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ZERO-HOPF BIFURCATION ANALYSIS FOR A KRAWIEC-SZYDLOWSKI MODEL OF BUSINESS CYCLES WITH TWO DELAYS

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. In this paper, we study a Krawiec-Szydlowski model of business cycles with delays in both the gross product and the capital stock. We investigate zero-Hopf singularities. The conditions under which the zero-Hopf bifurcation occurs are established. By performing center manifold reduction, the normal forms on the center manifold for zero-Hopf singularity are derived and the bifurcation diagrams as well as the direction and stability of the periodic solutions are obtained. Examples are given to confirm the theoretical results.

AMS (MOS) Subject Classification. 34K18.

1. PRELIMINARIES

Business cycles and economical fluctuations have long been observed and mathematical models that describe these behaviors have been established and studied by many researchers. Kaldor [9] is the first to construct a mathematical model, the Kaldor model, that uses a system of ordinary differential equations with nonlinear investment and saving functions so that cyclic behaviors or limit cycles were exhibited. The Kaldor model alone has drew a great attention since its publication in 1940, see [1, 2, 4, 7, 23, 26, 31]. Kalecki [10, 11] found that a time delay for investment after a business decision has been made can also cause such a behavior. Later, Krawiec and Szydlowski [14, 15, 16, 17, 18] incorporated the idea of Kalecki into the Kaldor model in a series of papers by proposing the following Kaldor-Kalecki model (we prefer to call it Krawiec-Szydlowski model as suggested by a referee for our paper [29]) of business cycles

$$(1.1)\qquad \begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t), K(t)) - S(Y(t), K(t))], \\ \frac{dK(t)}{dt} = I(Y(t-\tau), K(t)) - qK(t), \end{cases}$$

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where Y represents the gross product, K the capital stock. The parameters $\alpha > 0$ is the adjustment coefficient in the goods market, $q \in (0,1)$ is the depreciation rate of capital stock. Functions $I(Y,K)$ and $S(Y,K)$ are investment and saving functions, and $\tau \geq 0$ is a time lag representing delay for the investment due to the past investment decision. Since this model was established, it has been studied extensively by many authors. Not only have the cyclic behaviors been observed from the model analysis, but the formulation of the limit cycles, the direction and stability of the periodic solutions, along with many other bifurcation behaviors have been studied and established, see [17, 18, 22, 23, 24, 25, 27, 28, 29, 30, 32].

Considering that the past investment decisions [17] also influence the change in the capital stock, Kaddar and Talibi Alaoui [8] extended the model (1.1) by imposing delays in both the gross product and capital stock. Thus adding the same delay to the capital stock K in the investment function $I(Y,K)$ of the second equation of Sys. (1.1) leads to the following model of business cycles

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t),K(t)) - S(Y(t),K(t))], \\ \frac{dK(t)}{dt} = I(Y(t-\tau),K(t-\tau)) - qK(t). \end{cases}$$

Assuming that the investment time delay may only happen for the capital stock, Wu and Wang [29] proposed and studied the following Krawiec-Szydlowski type model.

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t),K(t)) - S(Y(t),K(t))], \\ \frac{dK(t)}{dt} = I(Y(t),K(t-\tau)) - qK(t). \end{cases} \tag{1.2}$$

For this model, the mathematical analysis is carried out using the bifurcation and normal form theory. Both simple-zero singularity and double-zero singularity analysis are established. Stability of the equilibrium point is established and bifurcation diagrams are obtained.

Observing that delays for the investment due to the past investment decisions may occur on either or both gross product and capital stock, and may happen at different times, that is, with different delays, recently, Wu and Wang [30] proposed and studied the following models with two delays.

$$\begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t),K(t)) - S(Y(t),K(t))], \\ \frac{dK(t)}{dt} = I(Y(t-\tau_1),K(t-\tau_2)) - qK(t). \end{cases} \tag{1.3}$$

For fixed $\tau_1 \geq 0$, using τ_2 as the bifurcation parameter, they carried out a Hopf bifurcation analysis and a detailed discussion for the distribution of eigenvalues of the linear part of Sys. (1.3) at the equilibrium point in the (τ_1,τ_2) plane for a special case, and as a result, they found the conditions for the Hopf bifurcation to occur. They also established the direction and the stability of the periodic solutions bifurcated from the Hopf bifurcation by using the normal form theory on the center manifold.

In this paper we continue our study of Sys. (1.3). As always, we take the investment and saving functions I and S, respectively, as following.

$$I(Y, K) = I(Y) - \beta K,\ S(Y, K) = \gamma Y,$$

where I is a nonlinear function of Y, and $\beta > 0$ and $\gamma \in (0, 1)$ are constants. We thus obtain the following system.

$$\tag{1.4} \begin{cases} \frac{dY(t)}{dt} = \alpha[I(Y(t)) - \beta K(t) - \gamma Y(t)], \\ \frac{dK(t)}{dt} = I(Y(t - \tau_1)) - \beta K(t - \tau_2) - qK(t). \end{cases}$$

We first investigate the characteristic equation of the linear part of Sys. (1.4) at an equilibrium point for different delays τ_1 and τ_2. We will focus on zero-Hopf singularity in this research. Because of the complexity, we only carry out, for a special case as we did in [30], a detailed discussion of the distribution of the eigenvalues of the characteristic equation and give the conditions that guarantee that the characteristic equation has a simple zero root and a pair of purely imaginary roots. We show that the zero-Hopf bifurcation may occur as (τ_1, τ_2) passes some critical curves in the (τ_1, τ_2)-plane. Furthermore, we use the normal form theory to derive the corresponding normal form from which we obtain the bifurcation diagrams and the stability of the bifurcating limit cycles.

Note that, in business cycles, it is well known that some random factors influence the business dynamics. Incorporating the random factors, the following coupled stochastic model of business cycles was studied by Mircea et al in [19, 20].

$$\tag{1.5} \begin{cases} dY(t) &=& \alpha[I(Y(t)) - \beta K(t) - \gamma Y(t)]dt \\ && +g(t, Y(t), K(t))dW(t), \\ dK(t) &=& [I(Y(t - \tau_1)) - \beta K(t - \tau_2) - qK(t)]dt \\ && +h(t, Y(t), K(t))dW(t). \end{cases}$$

Recently, some stochastic reaction-diffusion models have been proposed and studied for food webs and neural networks. For example, Kao and Wang [12] proposed and performed a stability analysis for a stochastic coupled reaction-diffusion system on networks (SCRDSNs) and obtained some novel stability principles which have a close relationship to the topological property of the networks. It may be worthy to generalize the business models to stochastic reaction-diffusion model and to see how the dynamics from these models differentiate from the existing models.

The following result by Ruan and Wei [21] will be used in this paper.

Lemma 1.1. *Consider the transcendental polynomial*

$$P(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}) = p(\lambda) + q_1(\lambda)e^{-\lambda\tau_1} + q_2(\lambda)e^{-\lambda\tau_2},$$

where p, q_1, q_2 *are real polynomials such that* $\max\{\deg q_1, \deg q_2\} < \deg(p)$ *and* $\tau_1, \tau_2 \geq 0$. *Then as* (τ_1, τ_2) *varies, the sum of the orders of the zeros of* P *in the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

The rest of this manuscript is organized as follows. In Section 2, a detailed discussion for a special case is given for the distribution of eigenvalues of the linear part of Sys. (1.4) at an equilibrium point in the (τ_1, τ_2)-parameter space. Fixing $\tau_1 \geq 0$, and letting τ_2 vary, conditions are found such that the there are a simple zero eigenvalue and a pair of purely imaginary eigenvalues as τ_2 passes some critical values. In Section 3, the theory of center manifold reduction for general delayed differential equations (DDEs) is used to derive the normal forms for Sys. (1.4) for zero-Hopf singularity on the center manifold. In Section 4, numerical simulations are presented to confirm the theoretical results. A conclusion of our results is given in Section 5. Finally, a shortcut of normal form of DDEs for zero-Hopf singularity and the bifurcation diagrams are put in Appendix.

2. DISTRIBUTION OF EIGENVALUES

Throughout the rest of this paper, we assume that (Y^*, K^*) is an equilibrium point of Sys. (1.4) and $I(s)$ is a nonlinear C^4 function. Let $I^* = I(Y^*)$, $u_1 = Y - Y^*, u_2 = K - K^*$, and $i(s) = I(s + Y^*) - I^*$. Then Sys. (1.4) can be transformed as

$$\begin{cases} \frac{du_1(t)}{dt} = \alpha[i(u_1(t)) - \beta u_2(t) - \gamma u_1(t)], \\ \frac{du_2(t)}{dt} = i(u_1(t - \tau_1)) - \beta u_2(t - \tau_2) - q u_2(t). \end{cases} \tag{2.1}$$

Let the Taylor expansion of i at 0 be

$$i(s) = ks + i^{(2)} s^2 + i^{(3)} s^3 + O(|s|^4)$$

where

$$k = i'(0) = I'(Y^*),\ i^{(2)} = \frac{1}{2} i''(0) = \frac{1}{2} I''(Y^*),\ i^{(3)} = \frac{1}{3!} i'''(0) = \frac{1}{3!} I'''(Y^*).$$

The linear part of Sys. (2.1) at $(0, 0)$ is

$$\begin{cases} \frac{du_1(t)}{dt} = \alpha[(k - \gamma) u_1(t) - \beta u_2(t)], \\ \frac{du_2(t)}{dt} = k u_1(t - \tau_1) - \beta u_2(t - \tau_2) - q u_2(t), \end{cases} \tag{2.2}$$

and the corresponding characteristic equation is

$$\Delta(\lambda) \equiv \lambda^2 + A\lambda + B + Ce^{-\lambda\tau_1} + \beta(\lambda + D)e^{-\lambda\tau_2} = 0, \tag{2.3}$$

where

$$A = q - \alpha(k - \gamma), \quad B = -\alpha q(k - \gamma), \quad C = \alpha\beta k, \quad D = -\alpha(k - \gamma).$$

Define

$$k_1 = \frac{\beta\gamma}{q} + \gamma.$$

We then have the following theorem distribution results for no delay system.

Theorem 2.1. *Let $\tau_1 = \tau_2 = 0$. If $k < k_1$, all roots of Eq. (2.3) have negative real parts, and hence (Y^*, K^*) is asymptotically stable. If $k = k_1$, Eq. (2.3) has a zero root and a negative root, and hence (Y^*, K^*) is neutrally stable. If $k > k_1$, Eq. (2.3) has one positive root and one negative root, and hence (Y^*, K^*) is unstable.*

In this paper we focus on the study of zero-Hopf bifurcation. To this end, we assume that $q = \beta$, and $k = k_1$. Then $k = k_1 = 2\gamma$ and $\Delta(\lambda)$ becomes

$$\Delta(\lambda) \equiv \lambda^2 + (\beta - \alpha\gamma)\lambda - \alpha\beta\gamma + 2\alpha\beta\gamma e^{-\lambda\tau_1} + \beta(\lambda - \alpha\gamma)e^{-\lambda\tau_2} = 0. \tag{2.4}$$

Let $i\omega(\omega > 0)$ be a root of Eq. (2.4). Plug it into Eq. (2.4) and separate the real and imaginary parts, we get

$$\begin{cases} -\omega^2 - \alpha\beta\gamma + 2\alpha\beta\gamma\cos\omega\tau_1 - \alpha\beta\gamma\cos\omega\tau_2 + \beta\omega\sin\omega\tau_2 &= 0, \\ (\beta - \alpha\gamma)\omega - 2\alpha\beta\gamma\sin\omega\tau_1 + \alpha\beta\gamma\sin\omega\tau_2 + \beta\omega\cos\omega\tau_2 &= 0. \end{cases} \tag{2.5}$$

Assuming first that $\tau_2 = 0$, Sys. (2.5) becomes

$$\begin{cases} \omega^2 + 2\alpha\beta\gamma &= 2\alpha\beta\gamma\cos\omega\tau_1, \\ (2\beta - \alpha\gamma)\omega &= 2\alpha\beta\gamma\sin\omega\tau_1. \end{cases}$$

Adding squares together yields

$$\omega^4 + (4\beta^2 + \alpha^2\gamma^2)\omega^2 = 0,$$

which has no positive roots for ω.

Assuming next that $\tau_1 = 0$, Sys. (2.5) then becomes

$$\begin{cases} \alpha\beta\gamma - \omega^2 &= \alpha\beta\gamma\cos\omega\tau_2 - \beta\omega\sin\omega\tau_2 \\ (\alpha\gamma - \beta)\omega &= \alpha\beta\gamma\sin\omega\tau_2 + \beta\omega\cos\omega\tau_2. \end{cases} \tag{2.6}$$

Again adding squares and the resulting equation for ω is

$$\omega^4 + (\alpha^2\gamma^2 - 4\alpha\beta\gamma)\omega^2 = 0,$$

which has only one positive root

$$\omega_2^+ = \sqrt{4\alpha\beta\gamma - \alpha^2\gamma^2}$$

if $4\beta > \alpha\gamma$ and has no positive roots if $4\beta \leq \alpha\gamma$. If $4\beta > \alpha\gamma$, from (2.6) with $\omega = \omega_2^+$, define, for each $j = 0, 1, 2, \ldots,$

$$\tau_2^j = \frac{1}{\sqrt{4\alpha\beta\gamma - \alpha^2\gamma^2}}\left(\arccos\frac{\alpha\gamma - 2\beta}{2\beta} + 2j\pi\right) > 0. \tag{2.7}$$

For $\tau_1 = 0$, let $\lambda_2(\tau_2) = \sigma_2(\tau_2) + i\omega_2(\tau_2)$ be the root of Eq. (2.4) such that $\sigma_2(\tau_2^j) = 0$ and $\omega_2(\tau_2^j) = \omega_2^+$, respectively. A long calculation shows

$$\mathrm{Re}\left(\frac{d\lambda_2}{d\tau_2}\right)\Bigg|_{\tau_2=\tau_2^j} = \frac{4\beta - \alpha\gamma}{4\beta^3} > 0.$$

We thus have the following results.

Theorem 2.2. *Assume that $q = \beta$ and $k = k_1$.*

1. *Let $\tau_2 = 0$, then except a simple zero root all other roots of Eq. (2.4) have negative real parts for all $\tau_1 \geq 0$. Therefore, (Y^*, K^*) is neutrally stable.*
2. *Let $\tau_1 = 0$, and let $\tau_2^j, j = 0, 1, 2, \ldots,$ be defined in (2.7).*
    - *If $4\beta \leq \alpha\gamma$, then except a simple zero root all other roots of Eq. (2.4) have negative real parts for all $\tau_2 \geq 0$. Therefore, (Y^*, K^*) is neutrally stable.*
    - *If $4\beta > \alpha\gamma$, then except a simple zero root all other roots of Eq. (2.4) have negative real part for all $0 \leq \tau_2 < \tau_2^0$. Therefore, (Y^*, K^*) is neutrally stable. Eq. (2.4) has a simple zero root, a pair of pure imaginary roots $\pm i\omega_2^+$, a total of $2(j+1)$ roots with positive real part, and all other roots have negative real part if $\tau_2 = \tau_2^j$. Eq. (2.4) has a simple zero root, a total of $2(j+1)$ roots with positive real part, and all other roots have negative real part if $\tau_2 \in (\tau_2^j, \tau_2^{j+1})$, $j = 0, 1, 2, \ldots$. Therefore, (Y^*, K^*) is unstable for all $\tau_2 > \tau_2^0$.*

Now we will discuss the existence of pure imaginary root $i\omega(\omega > 0)$ for both τ_1, $\tau_2 > 0$. Rewrite Sys. (2.5) as

$$\left\{\begin{array}{rcl} -\omega^2 - \alpha\beta\gamma + 2\alpha\beta\gamma\cos\omega\tau_1 & = & \alpha\beta\gamma\cos\omega\tau_2 - \beta\omega\sin\omega\tau_2, \\ (\beta - \alpha\gamma)\omega - 2\alpha\beta\gamma\sin\omega\tau_1 & = & -\alpha\beta\gamma\sin\omega\tau_2 - \beta\omega\cos\omega\tau_2. \end{array}\right. \tag{2.8}$$

Adding squares of both sides, we obtain

$$4\alpha\beta\gamma[(\omega^2 + \alpha\beta\gamma)\cos\omega\tau_1 + (\beta - \alpha\gamma)\omega\sin\omega\tau_1] = \omega^4 + \alpha^2\gamma^2\omega^2 + 4\alpha^2\beta^2\gamma^2, \tag{2.9}$$

which is equivalent to

$$\sin(\omega\tau_1 + \theta) = \frac{\omega^4 + \alpha^2\gamma^2\omega^2 + 4\alpha^2\beta^2\gamma^2}{4\alpha\beta\gamma\sqrt{(\omega^2 + \alpha\beta\gamma)^2 + (\beta - \alpha\gamma)^2\omega^2}}, \tag{2.10}$$

where θ is an angle depending on ω such that

$$\cos\theta = \frac{(\beta - \alpha\gamma)\omega}{\sqrt{(\omega^2 + \alpha\beta\gamma)^2 + (\beta - \alpha\gamma)^2\omega^2}},$$

or

$$\theta(\omega) = \arccos\frac{(\beta - \alpha\gamma)\omega}{\sqrt{(\omega^2 + \alpha\beta\gamma)^2 + (\beta - \alpha\gamma)^2\omega^2}}.$$

Now we will give a detail discussion about the existence of positive roots ω of equation (2.10) for different values of τ_1. Let

$$F(\omega) = \frac{\omega^4 + \alpha^2\gamma^2\omega^2 + 4\alpha^2\beta^2\gamma^2}{4\alpha\beta\gamma\sqrt{(\omega^2 + \alpha\beta\gamma)^2 + (\beta - \alpha\gamma)^2\omega^2}},$$

$$G(\omega) = \frac{\omega^2 + \alpha\beta\gamma}{\sqrt{(\omega^2 + \alpha\beta\gamma)^2 + (\beta - \alpha\gamma)^2\omega^2}},$$

and, for a given $\tau_1 \geq 0$, let

$$P(\omega) = \sin(\tau_1\omega + \theta(\omega)).$$

Note that $\sin(\theta(\omega)) = G(\omega)$, or $P(\omega)|_{\tau_1=0} = G(\omega)$. Calculations show that $F(0) = G(0) = P(0) = 1$, $F'(0) = G'(0) = P'(0) = 0$, and

$$F''(0) = -\frac{2\beta^2 + \alpha^2\gamma^2}{2\alpha^2\beta^2\gamma^2}, \quad G''(0) = -\frac{(\beta - \alpha\gamma)^2}{\alpha^2\beta^2\gamma^2}.$$

and

$$P''(0) = -\frac{(\beta - \alpha\gamma - \alpha\beta\gamma\tau_1)^2}{\alpha^2\beta^2\gamma^2}.$$

It can be also shown that $F''(0) = G''(0)$ if and only if $4\beta = \alpha\gamma$, and $F''(0) < G''(0)$ if and only if $4\beta > \alpha\gamma$. In addition, $F(\omega) > 0$ for all $\omega > 0$, and

$$\lim_{\omega\to\infty} F(\omega) = \infty.$$

As a matter of fact, from the definition of F, we can show that there exists a $\omega_m > 0$ such that the function F is decreasing from 0 to ω_m and increasing from ω_m to ∞. Therefore, we proved that there exists a unique $\omega^* > 0$ such that $F(\omega^*) = 1$, and $F(\omega) < 1$ for all $\omega \in (0, \omega^*)$. Notice that, if $4\beta > \alpha\gamma$ then

$$\tau_1^* = \frac{\beta - \alpha\gamma + \sqrt{\beta^2 + \alpha^2\gamma^2/2}}{\alpha\beta\gamma} > 0,$$

and $F''(0) < P''(0)$ if and only if $0 \leq \tau_1 < \tau_1^*$. Further analysis of functions $F(\omega)$, $G(\omega)$, and $P(\omega)$, and the fact that $\theta(0) = \pi/2$, leads to the following results for the existence of positive roots of Eq. (2.10) for different values of τ_1.

Theorem 2.3. *For $\tau_1 = 0$, Eq. (2.10) has a unique positive root $\omega = \omega_2^+ = \sqrt{4\alpha\beta\gamma - \alpha^2\gamma^2}$ if $4\beta > \alpha\gamma$, and it has no positive roots if $4\beta \leq \alpha\gamma$. For $\tau_1 > 0$, there are two cases:*

1. *Suppose $4\beta > \alpha\gamma$. If $0 < \tau_1 < \tau_1^*$ then Eq. (2.10) has one positive root; if $\tau_1 > \tau_1^*$ and $\omega^*\tau_1 + \theta(\omega^*) \leq 2\pi$, then Eq. (2.10) has no positive roots; if $\tau_1 > \tau_1^*$ and $\omega^*\tau_1 + \theta(\omega^*) \geq 5\pi/2$, then Eq. (2.10) has at least one positive root.*
2. *Suppose $4\beta < \alpha\gamma$. If $\tau_1 > 0$ and $\omega^*\tau_1 + \theta(\omega^*) \leq 2\pi$, then Eq. (2.10) has no positive roots; if $\tau_1 > 0$ and $\omega^*\tau_1 + \theta(\omega^*) \geq 5\pi/2$, then Eq. (2.10) has at least one positive root.*

Proof. From the discussion above about the functions F, G, and P, we get that if $F''(0) < P''(0)$ the graphs of F and P intersect at least once in $(0, \omega^*]$. Notice that $4\beta > \alpha\gamma$ implies $F''(0) < G''(0)$, and that $P''(0)|_{\tau_1=0} = G''(0)$. By continuity, for small values of $\tau_1 > 0$, $F''(0) < P''(0)$. It follows from that $F''(0) < P''(0)$ if $0 < \tau_1 < \tau_1^*$. When $\tau_1 > \tau_1^*$, it follows that $F''(0) > P''(0)$. The graph of P is below that of F if $\omega^*\tau_1 + \theta(\omega^*) \leq 2\pi$. Therefore, there are no intersections for the curves F

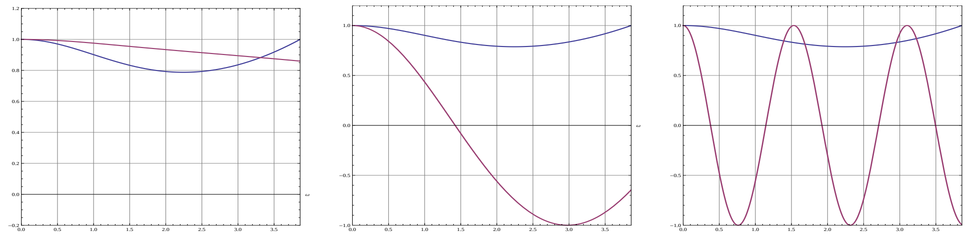

FIGURE 1. Left: $\tau_1 = 0.1 < \tau_1^*$. There is one intersection. Middle: $\tau_1 = 1 > \tau_1^*$, $\omega^*\tau_1 + \theta(\omega^*) = 5.58132 < 2\pi$. There are no intersections. Right: $\tau_1 = 4$, $\omega^*\tau_1 + \theta(\omega^*) = 17.1644 > 5\pi/2$. There is at least one intersection.

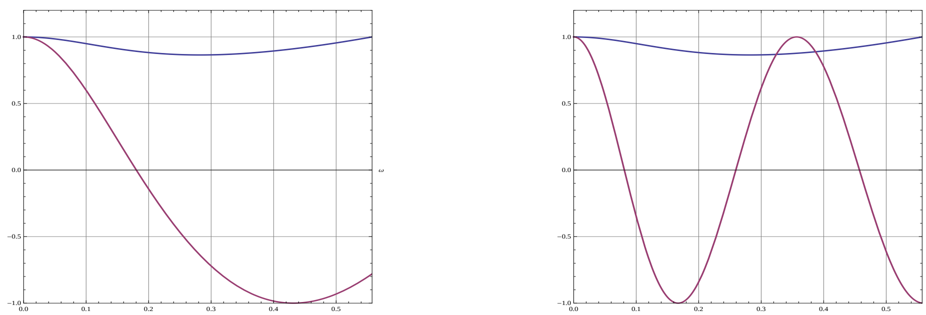

FIGURE 2. Left: $\tau_1 = 5$, $\omega^*\tau_1 + \theta(\omega^*) = 5.38817 < 2\pi$. There is no intersections. Right: $\tau_1 = 15$, $\omega^*\tau_1 + \theta(\omega^*) = 10.9633 > 5\pi/2$. There is at least one intersection.

and P in $(0, \omega^*]$. If $4\beta < \alpha\gamma$ then $F''(0) > G''(0)$, and from the expression of $P''(0)$ it follows that $F''(0) > P''(0)$ for all $\tau_1 \geq 0$. So the curve of P will be below that of F for ω small. In particular, if $\omega^*\tau_1 + \theta(\omega^*) \leq 2\pi$, there are no intersections for the curves F and P in $(0, \omega^*]$. Other results can be proved using that fact that the curve of P is a sine curve, completing the proof. $\square$

See Figures 1 and 2 for graphs of F and P for different values of $\tau_1 > 0$. In Figure 1, we choose $\alpha = 5$, $\beta = 2$, and $\gamma = 0.56$. Then $4\beta > \alpha\gamma$ and $\omega^* = 3.86103$, $\tau_1^* = 0.359687$, $F''(0) = -0.252551 < G''(0) = -0.020482$. In figure 2, we choose $\alpha = 5$, $\beta = 0.2$, and $\gamma = 0.56$. Then $4\beta < \alpha\gamma$ and $\omega^* = 0.557511$, $F''(0) = -12.6276 > G''(0) = -21.5561$.

Remark 2.4. We want to point out that for $\tau_1 > 0$ such that $2\pi < \omega^*\tau_1 + \theta(\omega^*) < 5\pi/2$, the graphs of F and P may or may not have intersections. See Figure 3, we choose the same parameter values as in Figure 1, but different values for τ_1. We actually can see that if τ_1 satisfies that $\omega^*\tau_1 + \theta(\omega^*)$ close to $5\pi/2$, then the two curves will intersect.

Remark 2.5. From Theorem 2.3, we can see that under the assumption of $4\beta > \alpha\gamma$, positive root exists and is unique when $0 \leq \tau_1 < \tau_1^*$. As τ_1 increases, the existence of positive roots is lost and then as τ_1 continues to increase, the existence of positive

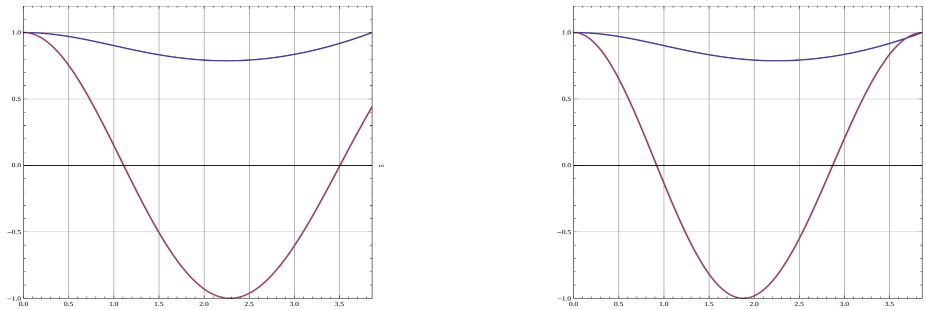

FIGURE 3. Left: $\tau_1 = 1.3$, $2\pi < \omega^*\tau_1 + \theta(\omega^*) = 6.73963 < 5\pi/2$. There are no intersections. Right: $\tau_1 = 1.585$, $2\pi < \omega^*\tau_1 + \theta(\omega^*) = 7.84003 < 5\pi/2$. There are two intersections.

roots comes back and remains. At the same time, under the assumption of $4\beta < \alpha\gamma$, no positive roots exist until τ_1 reaches a certain value, for instance, $\omega^*\tau_1 + \theta(\omega^*) > 2\pi$, or at least close to $5\pi/2$.

Solving Sys. (2.8) yields

$$\sin\omega\tau_2 = \frac{\alpha^2\gamma^2\omega + \omega^3 - 2\alpha\beta\gamma\omega\cos\omega\tau_1 + 2\alpha^2\beta\gamma^2\sin\omega\tau_1}{\alpha^2\beta\gamma^2 + \beta\omega^2} \equiv f(\omega),$$
$$\cos\omega\tau_2 = -\frac{\alpha^2\gamma^2 + \omega^2 - 2\alpha^2\gamma^2\cos\omega\tau_1 - 2\alpha\gamma\omega\sin\omega\tau_1}{\alpha^2\gamma^2 + \omega^2} \equiv g(\omega).$$

Let $\tau_1 > 0$. Suppose that Eq. (2.10) has n positive roots in $(0, \omega^*]$, say $\omega_1 < \omega_2 < \cdots < \omega_n$. Define, for each j, $j = 1, 2, \ldots, n$,

$$\tau_2^j = \begin{cases} \frac{1}{\omega_j}\arccos g(\omega_j) & \text{if } f(\omega_j) > 0, \\ \frac{1}{\omega_j}\left(2\pi - \arccos g(\omega_j)\right) & \text{if } f(\omega_j) \le 0, \end{cases}$$

and let

$$\tau_2^+ = \min\left\{\tau_2^j : j = 1, 2, \ldots, n\right\} > 0. \tag{2.11}$$

Let $\tau_1 > 0$ be such that Eq. (2.10) has at least one positive root. Define $\lambda(\tau_2) = \sigma(\tau_2) + i\omega(\tau_2)$ to be the root of Eq. (2.4) such that $\sigma(\tau_2^+) = 0$ and $\omega(\tau_2^+) = \omega_k$, $1 \le k \le n$, respectively. Differentiating both sides of Eq. (2.4) with respect to τ_2 gives

$$\left(\frac{d\lambda}{d\tau_2}\right)^{-1} = \frac{2\lambda + \beta - \alpha\gamma - 2\alpha\beta\gamma\tau_1 e^{-\lambda\tau_1} + \beta(1 + \alpha\beta\gamma\tau_2 - \lambda\tau_2)e^{-\lambda\tau_2}}{\beta\lambda(\lambda - \alpha\gamma)e^{-\lambda\tau_2}}$$

and at $\tau_2 = \tau_2^+$, it follows

$$\begin{aligned} \mathrm{Re}\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\Big|_{\tau=\tau_2^+} &= \sigma'(\tau_2^+) \\ &= \frac{\alpha^2\gamma^2\omega_k + 2\omega_k^3 - 2\alpha\beta\gamma(2 + \beta\tau_1 - \alpha\gamma\tau_1)\omega_k\cos\omega_k\tau_1}{\beta^2(\alpha^2\gamma^2\omega_k + \omega_k^3)} \end{aligned}$$

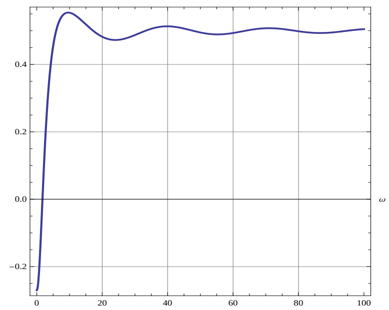

FIGURE 4. A typical curve of function Q for $\tau_1 = 0.2$.

$$+\frac{2\alpha\beta\gamma(-\beta+\alpha\gamma+\alpha\beta\gamma\tau_1+\tau_1\omega_k^2)\sin\omega_k\tau_1}{\beta^2(\alpha^2\gamma^2\omega_k+\omega_k^3)}.$$

For a given $\tau_1 > 0$, define

$$Q(\omega) = \frac{\alpha^2\gamma^2\omega + 2\omega^3 - 2\alpha\beta\gamma(2+\beta\tau_1-\alpha\gamma\tau_1)\omega\cos\omega\tau_1}{\beta^2(\alpha^2\gamma^2\omega+\omega^3)} + \frac{2\alpha\beta\gamma(-\beta+\alpha\gamma+\alpha\beta\gamma\tau_1+\tau_1\omega^2)\sin\omega\tau_1}{\beta^2(\alpha^2\gamma^2\omega+\omega^3)}. \tag{2.12}$$

A typical curve of function Q is given in Figure 4. The parameter values are the same as in figure 1. In this case, $G(\omega) < 0$ if $\omega < 1.73321$ and $Q(\omega) > 0$ if $\omega > 1.73321$.

The distribution of the roots of Eq. (2.4) in the (τ_1, τ_2) plane for $4\beta > \alpha\gamma$ is illustrated in Figure 5. Again we choose $\alpha = 5, \beta = 2$, and $\gamma = 0.56$. Then $4\beta > \alpha\gamma$ and $\tau_1^* = 0.359687$. As τ_1 increases from 0, the curves of F and P have a unique intersection until τ_1 reaches τ_1^*. The two curves no longer intersect as τ_1 continue to increase until τ_1 reaches $\tau_1^{**} = 1.5676$ where the curves regain the intersection, and actually there are multiple intersections as τ_1 continues to increase. Consequently, we have if (τ_1, τ_2) lies in Region I, Eq. (2.4) has a simple zero root and all other roots have negative real parts. If (τ_1, τ_2) lies on the curve L, which is given by τ_2^+ as a function of τ_1, Eq. (2.4) has a simple zero root, a pair of purely imaginary roots, and all other roots have negative real parts. If (τ_1, τ_2) lies in Region II, because of the unique intersection of F and P and $\sigma'(\tau_2^+) > 0$, (by calculation of Q) Eq. (2.4) has a simple zero root, a finite number of roots with positive real parts, and all the other roots have negative real parts. If (τ_1, τ_2) lies in Region III, because the curves of F and P may have multiple intersections, the distribution of roots of Eq. (2.4) is much more complicated.

We make the following assumptions based on parameters α, β, γ and the values of τ_1.

(H1) $4\beta > \alpha\gamma$, $0 < \tau_1 < \tau_1^*$;
(H2) $4\beta > \alpha\gamma$, $\tau_1 > \tau_1^*$ and $\omega^*\tau_1 + \theta(\omega^*) \le 2\pi$;

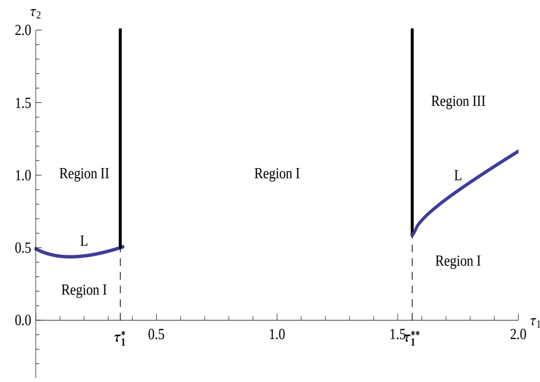

FIGURE 5. The distribution of roots of Eq. (2.4) in the (τ_1, τ_2) plane.

(H3) $4\beta > \alpha\gamma$, $\tau_1 > \tau_1^*$ and $\omega^*\tau_1 + \theta(\omega^*) \geq \frac{5\pi}{2}$;
(H4) $4\beta < \alpha\gamma$, $\tau_1 > 0$ and $\omega^*\tau_1 + \theta(\omega^*) \leq 2\pi$;
(H5) $4\beta < \alpha\gamma$, $\tau_1 > 0$ and $\omega^*\tau_1 + \theta(\omega^*) \geq 5\pi/2$.

We then have the following theorem.

Theorem 2.6. *Let $q = \beta$, $k = 2\gamma$, $\tau_1 > 0$, and τ_2^+ be defined above.*

1. *Under the assumption (H1), Eq.* (2.4) *has a simple zero root and all other roots have negative real parts for all $\tau_2 < \tau_2^+$.*
2. *Under the assumption (H1), Eq.* (2.4) *has a simple zero root, a pair of purely imaginary roots, and all other roots have negative real parts if $\tau_2 = \tau_2^+$.*
3. *Under the assumption (H2), Eq.* (2.4) *has a simple zero root and all other roots have negative real parts for all $\tau_2 \geq 0$.*
4. *Under the assumption (H3), Eq.* (2.4) *has a simple zero root and all other roots have negative real parts for all $\tau_2 < \tau_2^+$.*
5. *Under the assumption (H3), Eq.* (2.4) *has a simple zero root, a pair of purely imaginary roots, and all other roots have negative real parts if $\tau_2 = \tau_2^+$.*
6. *Under the assumption (H4), Eq.* (2.4) *has a simple zero root and all other roots have negative real parts for $\tau_2 \geq 0$.*
7. *Under the assumption (H5), Eq.* (2.4) *has a simple zero root and all other roots have negative real parts for all $\tau_2 < \tau_2^+$.*
8. *Under the assumption (H5), Eq.* (2.4) *has a simple zero root, a pair of purely imaginary roots, and all other roots have negative real parts if $\tau_2 = \tau_2^+$.*

Remark 2.7. In all cases, the purely imaginary roots are simple.

Theorem 2.8. *Suppose either the assumption (H1), (H3) or (H5) holds. Then if $\sigma'(\tau_2^+) > 0$, Sys.* (2.1) *exhibits a zero-Hopf bifurcation at $\tau_2 = \tau_2^+$*

3. COMPUTATION OF NORMAL FORM OF ZERO-HOPF SINGULARITY FOR $\tau_1, \tau_2 \geq 0$

In this section, we use the framework developed in [5, 6] (see the detail in Appendix I) to obtain the normal form for zero-Hopf singularity of Krawiec-Szydlowski model with delays. We always assume that $q = \beta$, $k = 2\gamma$. From Theorem 2.6 in Section 2, we know that, under the assumption of either (H1), (H3) or (H5), if $\tau_2 = \tau_2^+$, Eq. (2.4) has a simple zero root, a pair of purely imaginary roots, and all other roots have negative real parts. We assume that the pair of purely imaginary roots given by Eq. (2.9) are $\pm i\omega$. Now we use k, τ_2 as bifurcation parameters and fix $\tau_1 > 0$. Let $k = 2\gamma + \mu_1, \tau_2 = \tau_2^+ + \mu_2$. Then $\mu = (\mu_1, \mu_2)$ is the bifurcation parameter of the following system

$$(3.1) \qquad \begin{cases} \dot{u}_1(t) &= -\alpha\beta u_2(t) + \alpha i^{(2)} u_1^2(t) + \alpha i^{(3)} u_1^3(t) + \mathcal{O}(u_1^4), \\ \dot{u}_2(t) &= (2\gamma + \mu_1) u_1(t - \tau_1) - \beta u_2(t) - \beta u_2(t - \tau_2^+ - \mu_2) \\ & \quad + i^{(2)} u_1^2(t - \tau_1) + i^{(3)} u_1^3(t - \tau_1) + \mathcal{O}(u_1^4). \end{cases}$$

The linear part of Sys. (3.1) at (0,0) is

$$\begin{cases} \dot{u}_1 = -\alpha\beta u_2(t), \\ \dot{u}_2 = (2\gamma + \mu_1) u_1(t - \tau_1) - \beta u_2(t) - \beta u_2(t - \tau_2^+ - \mu_2). \end{cases}$$

Let

$$\eta(\theta, \mu) = A\delta(\theta) + B_1(\mu_1)\delta(\tau_1) + B_2\delta(\theta + \tau_2^+ + \mu_2)$$

where

$$A = \begin{pmatrix} 0 & -\alpha\beta \\ 0 & -\beta \end{pmatrix}, \quad B_1(\mu_1) = \begin{pmatrix} 0 & 0 \\ 2\gamma + \mu_1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & -\beta \end{pmatrix}.$$

Define

$$L(\mu)\varphi = \int_{-\tau_2^+}^{0} d\eta(\theta, \mu)\varphi(\theta), \quad \forall \varphi \in C,$$

where $C = C([-\tau_2^+, 0), \mathbb{C}^2)$ with norm $|\varphi|_\infty = \max\{|\varphi_1|_\infty, |\varphi_2|_\infty\}$ for $\varphi = (\varphi_1, \varphi_2)^T \in C$. Let $X = (u_1, u_2)^T$ and $F(X_t) = (F^1, F^2)^T$, where

$$F^1 = \alpha i^{(2)} u_1^2(t) + \alpha i^{(3)} u_1^3(t) + \mathcal{O}(u_1^4),$$
$$F^2 = i^{(2)} u_1^2(t - \tau_1) + i^{(3)} u_1^3(t - \tau_1) + \mathcal{O}(u_1^4).$$

Then Sys. (3.1) can be transformed into

$$(3.2) \qquad \dot{X}(t) = L(\mu)X_t + F(X_t).$$

Write the Taylor expansion of F as

$$F(\varphi) = \frac{1}{2}F_2(\varphi) + \frac{1}{3!}F_3(\varphi) + \mathcal{O}(\|\varphi\|^4).$$

Take the enlarged space of C

$$BC = \{\varphi : [-\tau_2^+, 0] \to \mathbb{C}^2 : \varphi \text{ is continuous on } [-\tau_2^+, 0),\ \exists \lim_{\theta\to 0^-} \varphi(\theta) \in \mathbb{C}^2\}.$$

Then the infinitesimal generator $\mathcal{A}_\mu : C^1 \to BC$ associated with L is given by

$$\begin{aligned}\mathcal{A}_\mu\varphi &= \dot\varphi + X_0[L(\mu)\varphi - \dot\varphi(0)]\\ &= \begin{cases}\dot\varphi, & \text{if } -\tau_2^+ \le \theta < 0,\\ \int_{-\tau_2^+}^0 d\eta(t,\mu)\varphi(t), & \text{if } \theta = 0,\end{cases}\end{aligned}$$

and its adjoint

$$\mathcal{A}_\mu^*\psi = \begin{cases}-\dot\psi, & \text{if } 0 < s \le \tau_2^+,\\ \int_{-\tau_2^+}^0 \psi(-t)d\eta(t,\mu), & \text{if } s = 0,\end{cases}$$

$\forall\psi \in C^{1*}$, where $C^{1*} = C^1((0,\tau_2^+], \mathbb{C}^{2*})$. Let $C' = C((0,\tau_2^+], \mathbb{C}^{2*})$ and the bilinear inner product between C and C' is given by

$$\begin{aligned}\langle\psi,\varphi\rangle &= \bar\psi(0)\varphi(0) - \int_{-\tau_2^+}^0\int_0^\theta \bar\psi(\xi-\theta)d\eta(\theta,0)\varphi(\xi)d\xi\\ &= \bar\psi(0)\varphi(0) + \int_{-\tau_1}^0 \bar\psi(\xi+\tau_1)B_1(0)\varphi(\xi)d\xi + \int_{-\tau_2^+}^0 \bar\psi(\xi+\tau_2^+)B_2\varphi(\xi)d\xi.\end{aligned}$$

From Section 2, we know that $\pm i\omega$ and 0 are eigenvalues of $\mathcal{A}_0$ and $\mathcal{A}_0^*$. Now we compute eigenvectors of $\mathcal{A}_0$ associated with $i\omega$ and 0 and an eigenvector of $\mathcal{A}_0^*$ associated with $-i\omega$ and 0. Let $q_1(\theta) = (\rho, 1)^T e^{i\omega\theta}$, $q_2 = (\sigma, 1)^T$ be eigenvectors of $\mathcal{A}_0$ associated with $i\omega$ and 0, respectively. Then $\mathcal{A}_0 q_1(\theta) = i\omega q(\theta)$, $\mathcal{A}_0 q_2 = 0$. It follows from the definition of $\mathcal{A}_0$ that

$$(i\omega I - A - B_1(0)e^{-i\tau_1\omega} - B_2 e^{-i\tau_2^+\omega})q_1(0) = 0, \quad (A + B_1(0) + B_2)q_2 = 0,$$

from which we obtain

$$\rho = \frac{\alpha\beta}{\alpha\gamma - i\omega}, \quad \sigma = \frac{\beta}{\gamma}.$$

Similarly, we compute eigenvectors of $\mathcal{A}_0^*$ associated with $-i\omega$ and 0. Let $p_1(s) = \frac{1}{D_1}(\delta, 1)e^{-i\omega s}, p_2 = \frac{1}{D_2}(\varepsilon, 1)$ be eigenvectors of $\mathcal{A}_0^*$ associated with $-i\omega$ and 0, respectively. Then $\mathcal{A}_0^* p_1(s) = -i\omega p_1(s)$, $\mathcal{A}_0^* p_2 = 0$. It follows from the definition of $\mathcal{A}_0^*$ that

$$p_1(0)(i\omega I - A - B_1(0)e^{-i\tau_1\omega} - B_2 e^{-i\tau_2^+\omega}) = 0, \quad p_2(A + B_1(0) + B_2) = 0.$$

from which we obtain

$$\delta = \frac{2e^{i\omega\tau_1}\gamma}{\alpha\gamma + i\omega}, \quad \varepsilon = -\frac{2}{\alpha}.$$

In order to assure $\langle p_1, q_1\rangle = 1$ and $\langle p_2, q_2\rangle = 1$, we have to determine factors D_1 and D_2. In fact

$$D_1 = 1 + \bar\rho(\sigma + 2e^{i\omega\tau_1}\gamma\tau_1) - e^{-i\omega\tau_2^+}\beta\tau_2^+, \quad D_2 = \frac{-2\beta + \alpha\gamma + \alpha\beta\gamma(2\tau_1 - \tau_2^+)}{\alpha\gamma}.$$

Let us compute $g_2^1(x,0,\mu)$ first. Since

$$\frac{1}{2}f_2^1(x,y,\mu)=\frac{1}{2}\Psi(0)F_j(\Phi x+y)$$

we have

$$\begin{aligned}\frac{1}{2}g_2^1(x,0,\mu)&=\frac{1}{2}\mathrm{Proj}_{S_1}\Psi(0)F_j(\Phi x)+\text{h.o.t.}\\&=\begin{pmatrix}(a_{11}\mu_1+a_{12}\mu_2)x_1+a_{13}x_1x_3\\(\bar a_{11}\mu_1+\bar a_{12}\mu_2)x_2+\bar a_{13}x_2x_3\\(a_{21}\mu_1+a_{22}\mu_2)x_3+a_{23}x_1x_2+a_{24}x_3^2\end{pmatrix}+\text{h.o.t.}\end{aligned}$$

where

$$\begin{aligned}a_{11}&=\frac{e^{-i\tau_1\omega}\rho}{\bar D_1},\quad a_{12}=\frac{ie^{-i\tau_2^+\omega}\beta\omega}{\bar D_1},\quad a_{13}=\frac{4i^{(2)}\beta\rho(2^{-i\omega\tau_1}+\alpha\bar\sigma)}{\gamma\bar D_1},\\a_{21}&=\frac{\beta}{2\gamma D_1},a_{22}=0,\quad a_{23}=\frac{-2i^{(2)}|\rho|^2}{D_2},\quad a_{24}=-\frac{i^{(2)}\beta^2}{\gamma^2D_2}.\end{aligned}$$

Next we compute $g_3^1(x,0,\mu)$. Since its computation is long, we compute it in three steps. Note that

$$\begin{aligned}\frac{1}{6}g_3^1(x,0,\mu)&=\frac{1}{6}\mathrm{Proj}_{\ker(M_2^1)}\tilde f_3^1(x,0,\mu)\\&=\frac{1}{6}\mathrm{Proj}_{S_2}\tilde f_3^1(x,0,0)+\mathcal{O}(|x||\mu|^2+|x|^2|\mu|),\\&=\frac{1}{6}\mathrm{Proj}_{S_2}f_3^1(x,0,0)+\frac{1}{4}\mathrm{Proj}_{S_2}[(D_xf_2^1)(x,0,0)U_2^1(x,0)\\&\quad+(D_yf_2^1)(x,0,0)U_2^2(x,0)]+\mathcal{O}(|\mu|^2|x|+|\mu||x|^2).\end{aligned}$$

Step 1: Compute $\frac{1}{6}\mathrm{Proj}_{S_2}f_3^1(x,0,0)$. Noting that

$$\frac{1}{6}f_3^1(x,0,0)=\begin{pmatrix}\bar Di^{(3)}q^3\tau_0[\alpha(x_1+x_2+x_3)^3+\bar\sigma(e^{-i\tau_0\omega}x_1+e^{i\tau_0\omega}x_2+x_3)^3]\\Di^{(3)}\tau_0[\alpha(x_1+x_2+x_3)^3+\sigma(e^{-i\tau_0\omega}x_1+e^{i\tau_0\omega}x_2+x_3)^3]\\D_1i^{(3)}\tau_0[\alpha(x_1+x_2+x_3)^3+\nu(e^{-i\tau_0\omega}x_1+e^{i\tau_0\omega}x_2+x_3)^3]\end{pmatrix},$$

we have

$$\frac{1}{6}\mathrm{Proj}_{S_2}f_3^1(x,0,0)=\begin{pmatrix}b_{11}x_1^2x_2+b_{12}x_1x_3^2\\\bar b_{11}x_1x_2^2+\bar b_{12}x_2x_3^2\\b_{21}x_1x_2x_3+b_{22}x_3^3\end{pmatrix},$$

where

$$\begin{aligned}b_{11}&=\frac{3}{\bar D_1}i^{(3)}\rho|\rho|^2(e^{-i\tau_1\omega}+\alpha\bar\delta),\quad b_{12}=\frac{3}{\gamma^2\bar D_1}i^{(3)}\rho\beta^2(e^{-i\tau_1\omega}+\alpha\bar\delta),\\b_{21}&=-\frac{6i^{(3)}\beta|\rho|^2}{\gamma D_2},\quad b_{22}=-\frac{i^{(3)}\beta^3}{\gamma^3D_2}.\end{aligned}$$

Step 2: Compute $\frac{1}{4}\mathrm{Proj}_{S_2}[(D_xf_2^1(x,0,0))U_2^1(x,0)]$. Since

$$U_2^1(x,0)=U_2^1(x,\mu)|_{\mu=0}=(M_2^1)^{-1}\mathrm{Proj}_{\mathrm{Im}(M_2^1)}f_2^1(x,0,0)$$

$$= (M_2^1)^{-1}\mathrm{Proj}_{\mathrm{Im}(M_2^1)} i^{(2)} \begin{pmatrix} \frac{1}{\bar{D}_1}[\alpha\bar{\delta}(\rho x_1+\bar{\rho}x_2+\frac{\beta}{\gamma}x_3)^2+(\rho e^{-i\omega\tau_1}x_1+\bar{\rho}e^{i\omega\tau_1}x_2+\frac{\beta}{\gamma}x_3)^2] \\ \frac{1}{D_1}[\alpha\delta(\rho x_1+\bar{\rho}x_2+\frac{\beta}{\gamma}x_3)^2+(\rho e^{-i\omega\tau_1}x_1+\bar{\rho}e^{i\omega\tau_1}x_2+\frac{\beta}{\gamma}x_3)^2] \\ \frac{1}{D_2}[-2(\rho x_1+\bar{\rho}x_2+\frac{\beta}{\gamma}x_3)^2+(\rho e^{-i\omega\tau_1}x_1+\bar{\rho}e^{i\omega\tau_1}x_2+\frac{\beta}{\gamma}x_3)^2] \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2ii^{(2)}e^{-2i\omega\tau_1}}{\omega\bar{D}_1}[-\rho^2(e^{2i\tau_1\omega}+\alpha\bar{\delta})x_1^2+\frac{\bar{\rho}^2}{3}(e^{2i\tau_1\omega}+\alpha\bar{\delta})x_2^2 \\ \quad +\frac{\beta^2(1+\alpha\bar{\delta})}{\gamma^2}x_3^2+2|\rho|^2(1+\alpha\bar{\delta})x_1x_2+\frac{\beta(e^{i\tau_1\omega}+\alpha\bar{\delta})\bar{\rho}}{\gamma}x_2x_3] \\ \frac{2ii^{(2)}e^{-2i\omega\tau_1}}{\omega D_1}[-\frac{1}{3}\rho^2(e^{-2i\tau_1\omega}+\alpha\bar{\delta})x_1^2+\bar{\rho}^2(e^{2i\tau_1\omega}+\alpha\delta)x_2^2 \\ \quad -\frac{\beta^2(1+\alpha\delta)}{\gamma^2}x_3^2-2|\rho|^2(1+\alpha\delta)x_1x_2-\frac{\beta(e^{-i\tau_1\omega}+\alpha\delta)\rho}{\gamma}x_1x_3] \\ \frac{ii^{(2)}}{\omega D_2}[(2-e^{-2i\tau_1\omega})\rho^2x_1^2+(-2+e^{2i\omega\tau_1})x_2^2 \\ \quad +\frac{4(2-e^{-i\omega\tau_1})\beta\rho}{\gamma}x_1x_3+\frac{4(-2+e^{i\omega\tau_1})\beta\bar{\rho}}{\gamma}x_2x_3] \end{pmatrix}$$

we have

$$\frac{1}{4}\mathrm{Proj}_{S_2}[(D_xf_2^1(x,0,0))U_2^1(x,0)] = \begin{pmatrix} c_{11}x_1^2x_2+c_{12}x_1x_3^2 \\ \bar{c}_{11}x_1x_2^2+\bar{c}_{12}x_2x_3^2 \\ c_{21}x_1x_2x_3e_3+c_{22}x_3^3 \end{pmatrix} + \text{h.o.t.}$$

where

$$\begin{aligned} c_{11} = &-\frac{2ie^{-2i\tau_1\omega}(i^{(2)})^2\rho|\rho|^2}{3\bar{D}_1|D_1|^2D_2\gamma\omega}[2\gamma(e^{2i\tau_1\omega}(7+(6+e^{2i\tau_1\omega})\alpha\delta) \\ &+\alpha(1+e^{2i\tau_1\omega}(6+7\alpha\delta))\bar{\delta})\bar{\rho}\bar{D}_1D_2-3D_1(2\gamma\rho(1+\alpha\bar{\delta})(1+e^{2i\tau_0\omega}\alpha\bar{\delta})D_2 \\ &+(-1+2e^{2i\tau_1\omega})\beta(e^{i\tau_1\omega}+\alpha\bar{\delta})\bar{D}_1)], \\ c_{12} = &-\frac{4ie^{-2i\tau_1\omega}(i^{(2)})^2\rho\beta^2}{\bar{D}_1|D_1|^2D_2\gamma^3\omega}[e^{i\tau_1\omega}\gamma(e^{i\tau_1\omega}(2+(1+e^{i\tau_1\omega})\alpha\delta) \\ &+\alpha(1+e^{i\tau_1\omega}(1+2\alpha\delta))\bar{\delta})\bar{\rho}\bar{D}_1D_2-(1+\alpha\bar{\delta})D_1(\gamma\rho(1+e^{i\tau_0\omega}\alpha\bar{\delta})D_2 \\ &+2e^{i\tau_1\omega}(-1+e^{i\tau_1\omega}\beta\bar{D}_1))], \\ c_{21} = &-\frac{4ie^{-2i\tau_1\omega}(i^{(2)})^2|\rho|^2\beta}{|D_1|^2D_2\gamma\omega}[\rho(e^{i\tau_1\omega}(-3+4e^{i\tau_1\omega})+(-1-2e^{i\tau_1\omega}+6e^{2i\tau_1\omega})\alpha\bar{\delta})D_1 \\ &+e^{i\tau_1\omega}(-2+e^{3i\tau_0\omega}\alpha\delta+e^{2i\tau_1\omega}(3+2\alpha\delta)-2e^{i\tau_1\omega}(2+3\alpha\delta))\bar{\rho}\bar{D}_1], \\ c_{22} = &-\frac{4ie^{-i\tau_0\omega}\beta^3(i^{(2)})^2}{|D_1|^2D_2\gamma^3\omega}[(-1+2e^{i\tau_1\omega})\rho(1+\alpha\bar{\delta})D_1+e^{i\tau_0\omega}(-2+e^{i\tau_1\omega})(1+\alpha\delta)\bar{\rho}\bar{D}_1]. \end{aligned}$$

Step 3: Compute $\frac{1}{4}\mathrm{Proj}_{S_2}[(D_yf_2^1(x,0,0))U_2^2(x,0)]$. This is the most difficult part for computing the terms with third order since the computation involves solving linear systems with singular coefficient matrices. Define $h = h(x)(\theta) = U_2^2(x,0)$, and write

$$h(\theta) = \begin{pmatrix} h^{(1)}(\theta) \\ h^{(2)}(\theta) \end{pmatrix} = h_{200}x_1^2+h_{020}x_2^2+h_{002}x_3^2+h_{110}x_1x_2+h_{101}x_1x_3+h_{011}x_2x_3,$$

where $h_{200}, h_{020}, h_{002}, h_{110}, h_{101}, h_{011} \in Q^1$. The coefficients of h are determined by $(M_2^2h)(x) = f_2^2(x,0,0)$, which is equivalent to

$$D_xhJx - A_{Q^1}(h) = (I-\pi)X_0F_2(\Phi x, 0).$$

Applying the definition of A_{Q^1} and π, we obtain

$$\begin{aligned}\dot{h} - D_x h J x &= \Phi(\theta)\Psi(0)F_2(\Phi x, 0),\\ \dot{h}(0) - Lh &= F_2(\Phi x, 0),\end{aligned}$$

where $\dot{h}$ denotes the derivative of $h(\theta)$ relative to θ. Let

$$F_2(\Phi x, 0) = A_{200}x_1^2 + A_{020}x_2^2 + A_{002}x_3^2 + A_{110}x_1x_2 + A_{101}x_1x_3 + A_{011}x_2x_3,$$

where $A_{ijk} \in \mathbb{C}^2$, $0 \le i, j, k \le 2$, $i + j + k = 2$. Comparing the coefficients of x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3, we have that $\bar{h}_{020} = h_{200}$, $\bar{h}_{011} = h_{101}$ and that h_{200}, h_{101}, h_{110}, h_{002} satisfy the following differential equations, respectively,

$$\begin{cases}\dot{h}_{200} - 2i\omega h_{200} = \Phi(\theta)\Psi(0)A_{200},\\ \dot{h}_{200}(0) - L(h_{200}) = A_{200},\end{cases} \tag{3.3}$$

$$\begin{cases}\dot{h}_{101} - i\omega h_{101} = \Phi(\theta)\Psi(0)A_{101},\\ \dot{h}_{101}(0) - L(h_{101}) = A_{101},\end{cases} \tag{3.4}$$

$$\begin{cases}\dot{h}_{110} = \Phi(\theta)\Psi(0)A_{110},\\ \dot{h}_{110}(0) - L(h_{110}) = A_{110},\end{cases} \tag{3.5}$$

$$\begin{cases}\dot{h}_{002} = \Phi(\theta)\Psi(0)A_{002},\\ \dot{h}_{002}(0) - L(h_{002}) = A_{002}.\end{cases} \tag{3.6}$$

Since

$$F_2(u_t, 0) = 2i^{(2)}\begin{pmatrix}\alpha u_1^2(0)\\ 2u_1^2(-\tau_1)\end{pmatrix},$$

we have

$$\begin{aligned}&f_2^1(x, y, 0) = \Psi(0)F_2(\Phi x + y, 0)\\ &= 2i^{(2)}\begin{pmatrix}\bar{D}[\alpha\bar{\delta}(\rho x_1 + \bar{\rho}x_2 + \frac{\beta}{\gamma}x_3 + y_1(0))^2 + (e^{-i\omega\tau_1}\rho x_1 + e^{i\omega\tau_1}\bar{\rho}x_2 + \frac{\beta}{\gamma}x_3 + y_1(-\tau_1))^2]\\ D[\alpha\bar{\delta}(\rho x_1 + \bar{\rho}x_2 + \frac{\beta}{\gamma}x_3 + y_1(0))^2 + (e^{-i\omega\tau_1}\rho x_1 + e^{i\omega\tau_1}\bar{\rho}x_2 + \frac{\beta}{\gamma}x_3 + y_1(-\tau_1))^2]\\ D_1[-2(\rho x_1 + \bar{\rho}x_2 + \frac{\beta}{\gamma}x_3 + y_1(0))^2 + (e^{-i\omega\tau_1}x_1 + e^{i\omega\tau_1}x_2 + x_3 + y_1(-\tau_1))^2]\end{pmatrix},\end{aligned}$$

which gives

$$\begin{aligned}&D_y f_2^1|_{y=0,\mu=0}(h) = 4i^{(2)}\\ &\begin{pmatrix}\bar{D}[\alpha\bar{\delta}(\rho x_1 + \bar{\rho}x_2 + \frac{\beta}{\gamma}x_3)h^{(1)}(0) + (e^{-i\omega\tau_1}\rho x_1 + e^{i\omega\tau_1}\bar{\rho}x_2 + \frac{\beta}{\gamma}x_3)h^{(1)}(-\tau_1)]\\ D[\alpha\bar{\delta}(\rho x_1 + \bar{\rho}x_2 + \frac{\beta}{\gamma}x_3)h^{(1)}(0) + (e^{-i\omega\tau_1}\rho x_1 + e^{i\omega\tau_1}\bar{\rho}x_2 + \frac{\beta}{\gamma}x_3)h^{(1)}(-\tau_1)]\\ D_1[-2(\rho x_1 + \bar{\rho}x_2 + \frac{\beta}{\gamma}x_3)h^{(1)}(0) + (e^{-i\omega\tau_1}x_1 + e^{i\omega\tau_1}x_2 + x_3)h^{(1)}(-\tau_1)]\end{pmatrix}.\end{aligned}$$

Thus

$$\frac{1}{4}\mathrm{Proj}_{S_2} D_y f_2^1|_{y=0,\mu=0} U_2^2 = \begin{pmatrix}d_{11}x_1^2x_2 + d_{12}x_1x_3^2\\ \bar{d}_{11}x_1x_2^2 + \bar{d}_{12}x_2x_3^2\\ d_{21}x_1x_2x_3 + d_{22}x_3^3\end{pmatrix}$$

where

$$
\begin{aligned}
d_{11} &= \frac{i^{(2)}}{\bar{D}_1}[e^{-i\omega\tau_1}\rho h_{110}^{(1)}(-\tau_1) + \alpha\bar{\delta}(\rho h_{110}^{(1)}(0) + \bar{\rho} h_{200}^{(1)}(0)) + e^{i\omega\tau_1}\bar{\rho} h_{200}^{(1)}(-\tau_1)],\\
d_{12} &= \frac{i^{(2)}}{\gamma\bar{D}_1}[e^{-i\omega\tau_1}\gamma\rho h_{002}^{(1)}(-\tau_1) + \alpha\bar{\delta}(\gamma\rho h_{002}^{(1)}(0) + \beta h_{101}^{(1)}(0)) + \beta h_{101}^{(1)}(-\tau_1)],\\
d_{21} &= -\frac{i^{(2)}}{\gamma D_2}[2\gamma\rho h_{011}^{(1)}(0) - e^{-i\omega\tau_1}\gamma\rho h_{011}^{(1)}(-\tau_1) + \gamma\bar{\rho}(2h_{101}^{(1)}(0) - e^{i\omega} h_{101}^{(1)}(-\tau_1))\\
&\quad + 2\beta h_{110}^{(1)}(0) - \beta h_{110}^{(1)}(-\tau_1)],\\
d_{22} &= \frac{i^{(2)}\beta}{\gamma D_2}[-2h_{002}^{(1)}(0) + h_{002}^{(1)}(-\tau_1)].
\end{aligned}
$$

The computation of $h_{ijk}^{(1)}(0)$ and $h_{ijk}^{(1)}(-\tau_1)$ will be carried out in the following lemmas. First we compute $h_{200}^{(1)}$.

Lemma 3.1. *From* (3.3)*, we have*

$$
\begin{aligned}
h_{200}^{(1)}(0) &= 2i^{(2)}\rho^2 e^{-2i\omega\tau_1}[\gamma(1 + e^{2i\omega\tau_1}\alpha\delta)(-3e^{2i\omega\tau_1}\alpha\beta\omega + (2i(-1 + e^{3i\omega\tau_1})\alpha\beta\gamma\\
&\quad + 3e^{2i\omega\tau_1}(\beta + 2i\omega)\omega)\bar{\rho})\bar{D}_1 D_2 + 3D_1((-1 + 2e^{2i\omega\tau_1})\beta(-ie^{2i\omega\tau_1}\omega(-i\beta + 2\omega)\\
&\quad + \alpha\gamma(-i(-1 + e^{2i\omega\tau_1})\beta + e^{2i\omega\tau_1}\omega))\bar{D}_1 + \gamma D_2(2i\alpha\beta\gamma\rho(-1 + e^{i\omega\tau_1})\\
&\quad + e^{2i\omega\tau_1}(-\alpha\beta\omega + \beta\rho\omega + 2i\rho\omega^2 - 2i\alpha^2\beta\gamma\rho\bar{\delta} + 2ie^{i\omega\tau_1}\alpha^2\beta\gamma\rho\bar{\delta} - e^{2i\omega\tau_1}\alpha^2\beta\omega\bar{\delta}\\
&\quad + e^{2i\omega\tau_1}\alpha\beta\rho\omega\bar{\delta} + 2ie^{i\omega\tau_1}\alpha\rho\omega^2\bar{\delta} + \alpha\beta\omega\bar{D}_1 - e^{2i\omega\tau_1}\alpha\beta\omega\bar{D}_1\\
&\quad - 2ie^{2i\omega\tau_1}\alpha\omega^2\bar{D}_1)))]/(3\Delta_1),\\
h_{200}^{(1)}(-\tau_1) &= i^{(2)}\rho^2 e^{-2i\omega\tau_1}[-2\gamma(1 + e^{2i\omega\tau_1}\alpha\delta)(3\alpha\beta\omega - (\beta + 2i\omega)(i(-1 + e^{3i\omega\tau_1})\alpha\gamma + \omega\\
&\quad + 2e^{3i\omega\tau_1}\omega)\bar{\rho})\bar{D}_1 D_2 + 3D_1((-1 + e^{2i\omega\tau_1})\beta(-2ie^{2i\omega\tau_1}\omega(-i\beta + 2\omega)\\
&\quad + \alpha\gamma(-i(-1 + e^{2i\omega\tau_1})\beta + 2e^{2i\omega\tau_1}\omega))\bar{D}_1 + 2\gamma D_2(-i\alpha\beta\gamma\rho(1 - e^{i\omega\tau_1})\\
&\quad - \alpha\beta\omega - \beta\rho\omega + 2e^{i\omega\tau_1}\beta\rho\omega + 2\alpha\gamma\rho\omega - 2e^{i\omega\tau_1}\alpha\gamma\rho\omega - 2i\rho\omega^2 + 4ie^{i\omega\tau_1}\rho\omega^2\\
&\quad - e^{2i\omega\tau_1}\alpha(-(-1 + 2e^{i\omega\tau_1}\rho(\beta + 2i\omega)\omega + \alpha(2(-1 + e^{i\omega\tau_1})\gamma\rho\omega\\
&\quad + \beta(-i(-1 + e^{i\omega\tau_1})\gamma\rho + \omega)))\bar{\delta}/(3\Delta_1),
\end{aligned}
$$

where

$$\Delta_1 = \gamma\omega(2e^{2i\tau_1\omega}\omega(-i\beta + 2\omega) + 2\alpha\gamma((-2 + e^{2i\omega\tau_1})\beta + 2ie^{2i\omega\tau_1}\omega)|D_1|^2 D_2).$$

Proof. From the first equation of (3.3), we have

$$h_{200}(\theta) = e^{2i\omega\theta}\int_0^\theta e^{-2i\omega t}\Phi(t)\Psi(0)A_{200}dt + ce^{2i\omega\theta}$$

where $c \in \mathbb{C}^2$ is a constant and hence

$$\dot{h}_{200}(0) = \Phi(0)\Psi(0)A_{200} + 2i\omega c$$

and

$$L(h_{200}) = B_1(0)\int_0^{-\tau_1} e^{-2i\omega t}\Phi(t)\Psi(0)A_{200}dt + B_2\int_0^{-\tau_2^+} e^{-2i\omega t}\Phi(t)\Psi(0)A_{200}dt + L(e^{2i\omega\theta})c.$$

From the second equation of (3.3), we have

$$\begin{aligned}(2i\omega I - L(e^{2i\omega\theta}))c &= (I-\Phi(0)\Psi(0))A_{200} + B_1(0)\int_{-\tau_1}^0 e^{-2i\omega t}\Phi(t)\Psi(0)A_{200}dt \\ &\quad + B_2\int_{-\tau_2^+}^0 e^{-2i\omega t}\Phi(t)\Psi(0)A_{200}dt \\ &\equiv RHS.\end{aligned}$$

Since $2i\omega$ is not an eigenvalue of L, the matrix $(2i\omega I - L(e^{2i\omega\theta}))$ is invertible. So we have

$$c = (2i\omega\tau_0 I - L(e^{2i\omega\theta}))^{-1}RHS.$$

After easy but long computation, we have the expressions of $h_{200}^{(1)}(0)$ and $h_{200}^{(1)}(-\tau_1)$. $\square$

In order to compute the rest of $h_{ijk}^{(1)}(0)$ and $h_{ijk}^{(1)}(-\tau_1)$, we have to use the following result from Kuznetsov [13].

Lemma 3.2. *For a linear system $Mw = v$ where M is a singular $n\times n$ matrix, there is a unique solution for solving the following bordered system*

$$\begin{pmatrix} M & q \\ p & 0\end{pmatrix}\begin{pmatrix} w \\ u\end{pmatrix} = \begin{pmatrix} v \\ 0\end{pmatrix}$$

where p, q satisfy the following conditions

$$Mq = 0, \quad pM = 0, \quad (p,q) = 1, \quad (p,v) = 0$$

where $(\cdot,\cdot)$ is defined by

$$(x,y) = \sum_{j=1}^n x_j y_j, \quad x = (x_1,\dots,x_n), \quad y = (y_1,\dots,y_n)^T.$$

We write the solution to the system as $w = M^{INV}v$.

Lemma 3.3. *From (3.4), we have*

$$\begin{aligned}h_{101}^{(1)}(0) &= 4i^{(2)}\beta\rho e^{-i\omega(\tau_1-\tau_2^+)}[\gamma(1+e^{i\omega\tau_1}\alpha\delta)(\rho\omega e^{i\omega\tau_1} \\ &\quad + (-i(-1+e^{2i\omega\tau_1})\gamma\rho - e^{i\omega\tau_1}\omega)\bar{D}_1D_2 \\ &\quad + D_1((-1+2e^{i\omega\tau_1})(-e^{-i\omega\tau_1}\gamma\rho\omega + \beta(2i(-1+e^{i\omega\tau_1})\gamma\rho + e^{i\omega\tau_1}\omega))\bar{D}_1 \\ &\quad + \gamma\omega D_2(2\gamma\rho^2(1+e^{i\omega\tau_1}\alpha\bar{\delta})\tau_1 + e^{i\omega\tau_1}(e^{i\omega\tau_1}\alpha-\rho)\bar{D}_1)))/(\gamma\Delta_1),\end{aligned}$$

$$h_{101}^{(1)}(-\tau_1) = \frac{2i^{(2)}\beta\rho}{\gamma^2\omega|D_1|^2D_2}[-i(-1+e^{-2i\omega\tau_1})\gamma(1+e^{i\omega\tau_1}\alpha\delta)\bar{\rho}\bar{D}_1D_2$$

$$
\begin{aligned}
&+ 2e^{-i\omega\tau_1} D_1(i(-1+e^{i\omega\tau_1})(-1+2e^{i\omega\tau_1})\beta\bar{D}_1 \\
&+ e^{-i\omega\tau_1}\gamma\omega D_2(-\rho\tau_1 - e^{i\omega\tau_1}\alpha\rho\bar{\delta}\tau_1 + (e^{i\omega\tau_2}(\gamma(1+e^{i\omega\tau_1}\alpha\delta)(e^{i\omega\tau_1}\rho\omega \\
&+ (-i(-1+e^{2i\omega\tau_1})\gamma\rho - e^{i\omega\tau_1}\omega)\bar{\rho})\bar{D}_1 D_2 \\
&+ D_1((-1+2e^{i\omega\tau_1})(-e^{i\omega\tau_1}\gamma\rho\omega + \beta(2i(-1+e^{i\omega\tau_1})\gamma\rho + e^{i\omega\tau_1}\omega))\bar{D}_1 \\
&+ \gamma\omega D_2(2\gamma\rho^2(1+e^{i\omega\tau_1}\alpha\bar{\delta})\tau_1 + e^{i\omega\tau_1}(e^{i\omega\tau_1}\alpha - \rho)\bar{D}_1))))))]/\Delta_1.
\end{aligned}
$$

Proof. From the first equation of (3.4), we have

$$h_{101}(\theta) = e^{i\omega\theta}\int_0^\theta e^{-i\omega t}\Phi(t)\Psi(0)A_{101}dt + ce^{i\omega\theta}$$

and hence

$$\dot{h}_{101}(0) = \Phi(0)\Psi(0)A_{101} + i\omega c$$

and

$$
\begin{aligned}
L(h_{101}) = B_1(0)&\int_0^{-\tau_1} e^{-i\omega t}\Phi(t)\Psi(0)A_{101}dt \\
&+ B_2\int_0^{-\tau_2^+} e^{-i\omega t}\Phi(t)\Psi(0)A_{101}dt + L(e^{i\omega\theta})c.
\end{aligned}
$$

From the second equation of (3.4), we have

$$(i\omega I - L(e^{i\omega\theta}))c = v$$

where

$$
\begin{aligned}
v = (I - \Phi(0)\Psi(0))A_{101} + B_1(0)\int_{-\tau_1}^0 e^{-i\omega\tau_0 t}\Phi(t)\Psi(0)A_{101}dt \\
+ \int_{-\tau_2^+}^0 e^{-i\omega\tau_0 t}\Phi(t)\Psi(0)A_{101}dt.
\end{aligned}
$$

Since $i\omega$ is an eigenvalue of L, the matrix $M \equiv (i\omega I - L(e^{i\omega\theta}))$ is not invertible, where

$$M = \begin{pmatrix} -\alpha\gamma + i\omega & \alpha\beta \\ -2e^{-i\omega\tau_1}\gamma & \beta(1+e^{-i\omega\tau_1}) + i\omega \end{pmatrix}.$$

Consider the following bordered system

$$\begin{pmatrix} M & q \\ p & 0 \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

It is not hard to get

$$q = (\rho, 1)^T, \quad p = d(\delta, 1), \quad d = \frac{1}{\rho\delta + 1}$$

such that $Mq = 0, pM = 0$, $(p, q) = 1$, and $(p, v) = 0$. Then we obtain the unique solution $c = M^{INV}v$. Replacing c in h_{101} and after long computation, we have the expressions of $h_{101}^{(1)}(0)$ and $h_{101}^{(1)}(-\tau_1)$. □

Lemma 3.4. *From* (3.5) *and* (3.6), *we have*

$$\begin{aligned}
h_{110}^{(1)}(0) &= \frac{4i^{(2)}\alpha|\rho|^2}{(-2\beta+\alpha\gamma)|D_1|^2}[|D_1|^2(\beta-\alpha\gamma)-\Re[(1+\alpha\delta)\omega(\beta-\gamma\bar{\rho})\bar{D}_1]],\\
h_{110}^{(1)}(-\tau_1) &= \frac{4i^{(2)}\alpha|\rho|^2e^{-i\omega\tau_1}}{\Delta_2}[e^{i\omega\tau_1}\gamma(1+\alpha\delta)(i\alpha\gamma(-1+e^{i\omega\tau_1}+i\omega)+(-4i(-1\\
&\quad+e^{i\omega\tau_1})\alpha\beta\gamma+\alpha\gamma(i-i\alpha\gamma+ie^{i\omega\tau_1}(-1+\alpha\gamma)+\omega))\bar{\rho})\bar{D}_1D_2\\
&\quad+D_1(e^{i\omega\tau_1}\beta(-2\beta+\alpha\gamma)^2\omega\tau_1\bar{D}_1+\gamma D_2(-i\alpha\beta+ie^{i\omega\tau_1}\alpha\beta-4i\beta^2\rho\\
&\quad+4ie^{i\omega\tau_1}\beta^2\rho+i\alpha\gamma\rho-ie^{i\omega\tau_1}\alpha\gamma\rho+4i\alpha\beta\gamma\rho-4ie^{i\omega\tau_1}\alpha\beta\gamma\rho-i\alpha^2\gamma^2\rho\\
&\quad+ie^{i\omega\tau_1}\alpha^2\gamma^2\rho-e^{i\omega\tau_1}\alpha\beta\omega+e^{i\omega\tau_1}\alpha\gamma\rho\omega+\alpha(4i(-1+e^{i\omega\tau_1})\beta^2\rho\\
&\quad+i(-1+e^{i\omega\tau_1})\alpha^2\gamma^2\rho+\alpha(\gamma\rho(i+e^{i\omega\tau_1}(-i+\omega))-\beta(i-4i\gamma\rho\\
&\quad+e^{i\omega\tau_1}(-i+4i\gamma\rho+\omega)))\bar{\delta}-e^{i\omega\tau_1})\alpha(-\beta+\alpha\gamma)\omega\bar{D}_1))]\\
h_{002}^{(1)}(0) &= \frac{2i^{(2)}\alpha\beta^2e^{-i\omega(2\tau_1+\tau_2^+)}}{\gamma\Delta_2}[-e^{i\omega(\tau_1+\tau_2^+)}(1+\alpha\delta)(\beta\omega-\gamma(2i(-1+e^{i\omega\tau_1})\beta\\
&\quad+\omega)\bar{\rho})\bar{D}_1D_2+e^{i\omega\tau_2}D_1(2e^{i\omega\tau_1}\beta^2\omega\tau_1\bar{D}_1+D_2(-2i\beta\gamma\rho+2ie^{i\omega\tau_1}\beta\gamma\rho\\
&\quad-e^{i\omega\tau_1}\beta\omega+e^{i\omega\tau_1}\gamma\rho\omega+\alpha(e^{i\omega\tau_1}\gamma\rho\omega+i\beta(2(-1+e^{i\omega\tau_1})\gamma\rho+ie^{i\omega\tau_1}\omega))\bar{\delta}\\
&\quad+e^{i\omega\tau_1}(\beta-\alpha\gamma)\omega\bar{D}_1))],\\
h_{002}^{(1)}(-\tau_1) &= \frac{2i^{(2)}\alpha\beta^2e^{-i\omega(2\tau_1+\tau_2^+)}}{\gamma^2\Delta_2}[e^{i\omega(\tau_1+\tau_2^+)}\gamma(1+\alpha\delta)(-\alpha\beta\omega+(4i(-1+e^{i\omega\tau_1})\beta^2\\
&\quad-2i(-1+e^{i\omega\tau_1})\alpha\beta\gamma+\alpha(i(-1+e^{i\omega\tau_1})\alpha\gamma+\omega))\bar{\rho}\bar{D}_1D_2\\
&\quad+D_1(e^{i\omega(\tau_1+\tau_2^+)}\beta(4\beta^2-2\alpha\beta\gamma+\alpha^2\gamma^2)\omega\tau_1\bar{D}_1\\
&\quad+e^{i\omega\tau_2}\gamma D_2(-4i\beta^2\rho+4ie^{i\omega\tau_1}\beta^2\rho+2i\alpha\beta\gamma\rho\\
&\quad-2ie^{i\omega\tau_1}\alpha\beta\gamma\rho-i\alpha^2\beta^2\rho+ie^{i\omega\tau_1}\alpha^2\beta^2\rho-e^{i\omega\tau_1}\alpha\beta\omega+e^{i\omega\tau_1}\alpha\gamma\rho\omega\\
&\quad+\alpha(4i(-1+e^{i\omega\tau_1})\beta^2\rho+\alpha\gamma\rho(i(-1+e^{i\omega\tau_1})\alpha\gamma+e^{i\omega\tau_1})\omega)\\
&\quad-\alpha\beta(2i(-1+e^{i\omega\tau_1})\gamma\rho+e^{i\omega\tau_1}\omega))\bar{\delta}-e^{i\omega\tau_1}\alpha(-\beta+\alpha\gamma)\omega\bar{D}_1))],
\end{aligned}$$

where

$$\Delta_2=\gamma(-2\beta+\alpha\gamma)^2\omega|D_1|^2D_2.$$

Proof. From the first equation of (3.5), we have

$$h_{110}(\theta)=\int_0^\theta\Phi(t)\Psi(0)A_{110}dt+c$$

and hence

$$\dot{h}_{110}(0)=\Phi(0)\Psi(0)A_{110}$$

and

$$L(h_{110})=L\int_0^\theta\Phi(t)\Psi(0)A_{110}dt+L(1)c.$$

From the second equation of (3.5), we have $L(1)c = v$ where

$$L(1) = \begin{pmatrix} -\alpha\gamma & -\alpha\beta \\ 2\gamma & -2\beta \end{pmatrix}, \quad v = L(h_{110}) - L\int_0^\theta \Phi(t)\Psi(0)A_{110}dt.$$

Since 0 is an eigenvalue of L, the matrix $L(1)$ is not invertible. Similarly, this difficulty can be overcome by solving the following bordered system

$$\begin{pmatrix} L(1) & \varphi_2(0) \\ \psi_2(0) & 0 \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

This system has a unique solution $w = L(1)^{INV}v$. Thus c can be determined by

$$c = (L(1))^{INV}v.$$

Replacing c in h_{110}, we have the expressions of $h_{110}^{(1)}(0)$ and $h_{110}^{(1)}(-\tau_1)$. Similarly we have the expressions of $h_{002}^{(1)}(0)$ and $h_{002}^{(1)}(-\tau_1)$. □

Thus we obtain

$$\frac{1}{6}g_3^1(x,0,\mu) = \begin{pmatrix} (b_{11}+c_{11}+d_{11})x_1^2x_2 + (b_{12}+c_{12}+d_{12})x_1x_3^2 \\ (\bar{b}_{11}+\bar{c}_{11}+\bar{d}_{11})x_1x_2^2 + (\bar{b}_{12}+\bar{c}_{12}+\bar{d}_{12})x_2x_3^2 \\ (b_{21}+c_{21}+d_{21})x_1x_2x_3 + (b_{22}+c_{22}+d_{22})x_3^3 \end{pmatrix} + \mathcal{O}(|\mu|^2|x| + |\mu||x|^2).$$

So we can express Sys. (A.4) as

$$\begin{cases} \dot{x}_1 = (a_{11}\mu_1 + a_{12}\mu_2)x_1 + a_{13}x_1x_3 + (b_{11}+c_{11}+d_{11})x_1^2x_2 \\ \qquad\qquad +(b_{12}+c_{12}+d_{12})x_1x_3^2 + \text{h.o.t.}, \\ \dot{x}_2 = (\bar{a}_{11}\mu_1 + \bar{a}_{12}\mu_2)x_2 + \bar{a}_{13}x_2x_3 + (\bar{b}_{11}+\bar{c}_{11}+\bar{d}_{11})x_1x_2^2 \\ \qquad\qquad +(\bar{b}_{12}+\bar{c}_{12}+\bar{d}_{12})x_2x_3^2 + \text{h.o.t.}, \\ \dot{x}_3 = (a_{21}\mu_1 + a_{22}\mu_2)x_3 + a_{23}x_1x_2 + a_{24}x_3^2 + (b_{21}+c_{21}+d_{21})x_1x_2x_3 \\ \qquad\qquad +(b_{22}+c_{22}+d_{22})x_3^3 + \text{h.o.t.}. \end{cases} \tag{3.7}$$

Since $x_1 = \bar{x}_2$, through the change of variables $x_1 = w_1 - iw_2$, $x_2 = w_1 + iw_2$, $x_3 = w_3$, and then a change to cylindrical coordinates according to $w_1 = r\cos\xi$, $w_2 = r\sin\xi$, $w_3 = \zeta$, Sys. (3.7) becomes

$$\begin{cases} \dot{r} = \alpha_1(\mu)r + \beta_{11}r\zeta + \beta_{30}r^3 + \beta_{12}r\zeta^2 + \text{h.o.t.}, \\ \dot{\zeta} = \alpha_2(\mu)\zeta + \gamma_{20}r^2 + \gamma_{02}\zeta^2 + \gamma_{21}r^2\zeta + \gamma_{03}\zeta^3 + \text{h.o.t.}, \\ \dot{\xi} = -\omega + (\text{Im}[a_{11}]\mu_1 + \text{Im}[a_{12}]\mu_2)\zeta + \text{h.o.t.}, \end{cases}$$

where

$$\alpha_1(\mu) = \text{Re}[a_{11}]\mu_1 + \text{Re}[a_{12}]\mu_2, \beta_{11} = \text{Re}[a_{13}], \beta_{30} = \text{Re}[b_{11}+c_{11}+d_{11}],$$
$$\beta_{12} = \text{Re}[b_{12}+c_{12}+d_{12}], \alpha_2(\mu) = a_{21}\mu_1, \gamma_{20} = a_{23}, \gamma_{02} = a_{24},$$
$$\gamma_{21} = b_{21}+c_{21}+d_{21}, \gamma_{03} = b_{22}+c_{22}+d_{22}.$$

Since the third equation describes a rotation around the ζ-axis, it is irrelevant to our discussion and we shall omit it. Hence we obtain a system in the plane (r, ζ), up to the third order,

$$\begin{cases} \dot{r} = \alpha_1(\mu)r + \beta_{11}r\zeta + \beta_{30}r^3 + \beta_{12}r\zeta^2 + \text{h.o.t.}, \\ \dot{\zeta} = \alpha_2(\mu)\zeta + \gamma_{20}r^2 + \frac{1}{2}\gamma_{20}\zeta^2 + \gamma_{21}r^2\zeta + \gamma_{03}\zeta^3 + \text{h.o.t.}. \end{cases} \tag{3.8}$$

Then we have the following result.

Theorem 3.5. *Suppose that $q = \beta$. Then under the assumption (H1), (H3) or (H5), near $k = 2\gamma$, $\tau_2 = \tau_2^+$, on the center manifold, Sys.* (3.1) *is equivalent to Sys.* (3.8).

4. NUMERICAL SIMULATIONS

In this section, we will provide some numerical examples to support our theoretical results obtained in the previous section. Let $\alpha = 5$, $\beta = 2$, $\gamma = 0.56$, and $q = \beta$ and $k = 2\gamma$. It is easy to check that

$$4\beta - \alpha\gamma = 5.2 > 0, \quad \tau_1^* = 0.359687 > 0.$$

Choose $\tau_1 = 0.2$ and hence the assumption (H1) holds. Easy calculation shows that

$$\omega = 2.452983540195543, \quad \tau_2^+ = 0.4431345588112307.$$

From the function Q in Section 2, we know that $Q(\omega) > 0$ at $\omega = 2.452983540195543$, which means $\sigma'(\tau_2^+) > 0$. Therefore, Sys. (2.1) exhibits a zero-Hopf singularity at τ_2^+. Choose the function $i(s)$ as

$$i(s) = (2\gamma + \mu_1)s + \frac{1}{10}s^2 - \frac{1}{6}s^3.$$

Hence $i^{(2)} = \frac{1}{10}$ and $i^{(3)} = -\frac{1}{6}$ and $(0, 0)$ is an equilibrium point. Using the algorithm in Section 3, we obtain the coefficients a_{ij}, b_{ij}, c_{ij} and d_{ij} in Sys. (3.7)

$$\begin{aligned}
&a_{11} = 3.54795 - 0.468034i, \quad a_{12} = 6.48558 + 0.807035i, \\
&a_{13} = -0.664779 - 2.46825i, \quad a_{21} = -3.46848, \quad a_{22} = 0, \quad a_{23} = 2.80339, \\
&a_{24} = 2.47749, \quad b_{11} = 3.35817 + 12.4685i, \quad b_{12} = 5.93553 + 22.038i, \\
&b_{21} = -50.0606, \quad b_{22} = -14.7469, \quad c_{11} = -1.88723 - 1.21601i, \\
&c_{12} = -8.8665 - 2.00165i, \quad c_{21} = 11.624, \quad c_{22} = 7.11144, \\
&d_{11} = -0.974609 - 3.17048i, \quad d_{12} = -2.0221 + 1.3593i, \\
&d_{21} = 11.9912, \quad d_{22} = -3.93954,
\end{aligned}$$

and hence the coefficients in Sys. (3.8)

$$\begin{aligned}
&\alpha_1 = 3.54795\mu_1 + 6.48558\mu_2, \quad \alpha_2 = -3.46848\mu_1, \\
&\beta_{11} = -0.664779, \quad \beta_{12} = -4.95306, \quad \beta_{30} = 0.496329,
\end{aligned}$$

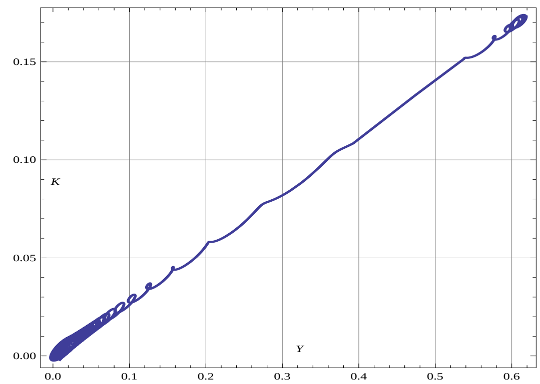

FIGURE 6. When $(\mu_1, \mu_2) = (0.001, -0.000474568)$, the solution curve converges to the nontrivial equilibrium point E_3.

$$\gamma_{20} = 2.80339, \quad \gamma_{02} = 2.47749, \quad \gamma_{21} = -26.4454.$$

Now we use the results in Appendix II to obtain the bifurcation diagrams. For small $\mu = (\mu_2, \mu_2)$, we compute $K_3 \approx -1.48196 \neq 0$. Note that $B = 0.268328 > 0$. Choose $(\mu_1, \mu_2) = (0.001, -0.000474568)$ and hence

$$k = 2\gamma + \mu_1, \quad \tau_2 = \tau_2^+ + \mu_2.$$

Easy calculation shows that $\chi_1 = 0.00142295, \chi_2 = 0.000853588$ and (χ_1, χ_2) lies between the curves N and M. Note that Sys. (3.1) has three equilibrium points

$$E_1(0,0), \quad E_2(-0.00983867, -0.00275483), \quad E_3(0.609839, 0.170755)$$

for this setting. Figure 6 shows that the orbit asymptotically approaches to E_3. This demonstrates the part (a) of Theorem A.1. However if we choose $(\mu_1, \mu_2) = (-0.005, 0.00239382)$, then

$$k = 2\gamma + \mu_1, \quad \tau_2 = \tau_2^+ + \mu_2.$$

Easy calculation shows that $\chi_1 = 1.77837 \times 10^{-16}$, $\chi_2 = 1.04034 \times 10^{-16}$ and (χ_1, χ_2) lies between the curves H and S. Figure 7 shows that an unstable periodic cycle. This demonstrates the part (b) of Theorem A.1.

5. CONCLUSIONS

Krawiec-Szydlowski or Kaldor-Kalecki business models with cycles have been studied extensively recently. Theoretically these models present a rather rich bifurcation phenomena. With one delay in either the investment or saving function, almost all bifurcations have been observed that include Hopf, Bautin, Bogdanov-Takens bifurcations. The study of these business models with two delays is scarce due to the extreme complexity of the analysis. Recently, the authors of this paper have performed a bifurcation analysis for a Kaldor-Kalecki business model with two delays and for a special case the conditions have been established for Hopf bifurcation to

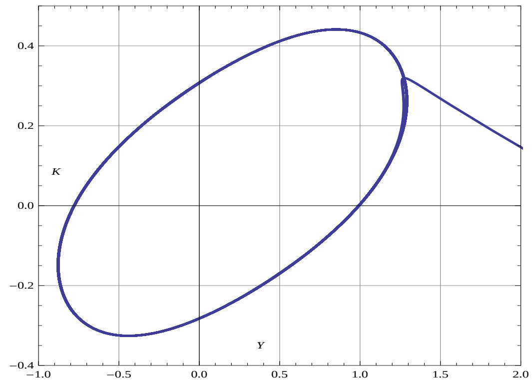

FIGURE 7. An unstable limit cycle is bifurcating from the origin when $(\mu_1, \mu_2) = (-0.005, 0.00239382)$.

occur. In this research, we investigate the zero-Hopf bifurcation for the same model with two positive delays. Again, for a special case we are able to obtain the conditions under which the zero-Hopf singularity occurs. By performing the center manifold reduction, we are able to derive the corresponding normal form on the center manifold and obtain the bifurcation diagram as well as the stability and the direction of the periodic solutions.

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Appendix

I: normal form of zero-Hopf singularity for general DDEs. This material is taken from [5, 6] and [27]. Let $C = C([-\tau, 0), \mathbb{R}^n)$ with norm $|\varphi|_\infty = \max_{1\le k\le n}\{|\varphi_k|_\infty\}$ for $\varphi = (\varphi_1, \ldots, \varphi_n)^T \in C$. Let $X = (u_1, \ldots, u_n)^T \in C$. Let L be a linear operator on C and $F : C \to C$ be of C^4 such that $F(0) = 0$. Consider the following system of DDEs

$$\dot{X}(t) = L(\mu)X_t + F(X_t). \tag{A.1}$$

Write the Taylor expansion of F as

$$F(\varphi) = \frac{1}{2}F_2(\varphi) + \frac{1}{3!}F_3(\varphi) + \mathcal{O}(\|\varphi\|^4).$$

Take the enlarged space of C

$$BC = \{\varphi : [-\tau, 0] \to \mathbb{R}^n : \ \varphi \text{ is continuous on } [-\tau, 0), \ \exists \lim_{\theta\to 0^-} \varphi(\theta) \in \mathbb{R}^n\}.$$

Then the elements of BC can be expressed as $\psi = \varphi + X_0\nu$, $\varphi \in C, \nu \in \mathbb{C}^n$ and

$$X_0(\theta) = \begin{cases} 0, & -\tau \le \theta < 0, \\ I, & \theta = 0, \end{cases}$$

where I is the identity matrix on C and the norm of BC is $|\varphi + X_0\nu| = |\varphi|_\infty + |\nu|$. Let $C^1 = C^1([-\tau, 0), \mathbb{R}^n)$. Then, by the Reisz Representation Theorem, there is η such that the infinitesimal generator $\mathcal{A}_\mu : C^1 \to BC$ associated with L is given by

$$\begin{aligned}\mathcal{A}_\mu\varphi &= \dot{\varphi} + X_0[L(\mu)\varphi - \dot{\varphi}(0)] \\ &= \begin{cases} \dot{\varphi}, & \text{if } -\tau \le \theta < 0, \\ \int_{-\tau}^0 d\eta(t, \mu)\varphi(t), & \text{if } \theta = 0, \end{cases}\end{aligned}$$

and its adjoint

$$\mathcal{A}_\mu^*\psi = \begin{cases} -\dot{\psi}, & \text{if } 0 < s \le \tau_2, \\ \int_{-\tau}^0 \psi(-t)d\eta(t, \mu), & \text{if } s = 0, \end{cases}$$

for $\forall\psi \in C^{1*}$, where $C^{1*} = C^1((0, \tau], \mathbb{R}^{n*})$. Let $C' = C((0, \tau], \mathbb{R}^{n*})$ and define a bilinear inner product between C and C' by

$$\langle\psi, \varphi\rangle = \bar{\psi}(0)\varphi(0) - \int_{-\tau}^0 \int_0^\theta \bar{\psi}(\xi - \theta)d\eta(\theta, 0)\varphi(\xi)d\xi.$$

Suppose that $L(0)$ has eigenvalues $\pm\omega i$ and 0 and other eigenvalues of L have negative real parts. Let $\Phi = (q_1, \bar{q}_1, q_2), \Psi = (\bar{p}_1, p_1, p_2)^T$ be such that

$$\mathcal{A}_0q_1 = i\omega q_1, \quad \mathcal{A}_0q_2 = 0, quad\mathcal{A}_0^*p_1 = -i\omega p_1, \quad \mathcal{A}_0^*p_2 = 0$$

and

$$\Phi'(\theta) = \Phi(\theta)J, \quad \Psi'(s) = -J\Psi(s), \quad \langle\Psi, \Phi\rangle = E$$

where $J = \mathrm{diag}(\omega i, -\omega i, 0), E = \mathrm{diag}(1,1,1)$. Let $P = \mathrm{span}\{q_1, \bar{q}_1, q_2\}$ and $P^* = \mathrm{span}\{\bar{p}_1, p_1, p_2\}$. Then C can be decomposed as

$$C = P \oplus Q \text{ where } Q = \{\varphi \in C : \langle \psi, \varphi \rangle = 0, \forall \psi \in P^*\}.$$

Let $Q^1 = Q \cap C^1$. Define the projection $\pi : BC \to P$ by

$$\pi(\varphi + X_0 \nu) = \Phi[(\Psi, \varphi) + \Psi(0)\nu].$$

Let $u = \Phi x + y$. Then Sys. (A.1) can be decomposed as

$$\begin{cases} \dot{x} = Jx + \Psi(0) F(\Phi x + y, \mu), \\ \dot{y} = A_{Q^1} y + (I - \pi) X_0 F(\Phi x + y, \mu), \end{cases}$$

which can be rewritten as

$$\begin{cases} \dot{x} = Jx + \frac{1}{2} f_2^1(x, y, \mu) + \frac{1}{3!} f_3^1(x, y, \mu) + \text{h.o.t.}, \\ \dot{y} = A_{Q^1} y + \frac{1}{2} f_2^2(x, y, \mu) + \frac{1}{3!} f_3^2(x, y, \mu) + \text{h.o.t.}, \end{cases} \tag{A.2}$$

where "h.o.t." represents the higher order terms and

$$f_j^1(x, y, \mu) = \Psi(0) F_j(\Phi x + y, \mu), \quad f_j^2(x, y, \mu) = (I - \pi) X_0 F_j(\Phi x + y, \mu).$$

Let Y be a normed space and $j, p \in \mathbb{N}$, and let

$$V_j^p(Y) = \left\{ \sum_{|q|=j} c_q x^q : q \in \mathbb{N}_0^q, c_q \in Y \right\}$$

with norm $|\sum_{|q|=j} c_q x^q| = \sum_{|q|=j} |c_q|_Y$. Define M_j to be the operator in $V_j^5(\mathbb{C}^3 \times \ker \pi)$ with the range in the same space by

$$M_j(p, h) = (M_j^1 p, M_j^2 h),$$

where

$$\begin{aligned} M_j^1 p &= M_j^1 \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \omega i \begin{pmatrix} x_1 \frac{\partial p_1}{\partial x_1} - x_2 \frac{\partial p_1}{\partial x_2} - p_1 \\ x_1 \frac{\partial p_2}{\partial x_1} - x_2 \frac{\partial p_2}{\partial x_2} + p_2 \\ x_1 \frac{\partial p_3}{\partial x_1} - x_2 \frac{\partial p_3}{\partial x_2} \end{pmatrix}, \\ M_j^2 h &= M_j^2 h(x, \mu) = D_x h(x, \mu) J x - A_{Q^1} h(x, \mu), \end{aligned} \tag{A.3}$$

with $p(x, \mu) \in V_j^5(\mathbb{C}^3)$, $h(x, \mu)(\theta) \in V_j^5(\ker \pi)$. It is easy to check that $V_j^5(\mathbb{C}^3) = \mathrm{Im}(M_j^1) \oplus \mathrm{Ker}(M_j^1)$ and

$$\ker(M_j^1) = \mathrm{span}\{\mu^p x^q e_k : (q, \bar{\lambda}) = \lambda_k,\ k = 1, 2, 3,\ p \in \mathbb{N}_0^2, q \in \mathbb{N}_0^3,\ |p| + |q| = j\}$$

with $\bar{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (\omega i, -\omega i, 0)$. Since J is a diagonal matrix, the operators M_j^1, $j \geq 2$, defined in $V_j^5(\mathbb{C}^3)$ have a diagonal representation relative to the canonical basis $\{\mu^p x^q e_k : k = 1, 2, 3,\ p \in \mathbb{N}_0^2, q \in \mathbb{N}_0^3,\ |p| + |q| = j\}$ of $V_j^5(\mathbb{C}^3)$ where $e_1 = (1, 0, 0)^T$, $e_2 = (0, 1, 0)^T$, $e_3 = (0, 0, 1)^T$.

On the center manifold, Sys. (A.2) can be transformed as the following normal form:

$$\dot{x} = Jx + \frac{1}{2}g_2^1(x,0,\mu) + \frac{1}{3!}g_3^1(x,0,\mu) + \text{h.o.t.} \tag{A.4}$$

where $g_j^1(x,0,\mu)$ are homogeneous polynomials of degree j in (x,μ) and

$$\begin{aligned} g_2^1(x,0,\mu) &= \mathrm{Proj}_{\ker(M_2^1)} f_2^1(x,0,\mu) = \mathrm{Proj}_{S_1} f_2^1(x,0,\mu) + \mathcal{O}(|\mu|^2), \\ g_3^1(x,0,\mu) &= \mathrm{Proj}_{\ker(M_3^1)} \tilde{f}_3^1(x,0,\mu) = \mathrm{Proj}_{S_2} \tilde{f}_3^1(x,0,0) + \mathcal{O}(|\mu|^2|x|). \end{aligned}$$

Here S_1 and S_2 (see [27] in detail) are spanned in $\mathbb{C}^3$, respectively, by

$$\mu_k x_1 e_1, \quad x_1x_3e_1, \quad \mu_k x_2 e_2, \quad x_2x_3e_2, \quad x_1x_2e_3, \quad \mu_k x_3 e_3, \quad x_3^2 e_3, \quad k = 1, 2,$$

and

$$x_1^2x_2e_1, \quad x_1x_3^2e_1, \quad x_1x_2^2e_2, \quad x_2x_3^2e_2, \quad x_1x_2x_3e_3, \quad x_3^3e_3.$$

and

$$\tilde{f}_3^1(x,0,\mu) = f_3^1(x,0,\mu) + \frac{3}{2}[(D_xf_2^1)(x,0,\mu)U_2^1(x) + (D_yf_2^1)(x,0,\mu)U_2^2(x)].$$

where

$$U_2^1(x,\mu)_{\mu=0} = (M_2^1)^{-1}\mathrm{Proj}_{\mathrm{Im}(M_2^1)} f_2^1(x,0,0) = (M_2^1)^{-1} f_2^1(x,0,0)$$

and $U_2^2(x,\mu)$ is determined by

$$(M_2^2U_2^2)(x,\mu) = f_2^2(x,0,\mu).$$

II: bifurcation diagrams. In order to use the bifurcation diagrams in [3], let us make a change of variables by $r \to r, \zeta \to \zeta + \eta$, where η will be determined later. Then, after truncation of the high order terms, Sys. (3.8) becomes

$$\left\{ \begin{aligned} \dot{r} &= (\alpha_1(\mu) + \beta_{11}\delta + \beta_{12}\eta^2)r + (\beta_{11} + 2\beta_{12}\eta)r\zeta + \beta_{30}r^3 + \beta_{12}r\zeta^2, \\ \dot{\zeta} &= (\alpha_2(\mu)\delta + \gamma_{02}\eta^2 + \gamma_{03}\eta^3) + (\alpha_2(\mu) + 2\gamma_{02}\eta + 3\gamma_{03}\eta^2)\zeta \\ &\quad + (\gamma_{20} + \gamma_{21}\eta)r^2 + (\gamma_{02} + 3\gamma_{03}\eta)\zeta^2 + \gamma_{21}r^2\zeta + \gamma_{03}\zeta^3. \end{aligned} \right. \tag{A.5}$$

Choose $\eta = \eta(\mu)$ such that

$$\alpha_2(\mu) + 2\gamma_{02}\eta + 3\gamma_{03}\eta^2 = 0.$$

For simplicity, we only discuss the case of $\gamma_{02} \neq 0, \gamma_{03} \neq 0$. Clearly, for small $\alpha_2(\mu)$, the equation above has two real roots. We take

$$\delta = \left\{ \begin{array}{ll} \frac{1}{3\gamma_{03}}\left[-\gamma_{02} + \sqrt{\gamma_{02}^2 - 3\gamma_{03}\alpha_2(\mu)}\right] & \text{if } \gamma_{02} > 0, \\ \frac{1}{3\gamma_{03}}\left[-\gamma_{02} - \sqrt{\gamma_{02}^2 - 3\gamma_{03}\alpha_2(\mu)}\right] & \text{if } \gamma_{02} < 0. \end{array} \right.$$

Then $\delta = \delta(\mu)$ is differentiable at $\mu = 0$ and $\delta(0) = 0$. Define

$$\kappa_1 = \alpha_1(\mu) + \beta_{11}\delta + \beta_{12}\delta^2, \quad \kappa_2 = \alpha_2(\mu)\delta + \gamma_{02}\delta^2 + \gamma_{03}\delta^3,$$

$$a = \beta_{11} + 2\beta_{12}\delta, \quad b = \gamma_{20} + \gamma_{21}\delta, \quad c = \gamma_{02} + 3\gamma_{03}\delta,$$

and let $x = r, y = \zeta$. Then (A.5) becomes

$$\begin{cases} \dot{x} = \kappa_1 x + axy + \beta_{30}x^3 + \beta_{12}xy^2, \\ \dot{y} = \kappa_2 + bx^2 + cy^2 + \gamma_{21}x^2y + \gamma_{03}y^3. \end{cases} \tag{A.6}$$

Let

$$x \to \sqrt{|c|}\, x, \quad y \to \sqrt{|b|}\, y, \quad t \to -c\sqrt{|b|}\, t$$

and

$$\chi_1 = -\frac{\kappa_1}{c\sqrt{|b|}}, \quad \chi_2 = -\frac{\kappa_2}{c|b|}.$$

Sys. (A.6) becomes

$$\begin{cases} \dot{x} = \chi_1 x + Bxy + d_1x^3 + d_2xy^2, \\ \dot{y} = \chi_2 + \nu x^2 - y^2 + d_3x^2y + d_4y^3, \end{cases} \tag{A.7}$$

where

$$B = -\frac{a}{c} \neq 0, \quad \nu = -\mathrm{sgn}(bc),$$

and

$$d_1 = -\frac{\beta_{30}|c|}{c\sqrt{|b|}}, \quad d_2 = \frac{\sqrt{|b|}\beta_{12}}{c}, \quad d_3 = -\frac{\gamma_{21}|c|}{c\sqrt{|b|}}, \quad d_4 = -\frac{\sqrt{|b|}\gamma_{03}}{c}.$$

If we assume

$$K_3 = \nu\left(\frac{2}{B} + 2\right) d_1 + \frac{2}{B} d_2 + \nu d_3 + 3d_4 \neq 0.$$

then the qualitative behavior of (A.7) near $(0,0)$ with small χ_1 and χ_2 is the same as that of the following system (see [3]),

$$\begin{cases} \dot{x} = \chi_1 x + Bxy + xy^2, \\ \dot{y} = \chi_2 + \eta x^2 - y^2. \end{cases} \tag{A.8}$$

Since $b = \gamma_{20} + \gamma_{21}\delta$ and $\gamma_{20} = 2\gamma_{02}$, we have $bc = \gamma_{20}\gamma_{02} + \mathcal{O}(|\mu_1|) = \frac{8(i^{(2)})^2\beta^2|\rho|^2}{\gamma^2 D_2^2} + \mathcal{O}(|\mu|)$ and hence $bc > 0$ for small μ if $\gamma_{02} \neq 0$. This implies that $\nu = -1$. Then Sys. (A.8) becomes

$$\begin{cases} \dot{x} = \chi_1 x + Bxy + xy^2, \\ \dot{y} = \chi_2 - x^2 - y^2. \end{cases} \tag{A.9}$$

Note that, for small χ_1 and χ_2, Sys. (A.9) has two trivial equilibrium points $E_{1,2} = (0, \pm\sqrt{\chi_2})$ if $\chi_2 > 0$, and two nontrivial equilibrium points

$$E_{3,4} = \left(\sqrt{\tfrac{1}{2}B\left(-B \pm \sqrt{B^2 - 4\chi_1}\right) + \chi_1 + \chi_2},\ \tfrac{1}{2}\left(-B \pm \sqrt{B^2 - 4\chi_1}\right)\right).$$

Only one of $E_{3,4}$ exists in a small neighborhood of the origin, denoted by E_3. The complete bifurcation diagrams of Sys. (A.9) can be found in [3].

Theorem A.1. *Let $B \neq 0$ be defined above.*

(a). *If* $B < 0$*, then the bifurcation diagram of Sys.* (A.9) *consists of the origin and the following curves:*

$$M = \{(\chi_1, \chi_2) : \chi_2 = 0,\ \chi_1 \neq 0\},$$
$$N = \left\{(\chi_1, \chi_2) : \chi_2 = \frac{1}{B^2}\chi_1^2 + \mathcal{O}(\chi_1^3),\ \chi_1 \neq 0\right\}.$$

Along M *and* N*, saddle-node and pitchfork bifurcations occur, respectively. Sys.* (A.9) *has no periodic orbits. Moreover, if* (χ_1, χ_2) *is in the region between* N *and* M*, the solution of Sys.* (A.9) *goes asymptotically to one of the equilibrium points* E_1, E_2*, and* E_3*.*

(b). *If* $B > 0$*, then the bifurcation diagram of Sys.* (A.9) *consists of the origin, the curves* M*,* N*, and the following curves:*

$$H = \{(\chi_1, \chi_2) : \chi_1 = 0,\ \chi_2 > 0\},$$
$$S = \{(\chi_1, \chi_2) : \chi_1 = -\frac{B}{3B+2}\chi_2 + \mathcal{O}(|\chi_2|^{3/2}),\ \chi_2 > 0\}.$$

Along M *and* N*, we have exactly the same bifurcation as in (a). Along* H *and* S*, Hopf bifurcation and heteroclinic bifurcation occur respectively. If* (χ_1, χ_2) *lies between the curves* H *and* S*, then* (A.9) *has a unique limit cycle which is unstable and becomes a heteroclinic orbit when* $(\chi_1, \chi_2) \in S$*.*