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THE CONNECTION BETWEEN VARIATIONAL EQUATIONS AND SOLUTIONS OF SECOND ORDER NONLOCAL INTEGRAL BOUNDARY VALUE PROBLEMS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In this paper, we make certain continuity and disconjugacy assumptions upon the second order boundary value problem with nonlocal integral boundary conditions, $y'' = f(x, y, y')$, $y(x_1) = y_1$, and $y(x_2) + \int_c^d ry(x)dx$, $a < x_1 < c < d < x_2 < b$, $y_1, y_2, r \in \mathbb{R}$. Then, supposing we have a solution, $y(x)$, of the boundary value problem, we differentiate the solution with respect to various boundary parameters. We show that the resulting function solves the associated variational equation of $y(x)$.

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1. INTRODUCTION

In this paper, our concern is characterizing derivatives of solutions to the second order nonlocal boundary value problem

$$y'' = f(x, y, y'), \quad a < x < b, \tag{1.1}$$

satisfying

$$y(x_1) = y_1, \quad y(x_2) + \int_c^d ry(x)dx = y_2, \tag{1.2}$$

where $a < x_1 < c < d < x_2 < b$, and $y_1, y_2, r \in \mathbb{R}$ with respect to the boundary parameters.

In particular, we show that under certain conditions solutions of (1.1) may be differentiatied with respect to the various parameters within the boundary conditions. The resulting function solves a linear second order differential equation called the variational equation.

Definition 1.1. Given a solution $y(x)$ of (1.1), we define the *variational equation along $y(x)$* by

$$z'' = \frac{\partial f}{\partial u_1}(x, y(x), y'(x))z + \frac{\partial f}{\partial u_2}(x, y(x), y'(x))z'. \tag{1.3}$$

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The result creates a nonlocal integral boundary value problem analogue of a theorem of Peano's discussed in [3] which concerns the differentiability of solutions of initial value problems with respect to the inital data.

Before we get to the main results of this work, we need to establish some background information and mention a few conditions that are imposed upon (1.1) throughout.

First, we require that

(i) $f(x, u_1, u_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous,
(ii) for $i = 1, 2$, $\partial f / \partial u_i(x, u_1, u_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and
(iii) solutions of initial value problems for (1.1) extend to (a, b).

Remark 1.2. We note that (iii) is not a necessary condition but lets us avoid continually making statements about maximal intervals of existence inside (a, b).

Uniqueness of solutions of (1.1) is a necessity to our results. To that end, we make the following assumption which is an analogue of disconjugacy for (1.1):

(iv) Given $a < x_1 < c < d < x_2 < b$ and $r \in \mathbb{R}$, if $y(x_1) = z(x_1)$ and $y(x_2) + \int_c^d ry(x)dx = z(x_2) + \int_c^d ry(x)dx$ where $y(x)$ and $z(x)$ are solutions of (1.1), then, on (a, b),
$$y(x) \equiv z(x).$$

The following final condition provides uniqueness of solutions of (1.3) along all solutions of (1.1):

(v) Given $a < x_1 < c < d < x_2 < b$ and $r \in \mathbb{R}$ and a solution $y(x)$ of (1.1), if $u(x_1) = 0$ and $u(x_2) + \int_c^d ry(x)dx = 0$, where $u(x)$ is a solution of (1.3) along $y(x)$, then, on (a, b),
$$u(x) \equiv 0.$$

In the last few decades, many authors have researched the connection between derivatives of solutions of (1.1) with respect to boundary data and solutions of (1.3) under conditions similar to those listed above. We refer to the works [2], [4], [5], [8], [9] and the references therein for examples. Outside the scope of this paper, authors have also done much in the same way with boundary value problems for difference equations, [1], [6], [7].

2. PRELIMINARY THEOREMS

In this section, we present two theorems which will be very useful in the proof of the main result of this work. First, we present the theorem of Peano for which we seek an analogue. This is the aforementioned theorem that was attributed to Peano in Hartman's book [3].

Theorem 2.1 (A Theorem of Peano)**.** *Assume that, with respect to* (1.1)*, conditions* (i)–(iii) *are satisfied. Let* $x_0 \in (a, b)$ *and* $y(x) := y(x, x_0, c_1, c_2)$ *denote the solution of* (1.1) *satisfying the initial conditions* $y(x_0) = c_1$, $y'(x_0) = c_2$. *Then,*

(*a*) *for* $i = 1, 2$, $\frac{\partial y}{\partial c_i}(x)$ *exists on* (a, b), *and* $\alpha_i(x) := \frac{\partial y}{\partial c_i}(x)$ *is the solution of the variational equation* (1.3) *along* $y(x)$ *satisfying the respective initial conditions*

$$\alpha_1(x_0) = 1, \quad \alpha_1'(x_0) = 0,$$

$$\alpha_2(x_0) = 0, \quad \alpha_2'(x_0) = 1.$$

(*b*) $\frac{\partial y}{\partial x_0}(x)$ *exists on* (a, b), *and* $\beta(x) := \frac{\partial y}{\partial x_0}(x)$ *is the solution of the variational equation* (1.3) *along* $y(x)$ *satisfying the initial conditions*

$$\beta(x_0) = -y'(x_0), \quad \beta'(x_0) = -y''(x_0).$$

(*c*) $\frac{\partial y}{\partial x_0}(x) = -y'(x_0)\frac{\partial y}{\partial c_1}(x) - y''(x_0)\frac{\partial y}{\partial c_2}(x)$.

Lastly, we will make much use of a slight modification of a continuous dependence result which is an application of the Brouwer Invariance of Domain Theorem. For a typical proof, we refer the avid reader to [5].

Theorem 2.2. *Assume* (i)–(iv) *are satisfied with respect to* (1.1)*. Let* $y(x)$ *be a solution of* (1.1) *on* (a, b), *and let* $a < \alpha < x_1 < c < d < x_2 < \beta < b$ *and* $y_1, y_2, r \in \mathbb{R}$ *be given. Then, there exists a* $\delta > 0$ *such that, for* $i = 1, 2$, $|x_i - t_i| < \delta$, $|c - \xi| < \delta$, $|d - \Delta| < \delta$, $|r - \rho| < \delta$, $|u(x_1) - y_1| < \delta$, *and* $|u(x_2) + \int_c^d ru(x)dx - y_2| < \delta$, *there exists a unique solution* $u_\delta(x)$ *of* (1.1) *such that* $u_\delta(t_1) = y_1$ *and* $u_\delta(t_2) + \int_\xi^\Delta \rho u_\delta(x)dx = y_2$ *and, for* $i = 1, 2$, $\{u_\delta^{(i)}(x)\}$ *converges uniformly to* $u^{(i)}(x)$ *as* $\delta \to 0$ *on* $[\alpha, \beta]$.

3. ANALOGUE OF PEANO'S THEOREM

In this section, we present our analogue to Theorem 2.1. The result is stated in five parts, but each proof is essentially the same. Therefore, we will not provide all proofs for all parts. In conclusion, we will present a corollary analogous to part (c) of Peano's Theorem.

Theorem 3.1. *Assume conditions* (i)–(v) *are satisfied. Let* $y(x)$ *be a solution of* (1.1) *on* (a, b). *Let* $a < x_1 < c < d < x_2 < b$ *and* $y_1, y_2, r \in \mathbb{R}$ *be given so that*

$$y(x) = y(x, x_1, x_2, y_1, y_2, c, d, r),$$

where

$$y(x_1) = y_1, \quad y(x_2) + \int_c^d ry(x)dx = y_2.$$

Then,

(a) *for* $i = 1, 2$, $u_i(x) := \frac{\partial y}{\partial y_i}(x)$ *exists on* (a, b) *and is the solution of the variational equation* (1.3) *along* $y(x)$ *satisfying the respective boundary conditions*

$$u_1(x_1) = 1 \quad \text{and} \quad u_1(x_2) + \int_c^d r u_1(x) dx = 0,$$

$$u_2(x_1) = 0 \quad \text{and} \quad u_2(x_2) + \int_c^d r u_2(x) dx = 1.$$

(b) *for* $i = 1, 2$, $z_i(x) := \frac{\partial y}{\partial x_i}(x)$ *exists on* (a, b) *and is the solution of the variational equation* (1.3) *along* $y(x)$ *satisfying the respective boundary conditions*

$$z_1(x_1) = -y'(x_1) \quad \text{and} \quad z_1(x_2) + \int_c^d r z_1(x) dx = 0,$$

$$z_2(x_1) = 0 \quad \text{and} \quad z_2(x_2) + \int_c^d r z_2(x) dx = -y'(x_2).$$

(c) $C(x) := \frac{\partial y}{\partial c}(x)$ *exists on* (a, b) *and is the solution of the variational equation* (1.3) *along* $y(x)$ *satisfying the boundary conditions*

$$C(x_1) = 0 \quad \text{and} \quad C(x_2) + \int_c^d r C(x) dx = -r y(c).$$

(d) $D(x) := \frac{\partial y}{\partial d}(x)$ *exists on and is the solution of the variational equation* (1.3) *along* $y(x)$ *satisfying the boundary conditions*

$$D(x_1) = 0 \quad \text{and} \quad D(x_2) + \int_c^d r D(x) dx = r y(d).$$

(e) $R(x) := \frac{\partial y}{\partial r}(x)$ *exists on* (a, b) *and is the solution of the variational equation* (1.3) *along* $y(x)$ *satisfying the boundary conditions*

$$R(x_1) = 0 \quad \text{and} \quad R(x_2) + \int_c^d r R(x) dx = -\int_c^d y(x) dx.$$

Proof. For part (a), we will provide the argument for $\partial y/\partial y_1$ as $\partial y/\partial y_2$ is quite similar. For brevity, we will denote $y(x, x_1, x_2, y_1, y_2, c, d, r)$ by $y(x, y_1)$ as y_1 is the parameter of concern. Let $\delta > 0$ be as in Theorem 2.2, $0 < |h| < \delta$ be given, and define the difference quotient

$$u_{1h}(x) = \frac{1}{h}[y(x, y_1 + h) - y(x, y_1)].$$

Note that for every $h \neq 0$,

$$u_{1h}(x_1) = \frac{1}{h}[y(x_1, y_1 + h) - y(x_1, y_1)] = \frac{1}{h}[y_1 + h - y_1] = 1$$

and

$$u_{1h}(x_2) + \int_c^d r u_{1h}(x)dx = \frac{1}{h}\big[y(x_2, y_1 + h) + \int_c^d r y(x, y_1 + h)dx$$
$$- y(x_2, y_1) - \int_c^d r y(x, y_1)dx\big]$$
$$= \frac{1}{h}[y_2 - y_2] = 0.$$

Now that we have established the boundary conditions, we will view $y(x)$ in terms of the solution of an initial value problem at x_1, and along with telescoping sums, the Mean Value Theorem, and Theorems 2.1, 2.2, we will show that $u_{1h}(x)$ is a solution of the variational equation (1.3). To that end, let

$$\mu = y'(x_1, y_1)$$

and

$$\nu = \nu(h) = y'(x_1, y_1 + h) - \mu.$$

Then $y(x) = u(x, x_1, y_1, \mu)$, and we have

$$u_{1h}(x) = \frac{1}{h}[u(x, x_1, y_1 + h, \mu + \nu) - u(x, x_1, y_1, \mu)].$$

Next, by utilizing a telescoping sum, we have

$$u_{1h}(x) = \frac{1}{h}[u(x, x_1, y_1 + h, \mu + \nu) - u(x, x_1, y_1, \mu + \nu)$$
$$+ u(x, x_1, y_1, \mu + \nu) - u(x, x_1, y_1, \mu)]$$

By Theorem 2.1 and the Mean Value Theorem, we obtain

$$u_{1h}(x) = \frac{1}{h}[\alpha_1(x, u(x, x_1, y_1 + \bar{h}, \mu + \nu))(y_1 + h - y_1)$$
$$+ \alpha_2(x, u(x, x_1, y_1, \mu + \bar{\nu}))(\mu + \nu - \mu)],$$

where for $i = 1, 2$, $\alpha_i(x, y(\cdot))$ is the solution of the variational equation (1.1) along $u(\cdot)$ satisfying respectively

$$\alpha_1(x_1) = 1, \quad \alpha_1'(x_1) = 0,$$

$$\alpha_2(x_1) = 0, \quad \alpha_2'(x_1) = 1.$$

Furthermore, $y_1 + \bar{h}$ is between y_1 and $y_1 + h$, and $\mu + \bar{\nu}$ is between μ and $\mu + \nu$. Simplifying,

$$u_{1h}(x) = \alpha_1(x, u(x, x_1, y_1 + \bar{h}, \mu + \nu)) + \frac{\nu}{h}\alpha_2(x, u(x, x_1, y_1, \mu + \bar{\nu})).$$

Thus, to show $\lim_{h\to 0} u_{1h}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\nu}{h}$ exists.

By hypothesis (v), since $\alpha_2(x, u(\cdot))$ is a nontrivial solution of (1.3) along $u(\cdot)$, and $\alpha_2(x_1, u(\cdot)) = 0$, we have $\alpha_2(x_2, u(\cdot)) + \int_c^d r\alpha_2(x, u(\cdot))dx \neq 0$.

Recall,

$$u_{1h}(x_2) + \int_c^d r u_{1h}(x)dx = 0.$$

Hence, we obtain

$$\begin{aligned}\lim_{h\to 0}\frac{\nu}{h} &= \frac{-\alpha_1(x_2, u(x, x_1, y_1, \mu)) - \int_c^d r\alpha_1(x, u(x, x_1, y_1, \mu))}{\alpha_2(x_2, u(x, x_1, y_1, \mu)) + \int_c^d r\alpha_2(x, u(x, x_1, y_1, \mu))dx} \\ &= \frac{-\alpha_1(x_2, y(\cdot)) - \int_c^d r\alpha_1(x, y(\cdot))}{\alpha_2(x_2, y(\cdot)) + \int_c^d r\alpha_2(x, y(\cdot))dx} \\ &:= U.\end{aligned}$$

Now let $u_1(x) = \lim_{h\to 0} u_{1h}(x)$, and note by construction of $u_{1h}(x)$,

$$u_1(x) = \frac{\partial y}{\partial y_1}(x).$$

Furthermore,

$$u_1(x) = \lim_{h\to 0} u_{1h}(x) = \alpha_1(x, y(x)) + U\alpha_2(x, y(x))$$

which is a solution of the variational equation (1.3) along $y(x)$.

In addition,

$$u_1(x_1) = \lim_{h\to 0} u_{1h}(x_1) = 1,$$

and

$$u_1(x_2) + \int_c^d r u_1(x)dx = \lim_{h\to 0}\left[u_{1h}(x_2) + \int_c^d r u_{1h}(x)dx\right] = 0.$$

This completes the argument for part (a).

For part (b), we will provide the argument for $\partial y/\partial x_1$ as the remaining part is nearly identical. For brevity, we will denote $y(x, x_1, x_2, y_1, y_2, c, d, r)$ by $y(x, x_1)$ as x_1 is the parameter of concern. Let $\delta > 0$ be as in Theorem 2.2, $0 < |h| < \delta$ be given, and define

$$z_{1h}(x) = \frac{1}{h}[y(x, x_1 + h) - y(x, x_1)].$$

Note that for every $h \neq 0$,

$$\begin{aligned}z_{1h}(x_1) &= \frac{1}{h}[y(x_1, x_1 + h) - y(x_1, x_1)] \\ &= \frac{1}{h}[y(x_1, x_1 + h) - y(x_1 + h, x_1 + h) + y(x_1 + h, x_1 + h) \\ &\quad - y(x_1, x_1)] \\ &= \frac{1}{h}[-y'(x_1 + \bar{h}, x_1 + h)(x_1 + h - x_1) + y_1 - y_1] \\ &= -y'(x_1 + \bar{h}, x_1 + h)\end{aligned}$$

where $x_1 + \bar{h}$ is between x_1 and $x_1 + h$, and

$$\begin{aligned} z_{1h}(x_2) + \int_c^d r z_{1h}(x)dx &= \frac{1}{h}\big[y(x_2, x_1 + h) + \int_c^d r y(x, x_1 + h)dx \\ &\quad - y(x_2, x_1) - \int_c^d r y(x, x_1)dx\big] \\ &= \frac{1}{h}[y_2 - y_2] = 0. \end{aligned}$$

Now that we have established the boundary conditions, we will view $y(x)$ in terms of the solution of an initial value problem at x_1 as in part (a), and along with telescoping sums, the Mean Value Theorem, and Theorems 2.1, 2.2, we will show that $z_{1h}(x)$ is a solution of the variational equation (1.3). To that end, let

$$\mu = y'(x_1, x_1)$$

and

$$\nu = \nu(h) = y'(x_1, x_1 + h) - \mu.$$

Then $y(x) = u(x, x_1, y_1, \mu)$, and we have

$$z_{1h}(x) = \frac{1}{h}[u(x, x_1 + h, y_1, \mu + \nu) - u(x, x_1, y_1, \mu)].$$

Next, by utilizing a telescoping sum, we have

$$\begin{aligned} z_{1h}(x) = \frac{1}{h}[&u(x, x_1 + h, y_1, \mu + \nu) - u(x, x_1, y_1, \mu + \nu) \\ &+ u(x, x_1, y_1, \mu + \nu) - u(x, x_1, y_1, \mu)] \end{aligned}$$

By Theorem 2.1 and the Mean Value Theorem, we obtain

$$\begin{aligned} z_{1h}(x) = \frac{1}{h}[&\beta(x, u(x, x_1 + \bar{h}, y_1, \mu + \nu))(x_1 + h - x_1) \\ &+ \alpha_2(x, u(x, x_1, y_1, \mu + \bar{\nu}))(\mu + \nu - \mu)], \end{aligned}$$

where $\beta(x, u(\cdot))$ and $\alpha_2(x, u(\cdot))$ are solutions of the variational equation (1.1) along $u(\cdot)$ satisfying respectively

$$\beta(x_1) = -y'(x_1), \quad \beta'(x_1) = -y''(x_1),$$

$$\alpha_2(x_1) = 0, \quad \alpha_2'(x_1) = 1.$$

Furthermore, $x_1 + \bar{h}$ is between x_1 and $x_1 + h$, and $\mu + \bar{\nu}$ is between μ and $\mu + \nu$. Simplifying,

$$z_{1h}(x) = \beta(x, u(x, x_1 + \bar{h}, y_1, \mu + \nu)) + \frac{\nu}{h}\alpha_2(x, u(x, x_1, y_1, \mu + \bar{\nu})).$$

Thus, to show $\lim_{h\to 0} z_{1h}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\nu}{h}$ exists.

By hypothesis (v), that $\alpha_2(x, u(\cdot))$ is a nontrivial solution of (1.3) along $u(\cdot)$, and $\alpha_2(x_1, u(\cdot)) = 0$, we have $\alpha_2(x_2, u(\cdot)) + \int_c^d r\alpha_2(x, u(\cdot))dx \neq 0$.

Recall,

$$z_{1h}(x_2) + \int_c^d r z_{1h}(x)dx = 0.$$

Hence, we obtain

$$\begin{aligned}
\lim_{h\to 0} \frac{\nu}{h} &= \frac{-\beta(x_2, u(x, x_1, y_1, \mu)) - \int_c^d r\beta(x_2, u(x, x_1, y_1, \mu))}{\alpha_2(x_2, u(x, x_1, y_1, \mu)) + \int_c^d r\alpha_2(x, u(x, x_1, y_1, \mu))dx} \\
&= \frac{-\beta(x_2, y(\cdot)) - \int_c^d r\beta(x_2, y(\cdot))}{\alpha_2(x_2, y(\cdot)) + \int_c^d r\alpha_2(x, y(\cdot))dx} \\
&:= U.
\end{aligned}$$

Now let $z_1(x) = \lim_{h\to 0} z_{1h}(x)$, and note by construction of $z_{1h}(x)$,

$$z_1(x) = \frac{\partial y}{\partial x_1}(x).$$

Furthermore,

$$z_1(x) = \lim_{h\to 0} z_{1h}(x) = \beta(x, y(x)) + U\alpha_2(x, y(x))$$

which is a solution of the variational equation (1.3) along $y(x)$.

In addition,

$$z_1(x_1) = \lim_{h\to 0} z_{1h}(x_1) = -y'(x_1),$$

and

$$z_1(x_2) + \int_c^d r z_1(x)dx = \lim_{h\to 0} \left[z_{1h}(x_2) + \int_c^d r z_{1h}(x, y(\cdot))dx \right] = 0.$$

Thus, part (b) is complete.

Parts (c), (d) and (e) are proven in much the same way. Therefore, we will provide the argument for $\partial y/\partial d$ and leave (c) and (e) to the active reader. For brevity, we will denote $y(x, x_1, x_2, y_1, y_2, c, d, r)$ by $y(x, d)$ as d is the parameter of concern. Let $\delta > 0$ be as in Theorem 2.2, $0 < |h| < \delta$ be given, and define

$$D_h(x) = \frac{1}{h}[y(x, d+h) - y(x, d)].$$

Note that for every $h \neq 0$,

$$D_h(x_1) = \frac{1}{h}[y(x_1, d+h) - y(x_1, d)] = \frac{1}{h}[y_1 - y_1] = 0$$

and

$$\begin{aligned}
D_h(x_2) + \int_c^d rD_h(x)dx &= \frac{1}{h}[y(x_2, d+h) + \int_c^d ry(x, d+h)dx \\
&\quad - y(x_2, d) - \int_c^d ry(x, d)dx] \\
&= \frac{1}{h}[y(x_2, d+h) \\
&\quad + \Big(\int_c^{d+h} ry(x, d+h)dx - \int_c^{d+h} ry(x, d+h)dx\Big) \\
&\quad + \int_c^d ry(x, d+h)dx - y_2] \\
&= \frac{1}{h}[y_2 - \int_c^d ry(x, d+h)dx - \int_d^{d+h} ry(x, d+h)dx \\
&\quad + \int_c^d ry(x, d+h)dx - y_2] \\
&= -\frac{1}{h}\int_d^{d+h} ry(x, d+h)dx \\
&= -\frac{1}{h}(d+h-d)ry(d+\bar{h}, d+h) \\
&= -ry(d+\bar{h}, d+h).
\end{aligned}$$

where $d + \bar{h}$ is between d and $d + h$.

Now that we have established the boundary conditions, we will view $y(x)$ in terms of the solution of an initial value problem at x_1 and along with telescoping sums, the Mean Value Theorem, and Theorems 2.1, 2.2, we will show that $D_h(x)$ is a solution of the variational equation (1.3). To that end, let

$$\mu = y'(x_1, d)$$

and

$$\nu = \nu(h) = y'(x_1, d+h) - \mu.$$

Then $y(x) = u(x, x_1, y_1, \mu)$, and we have

$$D_h(x) = \frac{1}{h}[u(x, x_1, y_1, \mu + \nu) - u(x, x_1, y_1, \mu)].$$

By Theorem 2.1 and the Mean Value Theorem, we obtain

$$D_h(x) = \frac{1}{h}[\alpha_2(x, u(x, x_1, y_1, \mu + \bar{\nu}))(\mu + \nu - \mu)],$$

where $\alpha_2(x, u(\cdot))$ is the solution of the variational equation (1.1) along $u(\cdot)$ satisfying

$$\alpha_2(x_1) = 0, \quad \alpha_2'(x_1) = 1.$$

Furthermore, $\mu + \bar{\nu}$ is between μ and $\mu + \nu$. Simplifying,

$$D_h(x) = \frac{\nu}{h}\alpha_2(x, u(x, x_1, y_1, \mu + \bar{\nu})).$$

Thus, to show $\lim_{h\to 0} D_h(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\nu}{h}$ exists.

By hypothesis (v), the fact that $\alpha_2(x, u(\cdot))$ is a nontrivial solution of (1.3) along $u(\cdot)$, and $\alpha_2(x_1, u(\cdot)) = 0$, we have

$$\alpha_2(x_2, u(\cdot)) + \int_c^d r\alpha_2(x, u(\cdot))dx \neq 0.$$

Recall,

$$D_h(x_2) + \int_c^d rD_h(x)dx = -ry(d + \bar{h}, d + h).$$

Hence, we obtain

$$\begin{aligned}\lim_{h\to 0} \frac{\nu}{h} &= \frac{-ry(d, d)}{\alpha_2(x_2, u(x, x_1, y_1, \mu)) + \int_c^d r\alpha_2(x, u(x, x_1, y_1, \mu))dx} \\ &= \frac{-ry(d, d)}{\alpha_2(x_2, y(\cdot)) + \int_c^d r\alpha_2(x, y(\cdot))dx} \\ &:= U.\end{aligned}$$

Now let $D(x) = \lim_{h\to 0} D_h(x)$, and note by construction of $D_h(x)$,

$$D(x) = \frac{\partial y}{\partial d}(x).$$

Furthermore,

$$D(x) = \lim_{h\to 0} D_h(x) = U\alpha_2(x, y(x))$$

which is a solution of the variational equation (1.3) along $y(x)$. In addition,

$$D(x_1) = \lim_{h\to 0} D_h(x_1) = 0,$$

and

$$D(x_2) + \int_c^d rD(x)dx = \lim_{h\to 0}\left[D_h(x_2) + \int_c^d rD_h(x)dx\right] = -ry(d).$$

This completes the argument for $\partial y/\partial d$, and hence the proof of the theorem. □

We conclude with a corollary to Theorem 3.1 which establishes an analogue to part (c) of Theorem 2.1 of Peano. The proof is a result of the dimensionality of the solution space for the variational equation.

Corollary 3.2. *Under the assumptions of Theorem* 3.1, *we have*

(a) $\frac{\partial y}{\partial x_i} = -y'(x_i)\frac{\partial y}{\partial y_i}$,
(b) $\frac{\partial y}{\partial c} = -\frac{y(c)}{y(d)} \cdot \frac{\partial y}{\partial d}$, *and*
(c) $\frac{\partial y}{\partial c} = \frac{y(c)}{\int_c^d y(x)dx} \cdot \frac{\partial y}{\partial r}$.

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