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HYERS-ULAM STABILITY OF HIGHER-ORDER CAUCHY-EULER DYNAMIC EQUATIONS ON TIME SCALES

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. We extend a recent result on third and fourth-order Cauchy-Euler equations by establishing the Hyers-Ulam stability of higher-order linear non-homogeneous Cauchy-Euler dynamic equations on time scales. That is, if an approximate solution of a higher-order Cauchy-Euler equation exists, then there exists an exact solution to that dynamic equation that is close to the approximate one. We generalize this to all higher-order linear non-homogeneous factored dynamic equations with variable coefficients.

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1. INTRODUCTION

Stan Ulam [27] posed the following problem concerning the stability of functional equations: give conditions in order for a linear mapping near an approximately linear mapping to exist. The problem for the case of approximately additive mappings was solved by Hyers [10], who proved that the Cauchy equation is stable in Banach spaces, and the result of Hyers was generalized by Rassias [24]. Obloza [19] appears to be the first author who investigated the Hyers-Ulam stability of a differential equation.

Since then there has been a significant amount of interest in Hyers-Ulam stability, especially in relation to ordinary differential equations, for example see [7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 25, 28]. Also of interest are many of the articles in a special issue guest edited by Rassias [23], dealing with Ulam, Hyers-Ulam, and Hyers-Ulam-Rassias stability in various contexts. Also see Popa et al [5, 20, 21, 22]. András and Mészáros [2] recently used an operator approach to show the stability of linear dynamic equations on time scales with constant coefficients, as well as for certain integral equations. Tunç and Biçer [26] proved the Hyers-Ulam stability of third and fourth-order Cauchy-Euler differential equations. Anderson et al [1, Corollary 2.6] proved the following concerning second-order non-homogeneous Cauchy-Euler equations on time scales:

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Theorem 1.1 (Cauchy-Euler Equation)**.** *Let $\lambda_1, \lambda_2 \in \mathbb{R}$ (or $\lambda_2 = \overline{\lambda_1}$, the complex conjugate) be such that*

$$t + \lambda_k \mu(t) \neq 0, \quad k = 1, 2$$

for all $t \in [a, \sigma(b)]_{\mathbb{T}}$, where $a \in \mathbb{T}$ satisfies $a > 0$. Then the Cauchy-Euler equation

$$x^{\Delta\Delta}(t) + \frac{1 - \lambda_1 - \lambda_2}{\sigma(t)}\, x^{\Delta}(t) + \frac{\lambda_1 \lambda_2}{t\sigma(t)}\, x(t) = f(t), \quad t \in [a, b]_{\mathbb{T}} \tag{1.1}$$

has Hyers-Ulam stability on $[a, b]_{\mathbb{T}}$. To wit, if there exists $y \in \mathrm{C}^{\Delta^2}_{\mathrm{rd}}[a, b]_{\mathbb{T}}$ that satisfies

$$\left| y^{\Delta\Delta}(t) + \frac{1 - \lambda_1 - \lambda_2}{\sigma(t)}\, y^{\Delta}(t) + \frac{\lambda_1 \lambda_2}{t\sigma(t)}\, y(t) - f(t) \right| \leq \varepsilon$$

for $t \in [a, b]_{\mathbb{T}}$, then there exists a solution $u \in \mathrm{C}^{\Delta^2}_{\mathrm{rd}}[a, b]_{\mathbb{T}}$ of (1.1) given by

$$u(t) = e_{\frac{\lambda_1}{t}}(t, \tau_2)\, y(\tau_2) + \int_{\tau_2}^{t} e_{\frac{\lambda_1}{t}}(t, \sigma(s)) w(s) \Delta s, \quad \text{any} \quad \tau_2 \in [a, \sigma^2(b)]_{\mathbb{T}},$$

where for any $\tau_1 \in [a, \sigma(b)]_{\mathbb{T}}$ the function w is given by

$$w(s) = e_{\frac{\lambda_2 - 1}{\sigma(s)}}(s, \tau_1) \left[y^{\Delta}(\tau_1) - \frac{\lambda_1}{\tau_1} y(\tau_1) \right] + \int_{\tau_1}^{s} e_{\frac{\lambda_2 - 1}{\sigma(s)}}(s, \sigma(\zeta)) f(\zeta) \Delta\zeta,$$

such that $|y - u| \leq K\varepsilon$ on $[a, \sigma^2(b)]_{\mathbb{T}}$ for some constant $K > 0$.

The motivation for this work is to extend Theorem 1.1 to the general nth-order Cauchy-Euler dynamic equation, and thus extend the results in [26] as well, with an approach different from [2]. We will show the stability in the sense of Hyers and Ulam of the equation

$$\sum_{k=0}^{n} \alpha_k M_k y(t) = f(t),$$

where

$$M_0 y(t) := y(t), \quad M_{k+1} y(t) := \varphi(t) \left(M_k y\right)^{\Delta}(t), \quad k = 0, 1, \ldots, n-1.$$

This is essentially [4, (2.14)] if $\varphi(t) = t$ and $f(t) = 0$. In the last section we will analyze the nth-order factored equation with differential operators D and I, where $Dy = y^{\Delta}$ and $Iy = y$, of the form

$$\prod_{k=1}^{n} \left(\varphi_k D - \psi_k I\right) y(t) = f(t), \quad t \in [a, b]_{\mathbb{T}},$$

for right-dense continuous functions φ_k and ψ_k, a more general dynamic equation with variable coefficients than the Cauchy-Euler equation. Throughout this work we assume the reader has a working knowledge of time scales as can be found in Bohner and Peterson [3, 4], originally introduced by Hilger [9].

2. HYERS-ULAM STABILITY FOR HIGHER-ORDER CAUCHY-EULER DYNAMIC EQUATIONS

In this section we establish the Hyers-Ulam stability of the higher-order non-homogeneous Cauchy-Euler dynamic equation on time scales of the form

$$\sum_{k=0}^{n} \alpha_k M_k y(t) = f(t), \tag{2.1}$$

where

$$M_0 y(t) := y(t), \quad M_{k+1} y(t) := \varphi(t) \left(M_k y\right)^{\Delta}(t), \quad k = 0, 1, \ldots, n-1$$

for given constants $\alpha_k \in \mathbb{R}$ with $\alpha_n \equiv 1$, and for functions $\varphi, f \in \mathrm{C}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$, using the following definition.

Definition 2.1 (Hyers-Ulam stability)**.** Let $\varphi, f \in \mathrm{C}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ and $n \in \mathbb{N}$. If whenever $M_k x \in \mathrm{C}^{\Delta}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ satisfies

$$\left| \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t) \right| \le \varepsilon, \quad t \in [a,b]_{\mathbb{T}}$$

there exists a solution u of (2.1) with $M_k u \in \mathrm{C}^{\Delta}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ for $k = 0, 1, \ldots, n-1$ such that $|x - u| \le K\varepsilon$ on $[a, \sigma^n(b)]_{\mathbb{T}}$ for some constant $K > 0$, then (2.1) has Hyers-Ulam stability $[a,b]_{\mathbb{T}}$.

Remark 2.2. Before proving the Hyers-Ulam stability of (2.1) we will need the following lemma, which allows us to factor (2.1) using the elementary symmetric polynomials [6] in the n symbols $\rho_1, \ldots, \rho_n$ given by

$$\begin{aligned}
s_1^n &= s_1(\rho_1, \ldots, \rho_n) = \sum_{i} \rho_i \\
s_2^n &= s_2(\rho_1, \ldots, \rho_n) = \sum_{i<j} \rho_i \rho_j \\
s_3^n &= s_3(\rho_1, \ldots, \rho_n) = \sum_{i<j<k} \rho_i \rho_j \rho_k \\
s_4^n &= s_4(\rho_1, \ldots, \rho_n) = \sum_{i<j<k<\ell} \rho_i \rho_j \rho_k \rho_\ell \\
&\vdots \\
s_t^n &= s_t(\rho_1, \ldots, \rho_n) = \sum_{i_1<i_2<\cdots<i_t} \rho_{i_1} \rho_{i_2} \ldots \rho_{i_t} \\
&\vdots \\
s_n^n &= s_n(\rho_1, \ldots, \rho_n) = \rho_1 \rho_2 \rho_3 \ldots \rho_n.
\end{aligned}$$

In general, we let s_i^j represent the ith elementary symmetric polynomial on j symbols. Then, given the α_k in (2.1), introduce the characteristic values $\lambda_k \in \mathbb{C}$ via the elementary symmetric polynomial s_t^n on the n symbols $-\lambda_1, \dots, -\lambda_n$, where $\alpha_n = s_0 \equiv 1$ and

$$\alpha_k = s_{n-k}^n = s_{n-k}(-\lambda_1, \dots, -\lambda_n) = \sum_{i_1<i_2<\cdots<i_{n-k}} (-1)^{n-k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n-k}}. \tag{2.2}$$

Lemma 2.3 (Factorization)**.** *Given $y, \varphi \in \mathrm{C}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ and $\alpha_k \in \mathbb{R}$ with $\alpha_n \equiv 1$, let $M_k y \in \mathrm{C}^{\Delta}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$, where $M_0 y(t) := y(t)$ and $M_{k+1}y(t) := \varphi(t)\left(M_k y\right)^{\Delta}(t)$ for $k = 0, 1, \dots, n-1$. Then we have the factorization*

$$\sum_{k=0}^{n} \alpha_k M_k y(t) = \prod_{k=1}^{n} \left(\varphi D - \lambda_k I\right) y(t), \qquad n \in \mathbb{N}, \tag{2.3}$$

where the differential operator D is defined via $Dx = x^{\Delta}$ for $x \in \mathrm{C}^{\Delta}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$, and I is the identity operator.

Proof. We proceed by mathematical induction on $n \in \mathbb{N}$, utilizing the substitution defined in (2.2). For $n = 1$,

$$\begin{aligned}\sum_{k=0}^{n} \alpha_k M_k y(t) &= \alpha_0 M_0 y(t) + \alpha_1 M_1 y(t) = s_1(-\lambda_1)y(t) + 1 \cdot \varphi(t) y^{\Delta}(t)\\ &= \left(\varphi D - \lambda_1 I\right) y(t)\end{aligned}$$

and the result holds. Assume (2.3) holds for $n \geq 1$. Then we have $\alpha_{n+1} \equiv 1$ and

$$\begin{aligned}\sum_{k=0}^{n+1} \alpha_k M_k y(t) &= \alpha_0 y(t) + \sum_{k=1}^{n} \alpha_k M_k y(t) + M_{n+1} y(t)\\
&= s_{n+1}^{n+1} y(t) + \sum_{k=1}^{n} s_{n+1-k}^{n+1} M_k y(t) + \varphi(t)\left(M_n y\right)^{\Delta}(t)\\
&= -\lambda_{n+1} s_n^n y(t) + \sum_{k=1}^{n} \left(s_{n+1-k}^n - \lambda_{n+1} s_{n-k}^n\right) M_k y(t) + \varphi(t) D\left(M_n y\right)(t)\\
&= -\lambda_{n+1}\left[s_n^n y(t) + \sum_{k=1}^{n} s_{n-k}^n M_k y(t)\right] + \sum_{k=1}^{n} s_{n+1-k}^n M_k y(t)\\
&\quad + \varphi(t) D\left(M_n y\right)(t)\\
&= -\lambda_{n+1} \sum_{k=0}^{n} s_{n-k}^n M_k y(t) + \varphi(t) D\left(\sum_{k=1}^{n} s_{n+1-k}^n M_{k-1} y(t) + M_n y\right)(t)\\
&= -\lambda_{n+1} \sum_{k=0}^{n} s_{n-k}^n M_k y(t) + \varphi(t) D\left(\sum_{k=0}^{n-1} s_{n-k}^n M_k y(t) + M_n y\right)(t)\\
&= -\lambda_{n+1} \sum_{k=0}^{n} s_{n-k}^n M_k y(t) + \varphi(t) D \sum_{k=0}^{n} s_{n-k}^n M_k y(t)\end{aligned}$$

$$= (\varphi(t)D - \lambda_{n+1}I) \sum_{k=0}^{n} s_{n-k}^{n} M_k y(t)$$

$$= (\varphi(t)D - \lambda_{n+1}I) \sum_{k=0}^{n} \alpha_k M_k y(t)$$

$$= (\varphi(t)D - \lambda_{n+1}I) \prod_{k=1}^{n} (\varphi D - \lambda_k I)\, y(t)$$

and the proof is complete. □

Theorem 2.4 (Hyers-Ulam Stability)**.** *Given $y, \varphi, f \in \mathrm{C}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ with $|\varphi| \geq A > 0$ for some constant A, and $\alpha_k \in \mathbb{R}$ with $\alpha_n \equiv 1$, consider* (2.1) *with $M_k y \in \mathrm{C}^{\Delta}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ for $k = 0, \ldots, n-1$. Using the λ_k from the factorization in Lemma* 2.3*, assume*

$$\varphi(t) + \lambda_k \mu(t) \neq 0, \quad k = 1, 2, \ldots, n \tag{2.4}$$

for all $t \in [a, \sigma^{n-1}(b)]_{\mathbb{T}}$. Then (2.1) *has Hyers-Ulam stability on $[a,b]_{\mathbb{T}}$.*

Proof. Let $\varepsilon > 0$ be given, and suppose there is a function x, with $M_k x \in \mathrm{C}^{\Delta}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$, that satisfies

$$\left| \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t) \right| \leq \varepsilon, \quad t \in [a,b]_{\mathbb{T}}.$$

We will show there exists a solution u of (2.1) with $M_k u \in \mathrm{C}^{\Delta}_{\mathrm{rd}}[a,b]_{\mathbb{T}}$ for $k = 0, 1, \ldots, n-1$ such that $|x - u| \leq K\varepsilon$ on $[a, \sigma^n(b)]_{\mathbb{T}}$ for some constant $K > 0$.

To this end, set

$$\begin{aligned}
g_1 &= \varphi x^{\Delta} - \lambda_1 x = (\varphi D - \lambda_1 I)\, x \\
g_2 &= \varphi g_1^{\Delta} - \lambda_2 g_1 = (\varphi D - \lambda_2 I)\, g_1 \\
&\vdots \\
g_k &= \varphi g_{k-1}^{\Delta} - \lambda_k g_{k-1} = (\varphi D - \lambda_k I)\, g_{k-1} \\
&\vdots \\
g_n &= \varphi g_{n-1}^{\Delta} - \lambda_n g_{n-1} = (\varphi D - \lambda_n I)\, g_{n-1}.
\end{aligned}$$

This implies by Lemma 2.3 that

$$g_n(t) - f(t) = \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t),$$

so that

$$|g_n(t) - f(t)| \leq \varepsilon, \quad t \in [a,b]_{\mathbb{T}}.$$

By the construction of g_n we have $\left|\varphi g_{n-1}^{\Delta} - \lambda_n g_{n-1} - f\right| \leq \varepsilon$, that is

$$\left| g_{n-1}^{\Delta} - \frac{\lambda_n}{\varphi} g_{n-1} - \frac{f}{\varphi} \right| \leq \frac{\varepsilon}{|\varphi|} \leq \frac{\varepsilon}{A}.$$

By [1, Lemma 2.3] and (2.4) there exists a solution $w_1 \in \mathrm{C}_{\mathrm{rd}}^{\Delta}[a,b]_{\mathbb{T}}$ of

$$w^{\Delta}(t) - \frac{\lambda_n}{\varphi(t)}w(t) - \frac{f(t)}{\varphi(t)} = 0, \quad \text{or} \quad \varphi(t)w^{\Delta}(t) - \lambda_n w(t) - f(t) = 0, \tag{2.5}$$

$t \in [a,b]_{\mathbb{T}}$, where w_1 is given by

$$w_1(t) = e_{\frac{\lambda_n}{\varphi}}(t,\tau_1)g_{n-1}(\tau_1) + \int_{\tau_1}^{t} e_{\frac{\lambda_n}{\varphi}}(t,\sigma(s))\frac{f(s)}{\varphi(s)}\Delta s, \quad \text{any} \quad \tau_1 \in [a,\sigma(b)]_{\mathbb{T}},$$

and there exists an $L_1 > 0$ such that

$$|g_{n-1}(t) - w_1(t)| \le L_1\varepsilon/A, \quad t \in [a,\sigma(b)]_{\mathbb{T}}.$$

Since $g_{n-1} = \varphi g_{n-2}^{\Delta} - \lambda_{n-1}g_{n-2}$, we have that

$$|\varphi g_{n-2}^{\Delta} - \lambda_{n-1}g_{n-2}(t) - w_1(t)| \le L_1\varepsilon/A, \quad t \in [a,\sigma(b)]_{\mathbb{T}}.$$

Again we apply [1, Lemma 2.3] to see that there exists a solution $w_2 \in \mathrm{C}_{\mathrm{rd}}^{\Delta}[a,\sigma(b)]_{\mathbb{T}}$ of

$$w^{\Delta}(t) - \frac{\lambda_{n-1}}{\varphi(t)}w(t) - \frac{w_1(t)}{\varphi(t)} = 0, \quad \text{or} \quad \varphi(t)w^{\Delta}(t) - \lambda_{n-1}w(t) - w_1(t) = 0,$$

$t \in [a,\sigma(b)]_{\mathbb{T}}$, where w_2 is given by

$$w_2(t) = e_{\frac{\lambda_{n-1}}{\varphi}}(t,\tau_2)g_{n-2}(\tau_2) + \int_{\tau_2}^{t} e_{\frac{\lambda_{n-1}}{\varphi}}(t,\sigma(s))\frac{w_1(s)}{\varphi(s)}\Delta s, \quad \text{any} \quad \tau_2 \in [a,\sigma^2(b)]_{\mathbb{T}},$$

and there exists an $L_2 > 0$ such that

$$|g_{n-2}(t) - w_2(t)| \le L_2L_1\varepsilon/A^2, \quad t \in [a,\sigma^2(b)]_{\mathbb{T}}.$$

Continuing in this manner, we see that for $k = 1,2,\dots,n-1$ there exists a solution $w_k \in \mathrm{C}_{\mathrm{rd}}^{\Delta}[a,\sigma^{k-1}(b)]_{\mathbb{T}}$ of

$$w^{\Delta}(t) - \frac{\lambda_{n-k+1}}{\varphi(t)}w(t) - \frac{w_{k-1}(t)}{\varphi(t)} = 0, \quad \text{or} \quad \varphi(t)w^{\Delta}(t) - \lambda_{n-k+1}w(t) - w_{k-1}(t) = 0,$$

$t \in [a,\sigma^{k-1}(b)]_{\mathbb{T}}$, where w_k is given by

$$w_k(t) = e_{\frac{\lambda_{n-k+1}}{\varphi}}(t,\tau_k)g_{n-k}(\tau_k) + \int_{\tau_k}^{t} e_{\frac{\lambda_{n-k+1}}{\varphi}}(t,\sigma(s))\frac{w_{k-1}(s)}{\varphi(s)}\Delta s, \quad \text{any} \quad \tau_k \in [a,\sigma^k(b)]_{\mathbb{T}}, \tag{2.6}$$

and there exists an $L_k > 0$ such that

$$|g_{n-k}(t) - w_k(t)| \le \prod_{j=1}^{k} L_j\varepsilon/A^k, \quad t \in [a,\sigma^k(b)]_{\mathbb{T}}.$$

In particular, for $k = n-1$,

$$|g_1(t) - w_{n-1}(t)| \le \prod_{j=1}^{n-1} L_j\varepsilon/A^{n-1}, \quad t \in [a,\sigma^{n-1}(b)]_{\mathbb{T}}$$

implies by the definition of g_1 that

$$\left|x^{\Delta}(t)-\frac{\lambda_1}{\varphi(t)}x(t)-\frac{w_{n-1}(t)}{\varphi(t)}\right|\le\prod_{j=1}^{n-1}L_j\varepsilon/A^n,\quad t\in[a,\sigma^{n-1}(b)]_{\mathbb{T}}.$$

Thus there exists a solution $w_n\in \mathrm{C}^{\Delta}_{\mathrm{rd}}[a,\sigma^{n-1}(b)]_{\mathbb{T}}$ of

$$w^{\Delta}(t)-\frac{\lambda_1}{\varphi(t)}w(t)-\frac{w_{n-1}(t)}{\varphi(t)}=0,\quad\text{or}\quad \varphi(t)w^{\Delta}(t)-\lambda_1 w(t)-w_{n-1}(t)=0,$$

$t\in[a,\sigma^{n-1}(b)]_{\mathbb{T}}$, where w_n is given by

$$w_n(t)=e_{\frac{\lambda_1}{\varphi}}(t,\tau_n)x(\tau_n)+\int_{\tau_n}^{t}e_{\frac{\lambda_1}{\varphi}}(t,\sigma(s))\frac{w_{n-1}(s)}{\varphi(s)}\Delta s,\quad\text{any}\quad \tau_n\in[a,\sigma^n(b)]_{\mathbb{T}},\tag{2.7}$$

and there exists an $L_n>0$ such that

$$|x(t)-w_n(t)|\le K\varepsilon:=\prod_{j=1}^{n}L_j\varepsilon/A^n,\quad t\in[a,\sigma^n(b)]_{\mathbb{T}}.\tag{2.8}$$

By construction,

$$\begin{aligned}(\varphi D-\lambda_1 I)\,w_n(t)&=w_{n-1}(t)\\ \prod_{k=1}^{2}(\varphi D-\lambda_k I)\,w_n(t)&=(\varphi D-\lambda_2 I)\,w_{n-1}(t)=w_{n-2}(t)\\ &\ \vdots\\ \prod_{k=1}^{n}(\varphi D-\lambda_k I)\,w_n(t)&=(\varphi D-\lambda_n I)\,w_1(t)\overset{(2.5)}{=}f(t)\end{aligned}$$

on $[a,\sigma^{n-1}(b)]_{\mathbb{T}}$, so that $u=w_n$ is a solution of (2.1), with $u\in\mathrm{C}^{\Delta}_{\mathrm{rd}}[a,\sigma^{n-1}(b)]_{\mathbb{T}}$ and $|x(t)-w_n(t)|\le K\varepsilon$ for $t\in[a,\sigma^n(b)]_{\mathbb{T}}$ by (2.8). Moreover, using (2.7) and (2.6), we have an iterative formula for this solution $u=w_n$ in terms of the function x given at the beginning of the proof. □

3. **EXAMPLE**

Letting $Dy=y^{\Delta}$ and I be the identity operator, consider the non-homogeneous fifth-order Cauchy-Euler dynamic equation

$$\left[(tD)^5+15(tD)^4+85(tD)^3+225(tD)^2+274tD+120I\right]y(t)=f(t)\tag{3.1}$$

for some right-dense continuous function f, on $[a,b]_{\mathbb{T}}$; in factored form it is

$$(tD+5I)(tD+4I)(tD+3I)(tD+2I)(tD+I)y(t)=f(t).$$

If $\mathbb{T}=\mathbb{R}$, this is equivalent to the non-homogeneous fifth-order Cauchy-Euler differential equation

$$t^5y^{(5)}+25t^4y^{(4)}+200t^3y'''+600t^2y''+600ty'+120y=f(t),$$

By Theorem 2.4 we have that (3.1) has Hyers-Ulam stability.

4. HIGHER-ORDER LINEAR NON-HOMOGENEOUS FACTORED DYNAMIC EQUATIONS WITH VARIABLE COEFFICIENTS

Generalizing away from higher-order Cauchy-Euler equations, we consider the following higher-order linear non-homogeneous factored dynamic equations with variable coefficients given by

$$\prod_{k=1}^{n} (\varphi_k D - \psi_k I)\, y(t) = f(t), \quad t \in [a,b)_{\mathbb{T}}, \tag{4.1}$$

where $\varphi_k, \psi_k, f \in \mathrm{C_{rd}}[a,b)_{\mathbb{T}}$ for $k = 1, 2, \ldots, n$, $Dy(t) = y^{\Delta}(t)$, I is the identity operator, and $|\varphi_k(t)| \geq A > 0$ for all $t \in [a,b)_{\mathbb{T}}$, for some constant $A > 0$. Here we allow for $b = \infty$ for those time scales that are unbounded above. Before our main result in this section we need the following lemma.

Lemma 4.1. *Let $\varphi, \psi, f \in \mathrm{C_{rd}}[a,b)_{\mathbb{T}}$ with $|\varphi(t)| \geq A > 0$ for some constant A, and assume*

$$\varphi(t) + \mu(t)\psi(t) \neq 0 \quad \text{and} \quad \int_a^t \left| e_{\frac{\psi}{\varphi}}(t, \sigma(\tau)) \frac{1}{\varphi(\tau)} \right| \Delta\tau < L \tag{4.2}$$

for all $t \in [a,b)_{\mathbb{T}}$, for some constant $0 < L < \infty$. Then the first-order dynamic equation

$$(\varphi D - \psi I)\, y - f = 0$$

has Hyers-Ulam stability on $[a,b)_{\mathbb{T}}$.

Proof. Suppose there exists a function x such that

$$|(\varphi D - \psi I)\, x(t) - f(t)| \leq \varepsilon$$

for some $\varepsilon > 0$, for all $t \in [a,b)_{\mathbb{T}}$. Set

$$q(t) = (\varphi D - \psi I)\, x(t) - f(t), \quad t \in [a,b)_{\mathbb{T}}.$$

Clearly $|q(t)| \leq \varepsilon$ for all $t \in [a,b)_{\mathbb{T}}$, and we can solve for x to obtain

$$x(t) = e_{\frac{\psi}{\varphi}}(t,a)x(a) + \int_a^t e_{\frac{\psi}{\varphi}}(t, \sigma(\tau)) \frac{q(\tau) + f(\tau)}{\varphi(\tau)} \Delta\tau.$$

Let y be the unique solution of the initial-value problem

$$(\varphi D - \psi I)\, y(t) - f(t) = 0, \quad y(a) = x(a).$$

Then y is given by

$$y(t) = e_{\frac{\psi}{\varphi}}(t,a)x(a) + \int_a^t e_{\frac{\psi}{\varphi}}(t, \sigma(\tau)) \frac{f(\tau)}{\varphi(\tau)} \Delta\tau,$$

and

$$|y(t) - x(t)| = \left| \int_a^t e_{\frac{\psi}{\varphi}}(t, \sigma(\tau)) \frac{q(\tau)}{\varphi(\tau)} \Delta\tau \right| \leq \varepsilon \int_a^t \left| e_{\frac{\psi}{\varphi}}(t, \sigma(\tau)) \frac{1}{\varphi(\tau)} \right| \Delta\tau \leq L\varepsilon$$

by condition (4.2). $\square$

Remark 4.2. The convergence condition on the integral in (4.2) is essentially the same as S1 and S2 in [2], and can be met for various functions. For example, if $\mathbb{T}=\mathbb{R}$, $\varphi(w)=w^2$, $\psi(w)=\sin w$, and $a=1$, then

$$\int_a^t \left| e_{\frac{\psi}{\varphi}}(t,\sigma(\tau))\frac{1}{\varphi(\tau)}\right| \Delta\tau = \int_1^t e_{\frac{\sin w}{w^2}}(t,\tau)\frac{1}{\tau^2}d\tau \in \left[1-e^{-1+\frac{1}{t}},\ -1+e^{1-\frac{1}{t}}\right]$$

for all $t \geq 1$, and so clearly converges on $[1,\infty)_{\mathbb{R}}$. If $\mathbb{T}=\mathbb{N}$, $\varphi(w)=w$, $\psi(w)=-5/2$, and $a=3$, then

$$\int_a^t \left| e_{\frac{\psi}{\varphi}}(t,\sigma(\tau))\frac{1}{\varphi(\tau)}\right| \Delta\tau = \sum_{\tau=3}^{t-1}\frac{\Gamma(1+\tau)\Gamma(t-5/2)}{\tau\Gamma(t)\Gamma(\tau-3/2)} = \frac{2}{5}-\frac{4\,\Gamma(t-5/2)}{5\sqrt{\pi}\,\Gamma(t)} \in [0,2/5)$$

for all integers $t \geq 3$, and so clearly converges on $[3,\infty)_{\mathbb{N}}$, where Γ is the gamma function.

Theorem 4.3 (Hyers-Ulam Stability). *Given $\varphi_k,\psi_k,f \in \mathrm{C}_{\mathrm{rd}}[a,b)_{\mathbb{T}}$ with $|\varphi_k(t)| \geq A>0$ for some constant A, assume*

$$\varphi_k(t)+\mu(t)\psi_k(t)\neq 0 \quad \textit{and} \quad \int_a^t \left| e_{\frac{\psi_k}{\varphi_k}}(t,\sigma(\tau))\frac{1}{\varphi_k(\tau)}\right| \Delta\tau < L_{k-1} \tag{4.3}$$

for $k=1,2,\dots,n$ and for all $t\in[a,b)_{\mathbb{T}}$, where $0<L_{k-1}<\infty$ is some constant. Then (4.1) has Hyers-Ulam stability on $[a,b)_{\mathbb{T}}$.

Proof. Suppose there exists a function x such that

$$\left|\prod_{k=1}^{n}(\varphi_k D-\psi_k I)\,x(t)-f(t)\right| \leq \varepsilon$$

for some $\varepsilon>0$, for all $t\in[a,b)_{\mathbb{T}}$. Define the new functions $x_0:=x$, $y_n:=f$, and

$$x_k := (\varphi_k D-\psi_k I)\,x_{k-1}, \quad k=1,\dots,n. \tag{4.4}$$

Then

$$x_k(t)=\varphi_k(t)x_{k-1}^{\Delta}(t)-\psi_k(t)x_{k-1}(t),$$

that can be solved to yield

$$x_{k-1}(t)=e_{\frac{\psi_k}{\varphi_k}}(t,a)x_{k-1}(a)+\int_a^t e_{\frac{\psi_k}{\varphi_k}}(t,\sigma(\tau))\frac{x_k(\tau)}{\varphi_k(\tau)}\Delta\tau$$

for $k=1,\dots,n$. Note that

$$|x_n-y_n|(t)=|x_n-f|(t)\leq\varepsilon,$$

so

$$|\varphi_n x_{n-1}^{\Delta}-\psi_n x_{n-1}-y_n| = |\varphi_n x_{n-1}^{\Delta}-\psi_n x_{n-1}-f| \leq \varepsilon.$$

Hyers-Ulam stability of this first-order equation by Lemma 4.1 implies there exists a function y_{n-1} such that

$$|x_{n-1}-y_{n-1}|\leq L_{n-1}\varepsilon$$

and

$$\varphi_n(t)y_{n-1}^{\Delta}(t) - \psi_n(t)y_{n-1}(t) = y_n(t) = f(t),$$

for some constant $L_{n-1} > 0$, where y_{n-1} is given by

$$y_{n-1}(t) = e_{\frac{\psi_n}{\varphi_n}}(t,a)y_{n-1}(a) + \int_a^t e_{\frac{\psi_n}{\varphi_n}}(t,\sigma(\tau))\frac{y_n(\tau)}{\varphi_n(\tau)}\Delta\tau.$$

Then

$$|\varphi_{n-1}x_{n-2}^{\Delta} - \psi_{n-1}x_{n-2} - y_{n-1}| \leq L_{n-1}\varepsilon,$$

so again Hyers-Ulam stability of the first-order equation implies there exists a function y_{n-2} such that

$$|x_{n-2} - y_{n-2}| \leq L_{n-2}L_{n-1}\varepsilon$$

and

$$\varphi_{n-1}(t)y_{n-2}^{\Delta}(t) - \psi_{n-1}(t)y_{n-2}(t) = y_{n-1}(t),$$

for some constant $L_{n-2} > 0$, where y_{n-2} is given by

$$y_{n-2}(t) = e_{\frac{\psi_{n-1}}{\varphi_{n-1}}}(t,a)y_{n-2}(a) + \int_a^t e_{\frac{\psi_{n-1}}{\varphi_{n-1}}}(t,\sigma(\tau))\frac{y_{n-1}(\tau)}{\varphi_{n-1}(\tau)}\Delta\tau.$$

Continuing in this way, we obtain a function y_0 such that

$$|x_0 - y_0| = |x - y_0| \leq \varepsilon \prod_{j=0}^{n-1} L_j \tag{4.5}$$

and

$$\varphi_1(t)y_0^{\Delta}(t) - \psi_1(t)y_0(t) = y_1(t),$$

for some constant $L_0 > 0$, where y_0 is given by

$$y_0(t) = e_{\frac{\psi_1}{\varphi_1}}(t,a)y_0(a) + \int_a^t e_{\frac{\psi_1}{\varphi_1}}(t,\sigma(\tau))\frac{y_1(\tau)}{\varphi_1(\tau)}\Delta\tau.$$

Note that by construction of y_0 and generally y_k, we have

$$\begin{aligned}
\prod_{k=1}^{n}\left(\varphi_k D - \psi_k I\right) y_0(t) &= \prod_{k=2}^{n}\left(\varphi_k D - \psi_k I\right) y_1(t) \\
&= \prod_{k=3}^{n}\left(\varphi_k D - \psi_k I\right) y_2(t) \\
&\vdots \\
&= \left(\varphi_n D - \psi_n I\right) y_{n-1}(t) = y_n(t) = f(t),
\end{aligned}$$

making y_0 a solution of (4.1). This fact, together with inequality (4.5), shows that (4.1) has Hyers-Ulam stability. $\square$

REFERENCES

[1] D. R. Anderson, B. Gates, and D. Heuer, Hyers-Ulam stability of second-order linear dynamic equations on time scales, *Communications Appl. Anal.* 16:3:281–292, 2012.

[2] S. András and A. R. Mészáros, Ulam-Hyers stability of dynamic equations on time scales via Picard operators *Appl. Math. Computation* 219:4853–4864, 2013.

[3] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston, 2001.

[4] M. Bohner and A. Peterson, Editors, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[5] J. Brzdęk, D. Popa, and B. Xu, Remarks on stability of linear recurrence of higher order, *Appl. Math. Lett.* Vol. 23, Issue 12: 1459–1463, 2010.

[6] P. B. Garrett, *Abstract Algebra*, Chapman and Hall/CRC, Boca Raton, 2007.

[7] P. Găvruţa and L. Găvruţa, A new method for the generalized Hyers-Ulam-Rassias stability, *Int. J. Nonlinear Anal. Appl.* 1 No.2: 11–18, 2010.

[8] P. Găvruţa, S. M. Jung, and Y. J. Li, Hyers-Ulam stability for second-order linear differential equations with boundary conditions, *Electronic J. Diff. Equations* Vol. 2011, No. 80:1–5, 2011.

[9] S. Hilger, Analysis on measure chains – a unified approach to continuous and discrete calculus, *Results Math.* 18:18–56, 1990.

[10] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.* 27:222–224, 1941.

[11] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order (I), *International J. Appl. Math. & Stat.* Vol. 7, No. Fe07:96–100, 2007.

[12] S.-M. Jung, Hyers-Ulam stability of linear differential equation of the first order (III), *J. Math. Anal. Appl.* 311:139–146, 2005.

[13] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order (II), *Appl. Math. Lett.* 19:854–858, 2006.

[14] S.-M. Jung, Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients, *J. Math. Anal. Appl.* 320:549–561, 2006.

[15] Y. J. Li and Y. Shen, Hyers-Ulam stability of linear differential equations of second order, *Appl. Math. Lett.* 23:306–309, 2010.

[16] Y. J. Li and Y. Shen, Hyers-Ulam stability of nonhomogeneous linear differential equations of second order, *International J. Math. & Mathematical Sciences* Vol. 2009, Article ID 576852, 7 pages, 2009.

[17] T. Miura, S. Miyajima, and S. E. Takahasi, A characterization of Hyers-Ulam stability of first order linear differential operators, *J. Math. Anal. Appl.* Vol. 286, Issue 1:136–146, 2003.

[18] T. Miura, S. Miyajima, and S. E. Takahasi, Hyers-Ulam stability of linear differential operator with constant coefficients, *Math. Nachr.* 258:90–96, 2003.

[19] M. Obloza, Hyers stability of the linear differential equation, *Rocznik Naukowo-Dydaktyczny Prace Matematyczne* 13:259–270, 1993.

[20] D. Popa, Hyers-Ulam stability of the linear recurrence with constant coefficients, *Adv. Difference Equations* 2005:2:101–107, 2005.

[21] D. Popa and I. Raşa, On the Hyers-Ulam stability of the linear differential equation, *J. Math. Anal. Appl.* 381:530–537, 2011.

[22] D. Popa and I. Raşa, Hyers-Ulam stability of the linear differential operator with nonconstant coefficients, *Appl. Math. Computation* 219:1562–1568, 2012.

[23] J. M. Rassias, special editor-in-chief, Special Issue on Leonhard Paul Eulers Functional Equations and Inequalities, *International J. Appl. Math. & Stat.*, Volume 7 Number Fe07, February 2007.

[24] Th. M. Rassias, On the stability of linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* 72:297–300, 1978.

[25] I. A. Rus, Ulam stability of ordinary differential equations, *Stud. Univ. Babeş-Bolyai Math.* 54:125–134, 2009.

[26] C. Tunç and E. Biçer, Hyers-Ulam stability of non-homogeneous Euler equations of third and fourth order, *Scientific Research and Essays* Vol. 8:5:220–226, 2013.

[27] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience, New York, 1960.

[28] G. Wang, M. Zhou, and L. Sun, Hyers-Ulam stability of linear differential equations of first order, *Appl. Math. Lett.* 21:1024–1028, 2008.