is a generalization of [13, 23]. Our existence results for (1.6) complements some ones in [7], where globally positive bounded solutions for equation with the one-dimensional curvature operator have been studied, in [11], where similar problems are investigated for equations with a general Φ and in [12], where the uniqueness of globally positive bounded solutions is examined when $\int_1^\infty a^{-1/\alpha}(t)dt < \infty$.

We close this section introducing the following notations. Denote by Φ^* the inverse map of Φ, that is $\Phi^*(u) = |u|^{1/\alpha}\mathrm{sgn}\, u$. We will use either the notation $u^{1/\alpha}(t)$ or $\Phi^*(u(t))$, according to the function u is nonnegative or it changes sign. Let b_+, b_- be respectively the positive and the negative part of b, i.e., $b_+(t) = \max\{b(t), 0\}$,

$b_-(t) = -\min\{b(t), 0\}$ and set

$$B_+ = \int_1^\infty b_+(r)dr, \quad B_- = \int_1^\infty b_-(r)dr.$$

If x is a solution of (1.1), then

$$x^{[1]}(t) = a(t)\Phi(x'(t))$$

denotes its quasiderivative. Finally, let

$$\underline{f}_0 = \lim_{u\to 0+}\left(\min_{t\in[T_1,T_2]} \frac{f(t,u)}{\Phi(u)}\right), \quad \overline{f}_0 = \lim_{u\to 0+}\left(\max_{t\in[0,1]} \frac{f(t,u)}{\Phi(u)}\right),$$

and

$$\underline{f}_\infty = \lim_{u\to \infty}\left(\min_{t\in[T_1,T_2]} \frac{f(t,u)}{\Phi(u)}\right), \quad \overline{f}_\infty = \lim_{u\to \infty}\left(\max_{t\in[0,1]} \frac{f(t,u)}{\Phi(u)}\right).$$

2. BVP ON $[0,1]$

In this section we consider the equation

$$\big(a(t)\Phi(x')\big)' + f(t,x) = 0, \quad t\in[0,1]. \tag{2.1}$$

Theorem 2.1. *Assume that f is either superlinear, i.e.*

$$\overline{f}_0 = 0 \quad \textit{and } \underline{f}_\infty = \infty, \tag{2.2}$$

or sublinear, i.e.

$$\underline{f}_0 = \infty \quad \textit{and} \quad \overline{f}_\infty = 0. \tag{2.3}$$

Then (2.1) *has two solutions y and w, positive on $(0,1)$ and such that*

$$y(0) = y'(1) = 0, \quad w(0) = w(1) = 0.$$

The proof of Theorem 2.1 is based on the Krasnoselskii fixed point theorem on cones. The argument is similar to the one given in [23, Theorem 3], see also [13, Theorem 1], with suitable changes. For reader's convenience, we sketch here only the proof for the solution y of (2.1) satisfying $y(0) = y'(1) = 0$, $y(t) > 0$ for $t\in(0,1)$ under the assumption that f is sublinear. In this particular case, the following Lemma is needed.

Lemma 2.2. *Let ϕ be a continuous function on $[0,1]\times[0,\infty)$, such that*

$$\lim_{u\to\infty}\left(\max_{t\in[0,1]}\phi(t,u)\right) = \infty.$$

Then for every $u_0 > 0$ fixed, there exists $u_1 \ge u_0$ such that

$$\phi(t,u) \le \max_{t\in[0,1]}\phi(t,u_1), \ \textit{for every } t\in[0,1] \textit{ and } u\in[0,u_1].$$

Proof. Let $\varphi(u) = \max_{t\in[0,1]} \phi(t,u)$, $u \geq 0$. Then φ is continuous and satisfies $\lim_{u\to\infty} \varphi(u) = \infty$. Let $u_0 > 0$ be fixed, and assume $\max_{u\in[0,u_0]} \varphi(u) = L > \varphi(u_0)$. Since φ is unbounded, there exists $u_1 > u_0$ such that $\varphi(u_1) = L$ and $\varphi(u) \leq L$ for $u \in [u_0, u_1]$. Then $\varphi(u) \leq \varphi(u_1)$ for all $u \in [0, u_1]$, that is

$$\max_{t\in[0,1]} \phi(t,u) \leq \max_{t\in[0,1]} \phi(t,x_1), \text{ for every } u \in [0,u_1],$$

and the assertion follows. □

Proof of Theorem 2.1. *Step 1.* Let $a(t) \equiv 1$. Let K be the cone in $C[0,1]$ of nonnegative concave functions, and for every $u \in K$ define the operator

$$\Psi(u)(t) = \int_0^t \left(\int_s^1 f(r,u(r))dr\right)^{1/\alpha} ds, \qquad 0 \leq t \leq 1.$$

Since the derivative of $\Psi(u)$ is nonincreasing on $(0,1)$, the function $\Psi(u)$ is nonnegative concave for every $u \in K$. Thus, Ψ maps K into itself. Further,

$$\Psi(u)(0) = \frac{d}{dt}\,\Psi(u)(t)|_{t=1} = 0$$

and every fixed point of Ψ is a solution of (2.1). Let us show that there exist two open sets Ω_1, Ω_2 in $C[0,1]$, $\bar{\Omega}_1 \subset \Omega_2$, such that $\|\Psi(u)\| \geq \|u\|$ for every $u \in K \cap \partial\Omega_1$, and $\|\Psi(u)\| \leq \|u\|$ for every $u \in K \cap \partial\Omega_2$, where $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Recalling that f is positive on $[T_1,T_2]$, let $\delta \in (0,1/2)$ be such that $[T_1,T_2] \subseteq [\delta, 1-\delta]$ and let $L = \int_{T_1}^{T_2} (T_2 - s)^{1/\alpha} ds$. Since f satisfies the sublinearity assumption (2.3), fixed $m > 1/(\delta L)$, there exists $R_1 > 0$ such that $f(t,u) \geq \Phi(mu)$ for every $u \in [0,R_1]$, $t \in [T_1,T_2]$. Put $\Omega_1 = \{u \in C[0,1] : \|u\| < R_1\}$. Fixed $u \in K$, $\|u\| = R_1$, since u is a concave function, we have $u(t) \geq \delta R_1$ for every $t \in [\delta, 1-\delta]$. Moreover,

$$\begin{aligned}\|\Psi(u)\| &= \int_0^1 \left(\int_s^1 f(r,u(r))dr\right)^{1/\alpha} ds \geq \int_{T_1}^{T_2} \left(\int_s^{T_2} f(r,u(r))dr\right)^{1/\alpha} ds \\ &\geq \int_{T_1}^{T_2} \left(\int_s^{T_2} \Phi(mu)(r)dr\right)^{1/\alpha} ds \geq m\delta R_1 \int_{T_1}^{T_2} (T_2-s)^{1/\alpha} ds = m\delta R_1 L > R_1,\end{aligned}$$

that is, $\|\Psi(u)\| > \|u\|$ for every $u \in K \cap \partial\Omega_1$.

Again in view of (2.3), fixed $0 < \varepsilon < 1$, there exists $R_0 > 0$ such that $f(t,u) \leq \Phi(\varepsilon u)$ for every $u \geq R_0$, $t \in [0,1]$.

In order to define the open set Ω_2, we distinguish the cases in which f is unbounded or bounded in $[0,1] \times [0,\infty)$.

If f is unbounded, then by Lemma 2.2, there exist $R_2 > R_0 + R_1$ such that $f(t,u) \leq \max_{t\in[0,1]} f(t,R_2)$ for every $u \in [0,R_2]$. Put $\Omega_2 = \{u \in C[0,1] : \|u\| < R_2\}$.

Fixed $u \in K$, $\|u\| = R_2$, we have

$$\|\Psi(u)\| \le \int_0^1 \left(\int_0^1 f(r,u(r))dr\right)^{1/\alpha} ds \le \left(\int_0^1 \max_{r\in[0,1]} f(r,R_2)dr\right)^{1/\alpha}$$
$$\le \left(\int_0^1 \Phi(\varepsilon R_2)dr\right)^{1/\alpha} = \varepsilon R_2 < R_2.$$

If f is bounded in $[0,1]\times[0,\infty)$, put $M = \sup f(t,u)$ for $(t,u) \in [0,1]\times[0,\infty)$ and let $R_2 > R_1 + M^{1/\alpha}$. Put $\Omega_2 = \{u \in C[0,1] : \|u\| < R_2\}$. Fixed $u \in K$, $\|u\| = R_2$ we have

$$\|\Psi(u)\| \le \int_0^1 \left(\int_0^1 f(r,u(r))dr\right)^{1/\alpha} ds \le M^{1/\alpha} < R_2.$$

Therefore, in both cases, we have $\|\Psi(u)\| < \|u\|$ for every $u \in K \cap \partial\Omega_2$.

A standard calculation yields the complete continuity of Ψ in K. Hence, the Krasnoselskii compression theorem on cones can be applied, leading to the existence of a fixed point y of Ψ in $K\cap(\bar{\Omega}_2\setminus\Omega_1)$. Since y is concave and $0 < R_1 \le \|y\| \le R_2$, we have $y(t) > 0$ in $(0,1]$ and, clearly, y is a solution of (2.1), satisfying $y(0) = y'(1) = 0$.

Step 2. Put

$$A(t) = \int_0^t \frac{1}{a^{1/\alpha}(t)},$$

and consider the change of the variable

$$\tau(t) = A(t)/A(1).$$

Then y is a solution of (2.1), satisfying $y(0) = y'(1) = 0$, if and only if z is a solution of the BVP

$$\begin{cases} (\Phi(z'))' + \bar{f}(\tau, z) = 0, & \tau \in [0,1] \\ z(0) = 0,\ z'(1) = 0, \end{cases} \tag{2.4}$$

where $z(\tau) = y(t(\tau))$, $\bar{f}(\tau, z(\tau)) = A(1)^\alpha f(t(\tau), y(t(\tau)))$ and $t(\tau)$ is the inverse function of τ. It is easy to see that $\bar{f}$ satisfies the assumption of Theorem 2.1, and from the first part of the proof we get that there exists a solution z of (2.4). Then $y(t) = z(\tau(t))$ is the desired solution of (2.1). □

3. BVP ON $[1,\infty)$

We start by considering (1.6). Clearly, if $\lambda = 0$, then $x(t) \equiv c$ is solution of (1.6). If $\lambda > 0$, $B_- = 0$ and $b(t) = b_+(t) \not\equiv 0$, the BVP (1.6) is related with the asymptotic behavior of the so-called Kneser solutions, see [18, Section 13]. In this case, it is well-known that (1.6) with $F(u) = |u|^\beta \operatorname{sgn} u$, $\beta \ne \alpha$, has a unique solution x, which satisfies $x^{[1]}(1) < 0$, for any $c > 0$ and $\lambda > 0$ if and only if (1.3), (1.4) are satisfied, see, e.g., [6, Theorem 1] for existence and [5, Theorem 4] for uniqueness.

When b changes sign, the following holds.

Theorem 3.1. *Assume* (1.3), (1.4). *Then for any* $c > 0$ *there exists* $\lambda_c > 0$ *such that for any positive* $\lambda \le \lambda_c$ *the BVP* (1.6) *has a unique solution* x *satisfying* $c/2 \le x(t) \le 3c/2$, *for all* $t \ge 1$. *Further,* x *is of bounded variation on* $[1, \infty)$.

In addition, if F *is nondecreasing and*

$$\frac{B_-}{B_+} \le \inf_{u>0} \frac{F(u)}{F(3u)}, \tag{3.1}$$

then $x^{[1]}(1) \le 0$.

To prove this theorem, the following result is useful.

Lemma 3.2. *Assume* (1.3). *Then any solution of* (1.6) *is a solution of the integral equation*

$$x(t) = c - \int_1^t \Phi^* \left(\frac{\lambda}{a(s)} \left(\int_s^\infty b(r) F(x(r)) dr \right) \right) ds.$$

Proof. Let x be a solution of (1.6). For $t_2 > t_1 \ge 1$ we have

$$\left| x^{[1]}(t_2) - x^{[1]}(t_1) \right| \le \lambda \int_{t_1}^{t_2} |b(s)| \, F(x(s)) ds.$$

Since x is bounded and $b \in L^1[1, \infty)$, there exists $\lim_{t\to\infty} x^{[1]}(t) = x_\infty^{[1]}$. Clearly, $x_\infty^{[1]} = 0$, otherwise, in view of (1.3), the solution x should be unbounded. Hence

$$x^{[1]}(1) = -\lambda \int_1^\infty b(r) F(x(r)) dr. \tag{3.2}$$

Thus, integrating the equation in (1.6), we get

$$x(t) = c + \int_1^t \Phi^* \left(\frac{1}{a(s)} \left(x^{[1]}(1) + \lambda \int_1^s b(r) F(x(r)) dr \right) \right) ds$$

and from (3.2) the assertion follows. $\square$

Proof of Theorem 3.1. Fixed $c > 0$, set

$$M_c = \max_{u \in I_c} F(u),$$

where I_c is the interval

$$I_c = \left[\frac{c}{2}, \frac{3c}{2} \right].$$

Since F is locally Lipschitz, there exists $H_c > 0$ such that for $u, v \in I_c$ we have

$$|F(u) - F(v)| \le H_c |u - v|. \tag{3.3}$$

Set

$$\lambda_c = \frac{1}{M_c} \min \left\{ \left(\frac{c}{2J} \right)^\alpha ; \left(\frac{\alpha M_c}{2 J H_c} \right)^\alpha \right\} \tag{3.4}$$

and consider the metric space BC of bounded continuous functions on $[1,\infty)$ with the metric $\delta(u,v)=\sup_{t\geq 1}|u(t)-v(t)|$. Let Ω be the subset of BC given by

$$\Omega=\left\{u\in C[1,\infty):\frac{c}{2}\leq u(t)\leq\frac{3}{2}c\right\},$$

and define in Ω the operator T as follows:

$$T(u)(t)=c-\int_1^t\Phi^*\left(\frac{\lambda_c}{a(s)}\left(\int_s^\infty b(r)F(u(r))dr\right)\right)ds. \tag{3.5}$$

In view of (3.4) we have

$$\begin{aligned}|T(u)(t)-c|&\leq\int_1^t\left(\frac{\lambda_c}{a(s)}\int_s^\infty|b(r)|F(u(r))dr\right)^{1/\alpha}ds\leq\\&\leq(\lambda_c M_c)^{1/\alpha}J\leq c/2.\end{aligned}$$

Then $T(\Omega)\subseteq\Omega$. Since Ω is a closed set of a complete metric space, Ω is itself a complete metric space. Let us show that T is a contraction in Ω with respect to δ. For any $u\in\Omega$ we have

$$\left|\int_s^\infty b(r)F(u(r))dr\right|\leq M_c\int_s^\infty|b(r)|dr.$$

Hence, taking into account that $0<\alpha\leq 1$, from the mean value theorem we obtain for $u,v\in\Omega$

$$\begin{aligned}&\left|\Phi^*\left(\int_s^\infty b(r)F(u(r))dr\right)-\Phi^*\left(\int_s^\infty b(r)F(v(r))dr\right)\right|\\&\leq\frac{1}{\alpha}\left(M_c\int_s^\infty|b(r)|dr\right)^{(1-\alpha)/\alpha}\left(\int_s^\infty|b(r)|\,|F(u(r))-F(v(r))|dr\right).\end{aligned}$$

Hence, using (3.3), we have

$$\begin{aligned}&\left|\Phi^*\left(\int_s^\infty b(r)F(u(r))dr\right)-\Phi^*\left(\int_s^\infty b(r)F(v(r))dr\right)\right|\\&\leq\frac{1}{\alpha}H_c\left(M_c\right)^{(1-\alpha)/\alpha}\left(\int_s^\infty|b(r)|dr\right)^{(1-\alpha)/\alpha}\int_s^\infty|b(r)|\,|u(r)-v(r)|dr\\&\leq\frac{1}{\alpha}H_c\left(M_c\right)^{(1-\alpha)/\alpha}\left(\int_s^\infty|b(r)|dr\right)^{1/\alpha}\delta(u,v).\end{aligned}$$

Thus

$$|T(u)(t)-T(v)(t)|\leq(\lambda_c)^{1/\alpha}\frac{1}{\alpha}H_c\left(M_c\right)^{(1-\alpha)/\alpha}J\delta(u,v),$$

and so, from (3.4) we have

$$\delta(T(u)-T(v))<\delta(u,v).$$

Hence the operator T is a contraction in Ω. Since $T(\Omega)\subset\Omega$, by applying the contraction theorem, we obtain the existence of a unique fixed point x of T in Ω.

Further, since

$$|x^{[1]}(s)| = \left| \int_s^\infty b(r)F(x(r))dr \right| \le M_c \int_s^\infty |b(r)|dr,$$

in view of (1.4), $x' \in L^1[1,\infty)$, x is of bounded variation on $[1,\infty)$ and $\lim_{t\to\infty} x(t)$ is finite. Hence, x is a solution of (1.6). Moreover, in view of Lemma 3.2, any solution of (1.6) in Ω is a fixed point of the operator T, given by (3.5). Thus, (1.6) with $\lambda = \lambda_c$ is uniquely solvable in Ω.

Now, assume F non decreasing and (3.1). Since $x(t) \in I_c$ for any $t \ge 1$, we have

$$\begin{aligned} x^{[1]}(1) = &- \lambda_c \int_1^\infty b(r)F(x(r))dr \\ &= \lambda_c \int_1^\infty b_-(r)F(x(r))dr - \lambda_c \int_1^\infty b_+(r)F(x(r))dr \\ &\le \lambda_c \left(F(3c/2)\ B_- - F(c/2)B_+\right). \end{aligned}$$

Thus, from (3.1), we obtain $x^{[1]}(1) \le 0$.

The same argument gives also the solvability of (1.6) for any positive $\lambda < \lambda_c$ and the proof is complete. □

The first part of Theorem 3.1 complements a similar result in [11, Theorem 3.3], where the existence of positive solutions tending to a nonzero constant as $t \to \infty$ for equations with general Φ-Laplacian has been studied, under additional assumptions on F. Theorem 3.1 also completes [7, Theorem 3.2], where the existence of globally positive solutions has been treated for equations with the curvature operator and a superlinear increasing nonlinearity.

Remark 3.3. The condition $B_+ > 0$ is necessary for existence of solutions x of (1.6) satisfying $x^{[1]}(1) \le 0$. Indeed, if $B_+ = 0$ and $b(t) = -b_-(t) \not\equiv 0$, for any solution x of (1.6) with $x^{[1]}(1) \le 0$, we have for large t

$$x^{[1]}(t) = x^{[1]}(1) - \lambda \int_1^t b_-(s)F(x(s))ds < x^{[1]}(1) \le 0.$$

Hence $x^{[1]}(t) < 0$ for large t and so, in view of (1.3), the solution x should be negative for large t, which is a contradiction.

Remark 3.4. It is easy to verify that condition (3.1) in Theorem 3.1 can be replaced by the weaker assumption that there exists $k > 1$ such that

$$\frac{B_-}{B_+} \le \inf_{u>0} \frac{F(u)}{F(ku)} \tag{3.6}$$

when F is nondecreasing, or by

$$\frac{B_-}{B_+} \le \inf_{c>0} \left(\frac{\min_{I_c} F(x)}{\max_{I_c} F(x)} \right)$$

when the monotonicity of F is not assumed. The details are left to the reader.

Remark 3.5. When $B_- = 0$, assumptions (3.1) or (3.6) are clearly satisfied. When $B_- > 0$, (3.1) or (3.6) require that there exists $k > 1$ such that

$$\inf_{u>0} \frac{F(u)}{F(ku)} > 0. \tag{3.7}$$

The class of continuous functions which satisfy (3.7) is sufficiently wide. For instance, $F(u) = u^\gamma, \gamma > 0$, or $F(u) = \log(1+u)$ satisfy (3.7). More generally, following the Karamata theory, any continuous function which is a regular varying function both at $u = 0$ and at $u = \infty$, of index $p \geq 0$ and $q \geq 0$ respectively, satisfies (3.7). Indeed, in this case

$$\lim_{u\to 0^+} \frac{F(u)}{F(ku)} = \frac{1}{k^p}, \qquad \lim_{u\to\infty} \frac{F(u)}{F(ku)} = \frac{1}{k^q},$$

for every $k > 0$. We refer to [2] for the definition and properties of regular varying functions.

Assume now that the assumptions of Theorem 3.1 are satisfied. For every $c > 0$ and $0 < \lambda \leq \lambda_c$, where λ_c is given by (3.4), denote by $x(t; c, \lambda)$ the unique solution of (1.6) with values in I_c. For any $0 < m \leq 1$, let $G_m : (0, \infty) \to (-\infty, 0]$ be the map that associates to $c > 0$ the value $x^{[1]}(1; c, m\lambda_c)$, i.e.

$$G_m(c) = x^{[1]}(1; c, m\lambda_c). \tag{3.8}$$

By Theorem 3.1, the map G_m is well defined. Let Γ_m be the set

$$\Gamma_m = \{(c, G_m(c), m\lambda_c) : c > 0\}. \tag{3.9}$$

The following result shows some properties of Γ_m.

Theorem 3.6. *Let F be nondecreasing, and assume* (1.3), (1.4) *and* (3.1). *Then for every $0 < m \leq 1$, the set Γ_m is an unbounded continuum. Further, it is contained in $\pi = \{(u, v, w) : u > 0, v \leq 0, w > 0\}$ and $\lim_{c\to 0^+} G_m(c) = 0$.*

Proof. Clearly, Γ_m is an unbounded set, and, from Theorem 3.1, is contained in $\pi = \{(u, v, w) : u > 0, v \leq 0, w > 0\}$. Fixed $\tilde{c} > 0$, let $\{c_n\}$ be a positive sequence converging to $\tilde{c}$. Set $\lambda_n = m\lambda_{c_n}$, $\tilde{\lambda} = \lambda_{\tilde{c}}$, $x_n(t) = x(t; c_n, \lambda_n)$ and $\widetilde{x}(t) = x(t; \tilde{c}, \tilde{\lambda})$. Choose n large so that $\tilde{c}/2 \leq c_n \leq 3\tilde{c}/2$. From here and Theorem 3.1, we have $\tilde{c}/4 \leq c_n/2 \leq x_n(t) \leq 3c_n/2 \leq 9\tilde{c}/4$, i.e. $\{x_n\}$ is equibounded on $[1, \infty)$. Since

$$x_n^{[1]}(1) = -\lambda_n \int_1^\infty b(r)F(x_n(r))dr,$$

using the Lebesgue dominated convergence theorem, and taking into account that, in view of (3.4), λ_c depends continuously on c, we get

$$\lim_n G_m(c_n) = \lim_n x_n^{[1]}(1) = -\tilde{\lambda} \int_1^\infty b(s)F(\tilde{x}(s))ds = \tilde{x}^{[1]}(1) = G_m(\tilde{c}).$$

Thus, G_m is a continuous map and so Γ_m is a continuum.

Now, let $\{c_n\}$ be a positive sequence such that $\lim_n c_n = 0$, and let $\Lambda_n = m\lambda_{c_n}$, $x_n(t) = x(t; c_n, \Lambda_n)$. Since $c_n/2 \le x_n(t) \le 3c_n/2$, from (3.4), we have

$$\Lambda_n F(x_n(t)) \le \lambda_{c_n} M_{c_n} = \min\left\{\left(\frac{c_n}{2J}\right)^{\alpha}; \left(\frac{\alpha M_{c_n}}{2JH_{c_n}}\right)^{\alpha}\right\}$$

and so

$$\lim_n \Lambda_n F(x_n(t)) = 0,$$

uniformly with respect to $t \ge 1$. Fixed $\varepsilon > 0$, choose N large so that $\Lambda_n F(x_n(t)) \le \varepsilon$ for $n \ge N$. Then

$$|G_m(c_n)| \le \Lambda_n \int_1^{\infty} |b(t)| F(x_n(t)) dt \le \varepsilon \int_1^{\infty} |b(t)| dt,$$

that is the assertion. □

4. MAIN RESULT: BVP (1.1), (1.2)

Now we are in position to give our main result.

Theorem 4.1. *Let F be nondecreasing. Assume* (1.3), (1.4), (3.1) *and either* (2.2) *or* (2.3).

Then, for infinitely many values of λ, the BVP (1.1)–(1.2) *has at least one solution which is of bounded variation on* $[0, \infty)$.

To prove this theorem, the following results will be needed. The first one is a generalization of the well known Kneser's theorem and reads as follows, see for instance [4, Section 1.3].

Proposition 4.2. *Consider the system*

$$z' = H(t, z), \quad (t, z) \in [t_1, t_2] \times \mathbb{R}^n$$

where H is continuous, and let K_0 be a continuum (i.e., compact and connected) subset of $\{(t, z) : t = t_1\}$ and $\mathcal{Z}(K_0)$ the family of all the solutions emanating from K_0. If any solution $z \in \mathcal{Z}(K_0)$ is defined on the interval $[t_1, t_2]$, then the cross-section $\mathcal{Z}(t_2; K_0) = \{z(t_2) : z \in \mathcal{Z}(K_0)\}$ is a continuum in $\mathbb{R}^n$.

Lemma 4.3. *Consider the Cauchy problem*

$$\left\{\begin{array}{c} \big(a(t)\Phi(x')\big)' + f(t, x_+) = 0 \quad t \in [0, 1] \\ x(0) = 0, \qquad x^{[1]}(0) = A > 0 \end{array}\right., \tag{4.1}$$

where $u_+ = \max\{u, 0\}$. Then:

i_1) *If x is a solution of* (4.1) *with $x(t_0) \le 0$ at some $t_0, 0 < t_0 \le 1$, then $x^{[1]}(t_0) < 0$.*

i_2) *Any solution of* (4.1) *is defined on the whole interval* $[0, 1]$.

Proof. Claim i_1). Let t_0 be the first zero of x, that is $x(t) > 0$ on $(0, t_0)$. Integrating the equation in (4.1) we have

$$0 = x(t_0) - x(0) = \int_0^{t_0} \Phi^* \left(\frac{1}{a(s)} \left(A - \int_0^s f(r, x_+(r)) dr \right) \right) ds$$
$$= \int_0^{t_0} \Phi^* \left(\frac{1}{a(s)} x^{[1]}(s) \right) ds.$$

Thus $x^{[1]}$ has to assume a negative value for $t = t_0$, because $x^{[1]}$ is nonincreasing and $x^{[1]}(0) = A > 0$. Clearly, x cannot have zeros greater than t_0, because for $t > t_0$ the equation in (4.1) becomes $\big(a(t)\Phi(x')\big)' = 0$. Then, Claim i_1) is proved.

Claim i_2). Since for any solution x of (4.1), the quasiderivative $x^{[1]}$ is nonincreasing, we have $x^{[1]}(t) \le A$. Thus x has an upper bound. If x becomes zero at some $t_0 > 0$, in virtue of Claim i_1), x is negative in a right neighborhood of t_0. Hence x cannot have points of minimum greater than t_0 and so x is negative on $(t_0, 1]$, which yields $x^{[1]}(t) = x^{[1]}(t_0) < 0$ on $(t_0, 1]$. Thus, we have for $t \ge t_0$

$$x(t) = \Phi^* \left(x^{[1]}(t_0) \right) \int_{t_0}^{t} \Phi^* \left(\frac{1}{a(s)} \right) ds,$$

that is x is also bounded from below. $\square$

Proof of Theorem 4.1. First, observe that every nonnegative solution of (4.1) is also solution of (2.1) in $[0, 1]$. Vice versa, if x is a solution of (2.1), with $x(0) = 0$ and $x(t) > 0$ for $t \in (0, 1)$, then x is also solution of (4.1) for a suitable $A > 0$. Indeed, assume by contradiction $x^{[1]}(0) = 0$. Hence $x'(t) \le 0$ for $t \in [0, 1]$, which, together with the condition $x(0) = 0$, contradicts the positivity of x in $(0, 1)$.

In virtue of Theorem 2.1, equation (2.1) has two solutions y and w, which are positive on $(0, 1)$ and satisfy $y(0) = 0, y^{[1]}(1) = 0$ and $w(0) = w(1) = 0$, respectively. Set $A_y = y^{[1]}(0), A_w = w^{[1]}(0)$. Then we have $A_y > 0$ and $A_w > 0$. Moreover, y, w are also solutions of (4.1) on $[0, 1]$ with $A = A_y$ and $A = A_w$, respectively. Assume, without loss of generality, $A_w < A_y$ and let

$$T = \{(x(1), x^{[1]}(1)) : x \text{ sol. of (4.1) s.t. } x^{[1]}(0) = A \in [A_w, A_y]\}.$$

In view of Lemma 4.3, any solution of (4.1) is defined on the whole interval $[0, 1]$. Hence, Proposition 4.2 assures that T is a continuum in $\mathbb{R}^2$, containing the points $(y(1), 0)$ and $(0, w^{[1]}(1))$. Notice that, again in view of Lemma 4.3, we have

$$y(1) > 0, \qquad w^{[1]}(1) < 0$$

and the set T does not contain any point $(0, c)$ with $c \ge 0$. Thus, a continuum $T_1 \subseteq T$ exists, such that $T_1 \subset \{(u, v) : u \ge 0, v \le 0\}$, $(0, 0) \notin T_1$, and there exists two points $P_1 = (p_1, 0)$, $P_2 = (0, -p_2)$ which belong to T_1.

Let $M_p = \max\{p : (p,q) \in T_1\}$, see Figure 1, and consider the immersion of T_1 into $\mathbb{R}^3$, that is the set

$$C_1 = \{(u,v,\lambda) : (u,v) \in T_1, \lambda > 0\}.$$

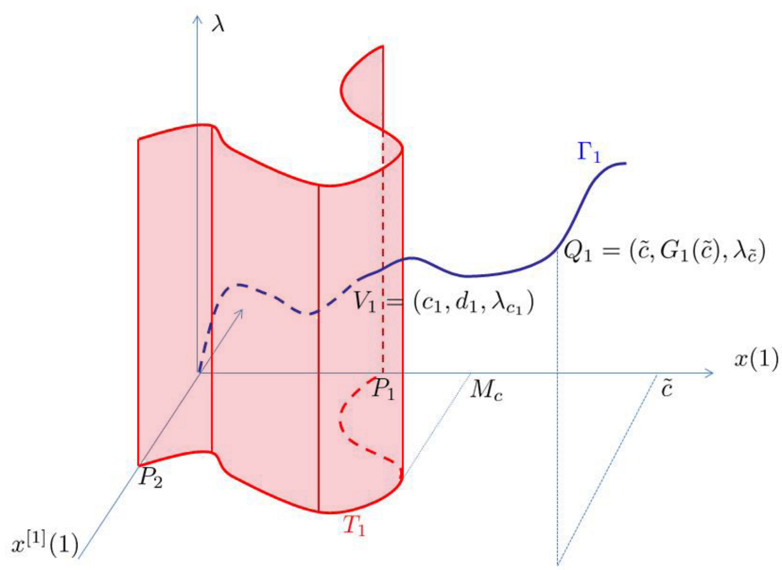

FIGURE 1. $C_1 \cap \Gamma_1 = V_1$.

Let $\widetilde{c} > M_p$. From Theorem 3.1, the point $Q_1 = (\widetilde{c}, G_1(\widetilde{c}), \lambda_{\widetilde{c}})$ belongs to Γ_1, where G_1, Γ_1 are defined in (3.8) and (3.9), respectively. Since, in view of Theorem 3.6, Γ_1 is a continuum which approaches, roughly speaking, the third axes as $c \to 0$, we obtain

$$C_1 \cap \Gamma_1 \neq \emptyset.$$

Now, let $V_1 = (c_1, d_1, \lambda_{c_1}) \in C_1 \cap \Gamma_1$ and let us prove that we can associate to the point V_1 a solution of the BVP (1.1)–(1.2), which implies that (1.1)–(1.2) is solvable for $\lambda = \lambda_{c_1}$. Since $(c_1, d_1) \in T_1 \subseteq T$, there exists a solution u of (4.1), for a suitable $A > 0$, such that $u(1) = c_1 > 0$ and $u^{[1]}(1) = d_1 < 0$. The inequality $u(1) > 0$ implies that u is positive on $(0,1]$, because, in view of Lemma 4.3, every solution of (4.1), which is negative at some point $T \in (0,1)$, is negative also for $t \in [T,1]$. Therefore u is solution of (1.1) for $t \in [0,1]$, with $u(0) = 0$, $u(t) > 0$ for $t \in (0,1]$. Further, as $V_1 \in \Gamma_1$, a solution v of (1.6) exists, such that $v(1) = c_1$, $v^{[1]}(1) = d_1$. Then v is a positive solution of (1.1) on $[1,\infty)$, and satisfies $\lim_{t\to\infty} v^{[1]}(t) = 0$. Hence, the function

$$x(t) = \begin{cases} u(t), & t \in [0,1], \\ v(t), & t > 1 \end{cases}$$

is a solution of the BVP (1.1)–(1.2) and it is of bounded variation on $[0,\infty)$.

Finally, let us shows that the BVP (1.1)–(1.2) is solvable for infinitely many values of the parameter λ. In virtue of Theorem 3.1, the BVP (1.1)–(1.2) with $c = \widetilde{c}$ is uniquely solvable for any $\lambda < \lambda_{\widetilde{c}}$. Thus, for $0 < m < 1$, consider the point $Q_m = (\widetilde{c}, G_m(\widetilde{c}), m\lambda_{\widetilde{c}})$. Hence $Q_m \in \Gamma_m$ and, again in view of Theorem 3.6, Γ_m is a continuum, which approaches, roughly speaking, the third axes as $c \to 0$. Thus

$$C_1 \cap \Gamma_m \neq \emptyset.$$

Now, let $V_m = (c_m, d_m, m\lambda_{c_m}) \in C_1 \cap \Gamma_m$ and let us show that we can associate to the point V_m another solution of the BVP (1.1)–(1.2), that is the BVP (1.1)–(1.2) is solvable also for $\lambda = m\lambda_{c_m}$. Clearly, $V_m \neq V_1$, because, if $c_m = c_1$, taking into account that $m \in (0, 1)$, we have $m\lambda_{c_m} = m\lambda_{c_1} \neq \lambda_{c_1}$. Hence, using the same argument as before, we obtain again a solution of (1.1)–(1.2). Moreover, since $V_m \neq V_1$, this solution is different to the one above found, and the assertion follows. □

Remark 4.4. When $b_- \equiv 0$, assumption (3.1) is satisfied and so, in this case, Theorem 4.1 complements [20, Theorem 1.1], in which the existence of solutions x satisfying the boundary conditions

$$x(0) = 0, \quad \lim_{t\to\infty} x(t) = 0, \quad x(t) > 0 \quad \text{for } t > 0$$

is studied for a subclass of equations of type (1.1).

5. APPLICATIONS AND CONCLUDING REMARKS

To illustrate our results, consider the following example.

Example 5.1. Consider the differential equation

$$x'' + p(t)|x|^\gamma \operatorname{sgn} x = \lambda b(t)|x|^2 \operatorname{sgn} x, \quad t \in [0, \infty), \tag{5.1}$$

where $\gamma \neq 1, p$ is a continuous function on $[0, \infty)$ such that $p(t) \geq 0$ for $t \in [0, 1]$, $\max_{t\in[0,1]} p(t) > 0$ and $p(t) \equiv 0$ for $t \geq 1$. Moreover, b is the function

$$b(t) = \begin{cases} 0 & t \in [0, 1] \\ e^{-\pi t} \sin(\pi t - \pi) & t \in [1, \infty) \end{cases}.$$

Then b is positive on $(1, 2) \cup (3, 4) \cup \cdots$ and negative on $(2, 3) \cup (4, 5) \cup \cdots$ and we obtain

$$B_+ = \sum_{n=0}^{\infty} \int_{2n+1}^{2n+2} b(t)dt, \quad B_- = -\sum_{n=1}^{\infty} \int_{2n}^{2n+1} b(t)dt.$$

Since for $t > 1$ we have

$$\frac{d}{dt}\left(\frac{e^{-\pi t}}{2\pi}(-\sin(\pi t - \pi) - \cos(\pi t - \pi))\right) = b(t),$$

an easy calculation gives

$$
\begin{aligned}
B_+ &= \frac{e^{-2\pi}+e^{-\pi}}{2\pi}\sum_{n=0}^{\infty} e^{-2\pi n} \\
&= \frac{e^{-2\pi}+e^{-\pi}}{2\pi}\frac{e^{2\pi}}{e^{2\pi}-1} = \frac{1}{2\pi(e^{\pi}-1)}, \\
B_- &= \frac{1+e^{-\pi}}{2\pi}\sum_{n=1}^{\infty} e^{-2\pi n} = \frac{1+e^{-\pi}}{2\pi}\frac{1}{e^{2\pi}-1}.
\end{aligned}
$$

Hence the condition (3.1) is valid, because

$$\frac{B_-}{B_+} = \frac{1+e^{-\pi}}{1+e^{\pi}} < 1/9$$

Thus, by Theorem 4.1, for infinitely many values of λ, equation (5.1) has solutions x of bounded variation on $[0,\infty)$, which satisfy the boundary conditions (1.2).

Concluding remarks. (i) Theorem 4 gives sufficient conditions for solvability of the nonlocal problem (1.1)–(1.2) in case (1.3) holds. The case

$$\int_1^{\infty} \frac{1}{a^{1/\alpha}(t)}dt < \infty$$

will be the object of our further consideration.

(ii) In this paper, we assumed that $\alpha \le 1$. This condition is useful to obtain the uniqueness of solutions of the BVP (1.6) which take values on I_c. This fact allows us to define the function G_m and to consider its "graph", that is the "path" Γ_m. From a tecnical point of view, in our argument the condition $\alpha \le 1$ arises in the proof of Theorem 3.1, when we use the mean value theorem. If $\alpha > 1$, it is an open problem whether the BVP (1.1)–(1.2) is solvable.

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