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SMALLEST EIGENVALUES, EXTREMAL POINTS, AND POSITIVE SOLUTIONS OF A FOURTH ORDER THREE POINT BOUNDARY VALUE PROBLEM

SARAH S. KING AND JEFFREY T. NEUGEBAUER

Department of Mathematics and Statistics, Eastern Kentucky University,
Richmond, Kentucky 40475 USA

Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. The existence of smallest positive eigenvalues is established for the linear differential equations $u^{(4)} + \lambda_1 q(t)u = 0$ and $u^{(4)} + \lambda_2 r(t)u = 0$, $0 \le t \le 1$, with each satisfying the boundary conditions $u(0) = u'(p) = u''(1) = u'''(1) = 0$ where $1 - \frac{\sqrt{3}}{3} \le p < 1$. A comparison theorem for smallest positive eigenvalues is then obtained. The existence of these smallest eigenvalues is then applied to characterize extremal points of the differential equation $u^{(4)} + q(t)u = 0$ satisfying boundary conditions $u(0) = u'(p) = u''(b) = u'''(b) = 0$ where $1 - \frac{\sqrt{3}}{3} \le p \le b \le 1$. These results are applied to show the existence of a positive solution to a nonlinear boundary value problem.

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1. Introduction

We begin this work by considering the eigenvalue problems

$$u^{(4)} + \lambda_1 q(t)u = 0, \quad 0 \le t \le 1, \tag{1.1}$$

$$u^{(4)} + \lambda_2 r(t)u = 0, \quad 0 \le t \le 1, \tag{1.2}$$

satisfying the boundary conditions

$$u(0) = u'(p) = u''(1) = u'''(1) = 0, \tag{1.3}$$

where $1 - \frac{\sqrt{3}}{3} \le p < 1$, and $q(t)$ and $r(t)$ are continuous nonnegative functions on $[0, 1]$, with neither $q(t)$ nor $r(t)$ vanishing identically on any nondegenerate compact subinterval of $[0, 1]$.

Using the theory of u_0-positive operators with respect to a cone in a Banach space, we establish the existence of smallest eigenvalues for (1.1), (1.3), and (1.2), (1.3), and then compare these smallest eigenvalues after assuming a relationship between $q(t)$ and $r(t)$.

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We will then consider first extremal points of the equation

$$u^{(4)} + q(t)u = 0, \quad 0 \le t \le 1, \tag{1.4}$$

satisfying the boundary conditions

$$u(0) = u'(p) = u''(b) = u'''(b) = 0, \tag{1.5_b}$$

where p is fixed with $1-\frac{\sqrt{3}}{3} \le p \le b \le 1$, and $q(t)$ is a continuous nonnegative function on $[0,1]$ that does not vanish identically on any nondegenerate compact subinterval of $[0,1]$. Throughout the paper, the reference to boundary conditions (1.5_k), where $k \in [p,1]$, signifies the boundary conditions $u(0) = u'(p) = u''(k) = u'''(k) = 0$.

We establish the existence of a largest interval, $[0,b)$, such that on any subinterval $[0,c]$ of $[0,b)$, there exists only the trivial solution of (1.4), (1.5_c). We accomplish this by characterizing the first extremal point through the existence of a nontrivial solution that lies in a cone by establishing the spectral radius of a compact operator. The establishment of the spectral radius relies heavily on the existence of the smallest eigenvalues of (1.1), (1.3), giving reason for why these two topics are presented together. We then apply these results to show the existence of a positive solution of a fourth order three point nonlinear boundary value problem.

The fourth order beam equation with various boundary conditions has been the topic of study by many authors recently. For example, in [13], Graef, Kong, and Yang studied the existence of positive solutions of the fourth order differential equation

$$u^{(4)}(t) + g(t)f(u(t)), \quad 0 \le t \le 1,$$

satisfying boundary conditions (1.3). Neugebauer [21] has studied the comparison of smallest eigenvalues for a fourth order equation with different three point boundary conditions. For more work done on the fourth order beam equation, see, for example, [1, 3, 4, 24].

The technique for the comparison of these eigenvalues involve the application of sign properties of the Green's function, followed by the application of u_0-positive operators with respect to a cone in a Banach space. These applications are presented in books by Krasnosel'skii [18] and by Krein and Rutman [19]. For a sample of recent work done on the subject, see [5, 9, 10, 11, 15, 16, 21, 22].

When characterizing first extremal points, we will be defining a family of Banach spaces, cones, and operators. Using the theory of Krein and Rutman [19], we show the existence of a first extremal point is equivalent to properties of the spectral radius of the operators and solutions of the boundary value problem existing in a cone. Using a fixed point theorem, we are able to use these results to show the existence of positive solutions to a nonlinear fourth order three point boundary value problem. For recent work done on extremal points, see [7, 8, 12, 14].

2. **Existence and Comparison of Smallest Eigenvalues**

We begin this section with some definitions and theorems that will be integral to our analysis.

Definition 2.1. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and
(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

Definition 2.2. A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^\circ$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

Remark 2.3. Krasnosel'skii [18] showed that every solid cone is reproducing.

Definition 2.4. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, $u \leq v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, $M \leq N$ with respect to $\mathcal{P}$ if $Mu \leq Nu$ for all $u \in \mathcal{P}$.

Definition 2.5. A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $u_0 \in \mathcal{P}\backslash\{0\}$ such that for each $u \in \mathcal{P}\backslash\{0\}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathcal{P}$.

The following two results are fundamental to our existence and comparison results for smallest eigenvalues and are attributed to Krasnosel'skii [18]. The proof of Theorem 2.6 can be found in Krasnosel'skii's book [18], and the proof of Theorem 2.7 is provided by Keener and Travis [17] as an extension of Krasonel'skii's results.

Theorem 2.6. *Let $\mathcal{B}$ be a real Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive, linear operator. Then L has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.7. *Let $\mathcal{B}$ be a real Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators and assume that at least one of the operators is u_0-positive. If $M \leq N$, $Mu_1 \geq \lambda_1 u_1$ for some $u_1 \in \mathcal{P}$ and some $\lambda_1 > 0$, and $Nu_2 \leq \lambda_2 u_2$ for some $u_2 \in \mathcal{P}$ and some $\lambda_2 > 0$, then $\lambda_1 \leq \lambda_2$. Futhermore, $\lambda_1 = \lambda_2$ implies u_1 is a scalar multiple of u_2.*

To derive our comparison results, we will define integral operators whose kernels are the Green's function for $-u^{(4)} = 0$ satisfying (1.3) and show these operators are

u_0-positive. This Green's function is given by

$$G(t,s) = \begin{cases} -t[\frac{p^2}{2} - ps] - \frac{t^2 s}{2} + \frac{t^3}{6}, & 0 \le t, p \le s \le 1, \\ -t[\frac{p^2}{2} - ps - \frac{(p-s)^2}{2}] - \frac{t^2 s}{2} + \frac{t^3}{6}, & 0 \le t \le s \le p \le 1, \\ -t[\frac{p^2}{2} - ps] - \frac{t^2 s}{2} + \frac{t^3}{6} - \frac{(t-s)^3}{6}, & 0 \le p \le s \le t \le 1, \\ -t[\frac{p^2}{2} - ps - \frac{(p-s)}{2}] - \frac{t^2 s}{2} + \frac{t^3}{6} - \frac{(t-s)^3}{6}, & 0 \le s \le t, p \le 1. \end{cases}$$

Now, $u(t)$ solves (1.1), (1.3) if and only if $u(t) = \lambda_1 \int_0^1 G(t,s)q(s)u(s)ds$, and $u(t)$ solves (1.2), (1.3) if and only if $u(t) = \lambda_2 \int_0^1 G(t,s)r(s)u(s)ds$.

It was shown in [13] that $G(t,s) \ge 0$ on $[0,1] \times [0,1]$ and $G(t,s) > 0$ on $(0,1] \times (0,1)$.

Lemma 2.8. *For* $0 < s < 1$, $\frac{\partial}{\partial t} G(t,s)|_{t=0} > 0$.

Proof. When $t = 0$, $t \le s$. Thus, we start by considering $\frac{\partial}{\partial t} G(t,s)|_{t=0}$ for $t \le s$ and $p \le s$. Then

$$\begin{aligned} \frac{\partial}{\partial t} G(t,s)\bigg|_{t=0} &= \frac{\partial}{\partial t} \left[-t \left[\frac{p^2}{2} - ps \right] - \frac{t^2 s}{2} + \frac{t^3}{6} \right] \bigg|_{t=0} \\ &= \left[-\left(\frac{p^2}{2} - ps \right) - ts + \frac{t^2}{2} \right] \bigg|_{t=0} \\ &= -\frac{p^2}{2} + ps \\ &\ge -\frac{p^2}{2} + p^2 \\ &= \frac{p^2}{2} \\ &> 0, \end{aligned}$$

for $0 < s < 1$.

Next, we consider $\frac{\partial}{\partial t} G(t,s)|_{t=0} > 0$ for $t \le s$ and $p \ge s$. Then

$$\begin{aligned} \frac{\partial}{\partial t} G(t,s)\bigg|_{t=0} &= \frac{\partial}{\partial t} \left[-t \left[\frac{p^2}{2} - ps - \frac{(p-s)^2}{2} \right] - \frac{t^2 s}{2} + \frac{t^3}{6} \right] \bigg|_{t=0} \\ &= \left[-\left(\frac{p^2}{2} - ps - \frac{(p-s)^2}{2} \right) - ts + \frac{t^2}{2} \right] \bigg|_{t=0} \\ &= -\frac{p^2}{2} + ps + \frac{(p-s)^2}{2} \\ &\ge -\frac{s^2}{2} + s^2 \\ &= \frac{s^2}{2} \\ &> 0, \end{aligned}$$

for $0 < s < 1$. Thus, for $t = 0$ and $0 < s < 1$, we have $\frac{\partial}{\partial t} G(t,s) > 0$. $\square$

To apply Theorems 2.6 and 2.7, we need to define a Banach space $\mathcal{B}$ and a cone $\mathcal{P} \subset \mathcal{B}$. Define the Banach space $\mathcal{B}$ by

$$\mathcal{B} = \{u \in C^1[0,1] \mid u(0) = 0\},$$

with the norm

$$\|u\| = \sup_{0 \le t \le 1} |u'(t)|.$$

Define the cone $\mathcal{P}$ to be

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(t) \ge 0 \text{ on } [0,1]\}.$$

Notice that for $u \in \mathcal{B}$, $0 \le t \le 1$,

$$|u(t)| = |u(t) - u(0)| = \left| \int_0^t u'(s)ds \right| \le \|u\|t \le \|u\|,$$

and so $\sup_{0 \le t \le 1} |u(t)| \le \|u\|$.

Lemma 2.9. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega = \{u \in \mathcal{B} \mid u(t) > 0 \text{ on } (0,1] \text{ and } u'(0) > 0\}.$$

Note $\Omega \subset \mathcal{P}$. We will show $\Omega \subset \mathcal{P}^\circ$. Let $u \in \Omega$. Since $u'(0) > 0$, there exists $\epsilon_1 > 0$ such that $u'(0) - \epsilon_1 > 0$, and so $u'(0) > \epsilon_1$. By the definition of the derivative, $u'(0) = \lim_{t \to 0^+} \frac{u(t)-u(0)}{t-0} > \epsilon_1$, and so there exists an $a \in (0,1)$ such that for all $t \in (0,a)$, $\frac{u(t)-u(0)}{t-0} > \epsilon_1$. It follows that for all $t \in (0,a)$, $u(t) > t\epsilon_1$. Also, since $u(t) > 0$ on $[a,1]$, there exists $\epsilon_2 > 0$ such that $u(t) - \epsilon_2 > 0$ for all $t \in [a,1]$.

Let $\epsilon = \min\{\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}\}$. Define $B_\epsilon(u) := \{v \in \mathcal{B} \mid \|u - v\| < \epsilon\}$. Let $v \in \mathcal{B}_\epsilon(u)$. So $|u'(0) - v'(0)| \le \|u - v\| < \epsilon$. Consequently, $v'(0) > 0$. Next, by the Mean Value Theorem, for $t \in (0,a)$, $|u(t) - v(t)| \le t\|u - v\|$. Thus $|u(t) - v(t)| < t\epsilon$ for $t \in (0,a)$. Thus $v(t) > u(t) - t\epsilon > t\epsilon_1 - t\frac{\epsilon_1}{2} = t\frac{\epsilon_1}{2} > 0$ for all $t \in (0,a)$. Lastly, for all $t \in [a,1]$, $|u(t) - v(t)| \le \|u - v\| < \epsilon$. We obtain that $v(t) > u(t) - \epsilon > \epsilon_2 - \frac{\epsilon_2}{2} > \frac{\epsilon_2}{2}$, and so $v(t) > 0$ on $(0,1]$. Thus, $v \in \Omega$, and so $\mathcal{B}_\epsilon(u) \subset \Omega$. Thus $\Omega \subset \mathcal{P}^\circ$. □

Next, we define our linear operators $M, N : \mathcal{B} \to \mathcal{B}$ by

$$Mu(t) = \int_0^1 G(t,s)q(s)u(s)ds, \quad 0 \le t \le 1,$$

and

$$Nu(t) = \int_0^1 G(t,s)r(s)u(s)ds, \quad 0 \le t \le 1.$$

A standard application of the Arzelà-Ascoli theorem shows M and N are compact operators.

Lemma 2.10. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

Proof. We will show that $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. First, let $u \in \mathcal{P}$. Since $u(t) \geq 0$ on $[0,1]$, $G(t,s) \geq 0$ on $[0,1] \times [0,1]$, and $q(t) \geq 0$ on $[0,1]$,

$$Mu(t) = \int_0^1 G(t,s)q(s)u(s)ds \geq 0,$$

for $0 \leq t \leq 1$. Thus $Mu \in \mathcal{P}$ and $M : \mathcal{P} \to \mathcal{P}$.

Next, let $u \in \mathcal{P}\backslash\{0\}$. Consequently, there exists a nondegenerate compact subinterval $[a,b] \subset (0,1)$ such that $u(t) > 0$ and $q(t) > 0$ on $[a,b]$. Since $G(t,s) > 0$ on $(0,1] \times (0,1)$,

$$Mu(t) = \int_0^1 G(t,s)q(s)u(s)ds \geq \int_a^b G(t,s)q(s)u(s)ds > 0,$$

for $0 < t \leq 1$. Also, $\frac{\partial}{\partial t}G(t,s)|_{t=0} > 0$ for $0 < s < 1$, so

$$(Mu)'(0) = \int_0^1 \frac{\partial}{\partial t}G(0,s)q(s)u(s)ds \geq \int_a^b \frac{\partial}{\partial t}G(0,s)q(s)u(s)ds > 0.$$

Hence, $Mu \in \Omega \subset \mathcal{P}^\circ$, and so $M : \mathcal{P}\backslash\{0\} \to \mathcal{P}^\circ$.

Now let $u \in \mathcal{P}\backslash\{0\}$ and choose a $u_0 \in \mathcal{P}^\circ$. It follows $Mu \in \mathcal{P}^\circ$. So there exists a $k_1 > 0$ sufficiently small so that $Mu - k_1u_0 \in \mathcal{P}^\circ$ and a $k_2 > 0$ sufficiently large so that $u_0 - \frac{1}{k_2}Mu \in \mathcal{P}^\circ$. This choice of k_1, k_2 insures that $k_1u_0 \leq Mu$ and $\frac{1}{k_2}Mu \leq u_0$ with respect to $\mathcal{P}$. Thus $k_1u_0 \leq Mu \leq k_2u_0$ with respect to $\mathcal{P}$. So M is u_0-positive. A similar argument can be made to show N is u_0-positive. □

Remark 2.11. Let Λ be an eigenvalue of M with eigenvector u. Notice that

$$\Lambda u = Mu = \int_0^1 G(t,s)q(s)u(s)ds,$$

if and only if

$$u(t) = \frac{1}{\Lambda}\int_0^1 G(t,s)q(s)u(s)ds,$$

if and only if

$$-u^{(4)}(t) = \frac{1}{\Lambda}q(t)u(t), \quad 0 \leq t \leq 1,$$

with

$$u(0) = u'(p) = u''(1) = u'''(1) = 0.$$

Thus the eigenvalues of (1.1), (1.3) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (1.2), (1.3) are reciprocals of eigenvalues N, and conversely.

Theorem 2.12. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Then M (and, by similar reasoning, N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

Proof. Since M is u_0-positive it has, from Theorem 2.6, an essentially unique eigenvector, namely $u \in \mathcal{P}$, and eigenvalue Λ with the properties stated above. Since $u \neq 0$, we have $Mu \in \Omega \subset \mathcal{P}^\circ$ and $\Lambda u = Mu$. Therefore, $u = \frac{1}{\Lambda}Mu = M(\frac{1}{\Lambda}u)$. Notice that $\frac{1}{\Lambda}u \neq 0$ and so $M(\frac{1}{\Lambda}u) \in \mathcal{P}^\circ$. It follows that $u \in \mathcal{P}^\circ$, completing the proof. $\square$

Theorem 2.13. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $q(t) \leq r(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues, defined in Theorem* 2.12, *associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $q(t) = r(t)$ on $[0,1]$.*

Proof. Let $q(t) \leq r(t)$ on $[0,1]$. Thus, for any $u \in \mathcal{P}$ and $t \in [0,1]$,

$$(Nu - Mu)(t) = \int_0^1 G(t,s)(r(s) - q(s))u(s)ds \geq 0,$$

and so $(Nu - Mu) \in \mathcal{P}$. Thus $Mu \leq Nu$ for all $u \in \mathcal{P}$, implying that $M \leq N$ with respect to $\mathcal{P}$. So, by Theorem 2.7, $\Lambda_1 \leq \Lambda_2$. If $q(t) = r(t)$, then $\Lambda_1 = \Lambda_2$.

Suppose now that $q(t) \neq r(t)$. Then there exists some subinterval $[a,b] \subset [0,1]$ such that $q(t) < r(t)$ for all $t \in [a,b]$. Through reasoning similar to the proof of Lemma 2.10, we have $N - M : \mathcal{P}\backslash\{0\} \to \Omega$. Therefore, $(N - M)u_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists some $\epsilon > 0$ such that $(N - M)u_1 - \epsilon u_1 \in \mathcal{P}$. Then $\epsilon u_1 \leq (N - M)u_1 = Nu_1 - Mu_1$ with respect to $\mathcal{P}$. We have $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$. This implies that $Nu \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, $\Lambda_1 + \epsilon \leq \Lambda_2$, thus $\Lambda_1 < \Lambda_2$. $\square$

By Remark 2.11, the following theorem is an immediate consequence of Theorems 2.12 and 2.13.

Theorem 2.14. *Assume the hypotheses of Theorem* 2.13. *Then there exist smallest positive eigenvalues λ_1 and λ_2 of* (1.1), (1.3) *and* (1.2), (1.3), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $q(t) = r(t)$ for $0 \leq t \leq 1$.*

3. **Characterization of Extremal Points**

We will begin this section with a few key definitions and theorems for classifying first extremal points of a boundary value problem.

Definition 3.1. We say that b_0 is the first extremal point of the boundary value problem (1.4), (1.5_b) if $b_0 = \inf\{b > p \mid (1.4), (1.5_b) \text{ has a nontrivial solution }\}$.

Definition 3.2. A bounded linear operator $N : \mathcal{B} \to \mathcal{B}$ is said to be positive with respect to the cone $\mathcal{P}$ if $N : \mathcal{P} \to \mathcal{P}$.

We denote the spectral radius of N by $r(N)$.

The following four theorems are crucial to our results. The first result can be found in [20] and the other three can be found in [2] or [18]. Assume in each of the following that $\mathcal{P}$ is a reproducing cone and $N, N_1, N_2 : \mathcal{B} \to \mathcal{B}$ are compact, linear, and positive with respect to $\mathcal{P}$.

Theorem 3.3. *Let N_b, $0 \le b \le 1$, be a family of compact, linear operators on a Banach space such that the mapping $b \to N_b$ is continuous in the uniform topology. Then the mapping $b \to r(N_b)$ is also continuous.*

Theorem 3.4. *Assume $r(N) > 0$. Then $r(N)$ is an eigenvalue of N, and there is a corresponding eigenvector in $\mathcal{P}$.*

Theorem 3.5. *If $N_1 \le N_2$ with respect to $\mathcal{P}$, then $r(N_1) \le r(N_2)$.*

Theorem 3.6. *Suppose there exist $\gamma > 0$, $u \in \mathcal{B}$, $-u \notin \mathcal{P}$ such that $\gamma u \le Nu$ with respect to $\mathcal{P}$. Then N has an eigenvector in $\mathcal{P}$ which corresponds to an eigenvalue λ with $\lambda \ge \gamma$.*

Now, we will characterize extremal points of the boundary value problem (1.4), (1.5_b). We will assume throughout that the boundary value problem (1.1), (1.5_p) has only the trivial solution for $\lambda \le 1$. For nonexistence results relating to this boundary value problem, we refer the reader to [13].

We define a Banach space $\mathcal{B}$ and cone $\mathcal{P} \subset \mathcal{B}$ in order to apply the above theorems. First, define the Banach space $\mathcal{B}$ to be

$$\mathcal{B} = \{u \in C^1[0,1] \mid u(0) = 0\},$$

with norm

$$\|u\| = \sup_{0 \le t \le 1} |u'(t)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ as

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(t) \ge 0 \text{ on } [0,1]\}.$$

From Lemma 2.9, we know $\Omega = \{u \in \mathcal{B} \mid u(t) > 0 \text{ on } (0,1] \text{ and } u'(0) > 0\} \subset \mathcal{P}^\circ$, and so $\mathcal{P}$ is reproducing.

Furthermore, for each $b \in [p,1]$, we define a family of Banach spaces $\mathcal{B}_b$ and cones $\mathcal{P}_b \subset \mathcal{B}_b$. Define the Banach space $\mathcal{B}_b$ by

$$\mathcal{B}_b = \{u \in C^1[0,b] \mid u(0) = 0\},$$

with norm

$$\|u\| = \sup_{0 \le t \le b} |u'(t)|.$$

Define the cone $\mathcal{P}_b \subset \mathcal{B}_b$ as

$$\mathcal{P}_b = \{u \in \mathcal{B}_b \mid u(t) \geq 0 \text{ on } [0, b]\}.$$

The proof of Lemma 2.9 can be easily adapted to show that for all $b \in [p, 1]$,

$$\Omega_b := \{u \in \mathcal{B}_b \mid u(t) > 0 \text{ on } (0, b] \text{ and } u'(t) > 0\} \subset \mathcal{P}_b^\circ.$$

Thus for each $b \in [p, 1]$, $\mathcal{P}_b$ is reproducing.

Notice that for $t \geq s$, $\frac{\partial^2}{\partial t^2} G(t, s) = \frac{\partial^3}{\partial t^3} G(t, s) = 0$. Thus, discounting the interval of existence, $G(t, s)$ is independent of the right endpoint chosen. So for each $b \in [p, 1]$, the Green's function for $-u^{(4)} = 0$, (1.5_b) is given by

$$G(b; t, s) = \begin{cases} -t[\frac{p^2}{2} - ps] - \frac{t^2 s}{2} + \frac{t^3}{6}, & 0 \leq t, p \leq s \leq b, \\ -t[\frac{p^2}{2} - ps - \frac{(p-s)^2}{2}] - \frac{t^2 s}{2} + \frac{t^3}{6}, & 0 \leq t \leq s \leq p \leq b, \\ -t[\frac{p^2}{2} - ps] - \frac{t^2 s}{2} + \frac{t^3}{6} - \frac{(t-s)^3}{6}, & 0 \leq p \leq s \leq t \leq b, \\ -t[\frac{p^2}{2} - ps - \frac{(p-s)}{2}] - \frac{t^2 s}{2} + \frac{t^3}{6} - \frac{(t-s)^3}{6}, & 0 \leq s \leq t, p \leq b. \end{cases}$$

Notice for $p \leq b_1 < b_2$, $G(b_1; t, s) = G(b_2; t, s)$ for $0 \leq t \leq b_1$.

For each $b \in [p, 1]$, define the family of linear operators N_b by

$$N_b u(t) = \begin{cases} \int_0^b G(b; t, s) q(s) u(s) ds, & 0 \leq t \leq b, \\ \int_0^b G(b; b, s) q(s) u(s) ds + (t - b) \int_0^b \frac{\partial}{\partial t} G(b; b, s) q(s) u(s), & b \leq t \leq 1. \end{cases}$$

From how N_b is defined, $N_b u \in C^1[0, 1]$ for $u \in C^1[0, 1]$, and $N_b u(0) = 0$. This yields $N_b : \mathcal{B} \to \mathcal{B}$. Also, when N_b is restricted to $\mathcal{B}_b$, $N_b : \mathcal{B}_b \to \mathcal{B}_b$ by $N_b u(t) = \int_0^b G(b; t, s) q(s) u(s) ds$, and so $u(t)$ is a solution to (1.4), (1.5_b) if and only if $u(t) = N_b u(t) = \int_0^b G(b; t, s) q(s) u(s) ds$ for $t \in [0, b]$.

The proof of the following lemma is similar to the proof of Lemma 2.10 and is therefore omitted.

Lemma 3.7. *For all $b \in [p, 1]$, the linear operator N_b is positive with respect to $\mathcal{P}$ and $\mathcal{P}_b$. Also, $N_b : \mathcal{P}_b \backslash \{0\} \to \mathcal{P}_b^\circ$.*

The following lemma gives, after applying Theorem 3.3, that the mapping $b \to r(N_b)$ is continuous.

Lemma 3.8. *The map $b \to N_b$ is continuous in the uniform topology.*

Proof. First, note from earlier that $\sup_{0 \leq t \leq 1} |u(t)| \leq \|u\|$. Consider the function $f : [p, 1] \to \{N_b\}$, $b \in [p, 1]$, defined by $f(b) = N_b$. Let $p \leq b_1 < b_2 \leq 1$. Let $\epsilon > 0$. Now

$$\begin{aligned} \|f(b_2) - f(b_1)\| &= \|N_{b_2} - N_{b_1}\| \\ &= \sup_{\|u\|=1} \|N_{b_2} u - N_{b_1} u\| \end{aligned}$$

$$= \sup_{\|u\|=1} \{ \sup_{t\in[0,1]} |(N_{b_2}u)'(t) - (N_{b_1}u)'(t)| \}.$$

Since $\frac{\partial}{\partial t}G(b;t,s)$ and $q(t)$ are continuous functions in t for $0 \le t \le b$, they are bounded above for $0 \le t \le b$. Choose K and Q such that $|\frac{\partial}{\partial t}G(b;t,s)| \le K$ for all $b \in [p,1]$ and $|q(t)| \le Q$ for $0 \le t \le 1$. Since $G(b;t,s) \in C^1[0,b]$ in t, there exists a $\delta > 0$ with $\delta < \frac{\epsilon}{2KQ}$ such that for $|t_2 - t_1| < \delta$, $|\frac{\partial}{\partial t}G(b;t_2,s) - \frac{\partial}{\partial t}G(b;t_1,s)| < \frac{\epsilon}{2KQ}$.

Suppose $0 \le t \le b_1$. Then for $|b_2 - b_1| < \delta$,

$$\begin{aligned}
|(N_{b_2}u)'(t) - (N_{b_1}u)'(t)| &= \Big| \int_0^{b_2} \frac{\partial}{\partial t}G(b_2;t,s)q(s)u(s)ds \\
&quad - \int_0^{b_1} \frac{\partial}{\partial t}G(b_1;t,s)q(s)u(s)ds \Big| \\
&= \Big| \int_{b_1}^{b_2} \frac{\partial}{\partial t}G(b_2;t,s)q(s)u(s)ds \Big| \\
&\le \int_{b_1}^{b_2} \Big| \frac{\partial}{\partial t}G(b_2;t,s) \Big| |q(s)||u(s)|ds \\
&\le \int_{b_1}^{b_2} KQds \\
&\le KQ|b_2 - b_1| \\
&< KQ\frac{\epsilon}{2KQ} < \epsilon.
\end{aligned}$$

Now suppose $b_1 < t \le b_2$. Thus for $|b_2 - b_1| < \delta$,

$$\begin{aligned}
|(N_{b_2}u)'(t) - (N_{b_1}u)'(t)| &= \Big| \int_0^{b_2} \frac{\partial}{\partial t}G(b_2;t,s)q(s)u(s)ds \\
&\quad - \int_0^{b_1} \frac{\partial}{\partial t}G(b_1;b_1,s)q(s)u(s)ds \Big| \\
&\le \int_{b_1}^{b_2} \Big| \frac{\partial}{\partial t}G(b_2;t,s) \Big| |q(s)||u(s)|ds \\
&\quad + \int_0^{b_1} \Big| \frac{\partial}{\partial t}G(b_1;t,s) - \frac{\partial}{\partial t}G(b_1;b_1,s) \Big| |q(s)||u(s)|ds \\
&< \int_{b_1}^{b_2} KQds + \int_0^{b_1} \frac{\epsilon}{2Q}Qds \\
&= KQ|b_2 - b_1| + \frac{\epsilon}{2Q}Qb_1 \\
&< KQ\frac{\epsilon}{2KQ} + \frac{\epsilon}{2Q}Q \\
&= \epsilon.
\end{aligned}$$

Now suppose $b_2 < t \le 1$. So for $|b_2 - b_1| < \delta$,

$$\begin{aligned}
|(N_{b_2}u)'(t) - (N_{b_1}u)'(t)| &= \left| \int_0^{b_2} \frac{\partial}{\partial t} G(b_2; b_2, s) q(s) u(s) ds \right. \\
&\quad \left. - \int_0^{b_1} \frac{\partial}{\partial t} G(b_1; b_1, s) q(s) u(s) ds \right| \\
&\le \int_{b_1}^{b_2} \left| \frac{\partial}{\partial t} G(b_2; b_2, s) \right| |q(s)||u(s)| ds \\
&\quad + \int_0^{b_1} \left| \frac{\partial}{\partial t} G(b_2; b_2, s) - \frac{\partial}{\partial t} G(b_2; b_1, s) \right| |q(s)||u(s)| ds \\
&< \int_{b_1}^{b_2} KQ ds + \int_0^{b_1} \frac{\epsilon}{2Q} Q ds \\
&= KQ|b_2 - b_1| + \frac{\epsilon}{2Q} Q b_1 \\
&< KQ \frac{\epsilon}{2KQ} + \frac{\epsilon}{2Q} Q \\
&= \epsilon.
\end{aligned}$$

Thus we have that $\sup_{\|u\|=1}\{\sup_{t\in[0,1]} |(N_{b_2}u)'(t) - (N_{b_1}u)'(t)|\} < \epsilon$ for $|b_2 - b_1| < \delta$. Then $\|f(b_2) - f(b_1)\| < \epsilon$ for $|b_2 - b_1| < \delta$, establishing that f is continuous. □

Theorem 3.9. *For $p \le b \le 1$, $r(N_b)$ is strictly increasing as a function of b.*

Proof. It was previously shown in Theorem 2.12 that if $b = 1$, there is a $\lambda > 0$ and $u \in \mathcal{P}_b \backslash \{0\}$ such that $N_b u(t) = \lambda u(t)$ for $t \in [0, b]$. Similarly, one can show that for $b \in [p, 1)$, there is a $\lambda > 0$ and $u \in \mathcal{P}_b \backslash \{0\}$ such that $N_b u(t) = \lambda u(t)$ for $t \in [0, b]$. Extend this u to $[b, 1]$ by $\lambda u(t) = \int_0^b G(b; b, s) q(s) u(s) ds + (t - b) \int_0^b \frac{\partial}{\partial t} G(b; b, s) q(s) u(s) ds$. Then for $t \in [0, 1]$, $N_b u(t) = \lambda u(t)$. Thus for $p \le b \le 1$, $r(N_b) \ge \lambda > 0$.

Now let $p \le b_1 < b_2 \le 1$. Since $r(N_{b_1}) > 0$, then by Theorem 3.4, there exists a $u_0 \in \mathcal{P}_{b_1} \backslash \{0\}$ such that $N_{b_1} u_0 = r(N_{b_1}) u_0$. Let $u_1 = N_{b_1} u_0$ and $u_2 = N_{b_2} u_0$. Then for $t \in (0, b_1]$,

$$(u_2 - u_1)(t) = \int_{b_1}^{b_2} G(b_2; t, s) q(s) u_0(s) ds > 0.$$

Also,

$$(u_2 - u_1)'(0) = \int_{b_1}^{b_2} \frac{\partial}{\partial t} G(b_2; 0, s) q(s) u_0(s) ds > 0.$$

Thus the restriction of $u_2 - u_1$ to $[0, b_1]$ belongs to Ω_{b_1}, so there exists $\delta > 0$ such that $u_2 - u_1 \ge \delta u_0$ with respect to $\mathcal{P}_{b_1}$. Since $u_2 \in \mathcal{P}$, it follows that $u_2 - u_1 \ge \delta u_0$ with respect to $\mathcal{P}$. Thus

$$u_2 \ge u_1 + \delta u_0 = r(N_{b_1}) u_0 + \delta u_0 = (r(N_{b_1}) + \delta) u_0$$

with respect to $\mathcal{P}$. So $N_{b_2} u_0 \ge (r(N_{b_1}) + \delta) u_0$ with respect to $\mathcal{P}$, and so by Theorem 3.6, $r(N_{b_2}) \ge r(N_{b_1}) + \delta$. Then $r(N_{b_2}) > r(N_{b_1})$ and $r(N_b)$ is strictly increasing. □

Theorem 3.10. *The following are equivalent:*

(i) b_0 *is the first extremal point of the boundary value problem corresponding to* (1.4), (1.5_b)*;*

(ii) *there exists a nontrivial solution* u *of the boundary value problem* (1.4), (1.5_{b_0}) *such that* $u \in \mathcal{P}_{b_0}$*;*

(iii) $r(N_{b_0}) = 1$.

Proof. First, we show (iii $\to$ ii). Assume $r(N_{b_0}) = 1$. By Theorem 3.4, 1 is an eigenvalue of N_{b_0}, so there exists a $u \in \mathcal{P}_{b_0}$ such that $N_{b_0}u = 1u$. So u solves (1.4), (1.5_{b_0}).

Next, we show (ii $\to$ i). Let u be a nontrivial solution to (1.4), (1.5_{b_0}) with $u \in \mathcal{P}_{b_0}$. Extend u to $[b_0, 1]$ by

$$u(t) = \int_0^{b_0} G(b_0; b_0, s)q(s)u(s)ds + (t - b_0)\int_0^{b_0} \frac{\partial}{\partial t}G(b_0; b_0, s)q(s)u(s)ds,$$

and so $N_{b_0}u(t) = u(t)$ for $t \in [0,1]$. Therefore, 1 is an eigenvalue of N_{b_0}. Thus $r(N_{b_0}) \geq 1$. If $r(N_{b_0}) = 1$, then for all $b \in [p, b_0]$, $r(N_b) < 1$. Hence b_0 is the first extremal point of (1.4), (1.5_b).

Now assume that $r(N_{b_0}) > 1$. By Theorem 3.4, there exists a $w \in \mathcal{P}_{b_0}\backslash 0$ such that $N_{b_0}w = r(N_{b_0})w$. By Lemma 3.7, $N_{b_0}w \in \Omega_{b_0}$. Therefore $r(N_{b_0})w \in \Omega_{b_0}$, and so $u - \delta w \in \mathcal{P}_{b_0}$ for some $\delta > 0$.

For $t \in [b_0, 1]$, extend $w(t)$ by letting $w(t) = \int_0^{b_0} G(b_0; b_0, s)q(s)w(s)ds + (t - b_0)\int_0^{b_0} \frac{\partial}{\partial t}G(b_0; b_0, s)q(s)w(s)ds$. Then $u - \delta w \in \mathcal{P}$, and so $u \geq \delta w$ with respect to $\mathcal{P}$. Assume δ is maximal such that this inequality holds. Then $u = N_{b_0}u \geq N_{b_0}(\delta w) = \delta N_{b_0}w = \delta r(N_{b_0})w$. Since $r(N_{b_0}) > 1$, $\delta r(N_{b_0}) > \delta$. However, $u \geq \delta r(N_{b_0})w$, which contradicts the maximality of δ. Thus $r(N_{b_0}) = 1$.

Lastly, we show (i $\to$ iii). Assuming (i), there exists a $u \in \mathcal{P}_{b_0}\backslash\{0\}$ such that $u = N_{b_0}u$. Thus $r(N_{b_0}) \geq 1$. We claim $r(N_{b_0}) = 1$. By way of contradiction, assume $r(N_{b_0}) > 1$. Following in the way of Remark 2.11, we can show that if Λ is an eigenvalue of N_p, then $\frac{1}{\Lambda}$ is an eigenvalue of (1.1), (1.5_p). By our assumption, (1.1), (1.5_p) has only the trivial solution for $\lambda \leq 1$. Thus if (1.1), (1.5_p) has a nontrivial solution, $\Lambda < 1$. So $r(N_p) < 1$. Since $r(N_b)$ is continuous with respect to b, by the Intermediate Value Theorem, there exists an $\alpha \in (p, b_0)$ such that $r(N_\alpha) = 1$. So there exists a nontrivial solution v to (1.4), (1.5_α) with $v \in \mathcal{P}_\alpha\backslash\{0\}$, which is a contradiction since b_0 is the first extremal point of (1.4), (1.5_α). Therefore $r(N_{b_0}) = 1$. $\square$

4. **Positive Solutions of the Nonlinear Problem**

In this section, we consider the nonlinear boundary value problem

$$u^{(4)} + f(t, u) = 0, \; 0 \leq t \leq 1, \tag{4.1}$$

satisfying boundary conditions (1.5_b), where $f(t,u) : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and $f(t,0) \equiv 0$.

Assume $q(t) \equiv \frac{\partial f}{\partial u}(t,u)\big|_{u=0}$ exists, is a nonnegative continuous function on $[0,1]$, and does not vanish identically on any nondegenerate compact subinterval of $[0,1]$. Then the variational equation along the zero solution of (4.1) is

$$u^{(4)} + q(t)u = 0. \tag{4.2}$$

For the existence of nontrivial solutions of the boundary value problem (4.1), (1.5_b), we apply the following fixed point theorem for nonlinear operator equations; see Deimling [6] or Schmitt and Smith [23].

Lemma 4.1. *Let $\mathcal{B}$ be a Banach space and $\mathcal{P} \subset \mathcal{B}$ a reproducing cone. Let $M : \mathcal{B} \to \mathcal{B}$ be a completely continuous, nonlinear operator such that $M : \mathcal{P} \to \mathcal{P}$ and $M(0) = 0$. Let M be Fréchet differentiable at $u = 0$ whose Fréchet derivative $N = M'(0)$ has the property:*

(A) *There exist $w \in \mathcal{P}$ and $\mu > 1$ such that $Nw = \mu w$, and $Nu = u$ implies that $u \notin \mathcal{P}$. Further, there exists $\rho > 0$ such that, if $u = (1/\lambda)Mu$, $u \in \mathcal{P}$ and $\|u\| = \rho$, then $\lambda \le 1$.*

Then, the equation $u = Mu$ has a solution $u \in \mathcal{P}\backslash\{0\}$.

Theorem 4.2. *Assume b_0 is the first extremal point of* (4.2), (1.5_b). *Assume also the following condition holds:*

(A') *There exists a $\rho(b) > 0$ such that, if $u(t)$ is a nontrivial solution of $u^{(4)} + (1/\lambda)f(t,u) = 0$ satisfying (1.5_b), and if $u \in \mathcal{P}$, with $\|u\| = \rho(b)$, then $\lambda \le 1$.*

Then, for all b satisfying $b_0 < b \le 1$, the boundary value problem (4.1), (1.5_b) *has a solution $u \in \mathcal{P}\backslash\{0\}$.*

Proof. For each b satisfying $b_0 < b \le 1$, let N_b be defined as in the previous section with respect to $q(t) \equiv \frac{\partial f}{\partial u}(t,u)\big|_{u=0}$. Define the linear operator M_b by

$$M_b u(x) = \begin{cases} \int_0^b G(t,s)q(s)f(s,u(s))ds, & 0 \le t \le b, \\ \int_0^b G(b,s)q(s)f(s,u(s))ds & \\ \quad +(t-b)\int_0^b \frac{\partial}{\partial t}G(b,s)q(s)f(s,u(s))ds, & b \le t \le 1. \end{cases}$$

Then M_b is Fréchet differentiable at $u = 0$ and $M_b'(0) = N_b$.

From Theorem 3.9 and Theorem 3.10, it follows that $r(N_{b_0}) = 1$ and $r(N_b) > 1$ for $b > b_0$. Moreover, since b_0 is the first extremal point of (4.2) corresponding to (1.5_b), it follows from Theorem 3.10 that, for $b > b_0$, if $N_b u = u$ and u is nontrivial, then $u \notin \mathcal{P}$. Thus, (A') and Lemma 4.1 imply there exists a $u \in \mathcal{P}\backslash\{0\}$ such that $M_b u = u$. So u is a nontrivial solution of (4.1), (1.5_b), with $u \in \mathcal{P}\backslash\{0\}$. $\square$

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