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RIEMANN LIOUVILLE AND CAPUTO FRACTIONAL DIFFERENTIAL AND INTEGRAL INEQUALITIES

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Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. Differential and integral inequalities have played a dominant role in the qualitative study of differential and integral equations. In this work, we will study fractional differential and integral inequalities. The fractional differential and integral inequalities will include both the Riemann Liouville type as well as Caputo type. These inequalities are useful in proving theoretical existence and uniqueness results for nonlinear fractional differential and integral equations. It is also useful in developing iterative techniques which are both theoretical and computational. We can prove the existence and compute the minimal and maximal solutions or coupled minimal and maximal solutions of the nonlinear fractional equations by the iterative technique. Further, if uniqueness conditions are satisfied, we can prove the existence of a unique solution which can be computed numerically.

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1. Introduction

Nonlinear problems (nonlinear dynamic systems or nonlinear differential equations) naturally occur, in mathematical models in various branches of science, engineering, finance, economics, etc. So far, in literature, most models are differential equations with integer derivatives. A vast literature for the qualitative study of dynamic systems with integer order is available, see [5, 8]. However, the qualitative and quantitative study of fractional differential and integral equations has gained importance recently due to its applications. See [1, 3, 4, 6, 9, 11] for details of the study of fractional integral and differential equations of both Riemann Liouville and Caputo

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type. Many practical applications of fractional differential and integral equations have also been provided in the references of the monographs cited above. The qualitative study of fractional differential and integral equations of various types has been established in [2, 3, 4, 6, 10, 11, 12, 13]. Among the type of fractional dynamic systems, the study of Riemann Liouville and Caputo type of fractional dynamic systems has gained more importance.

Integral and differential inequalities have played a significant role in the qualitative study of dynamic systems which can be seen in [5, 6]. In this work we recall some known results of integral and differential inequalities of both integer and fractional order for the scalar equations. In this work we have developed results for coupled fractional and ordinary integral inequalities where the nonlinear function is the sum of an increasing and a decreasing function. We have also developed the corresponding coupled fractional and ordinary differential inequality results without requiring the increasing or decreasing nature of the nonlinear function. In fact, we have developed results for coupled differential and integral inequalities for both ordinary and fractional equations. These results will be useful in the qualitative study of ordinary and fractional dynamic systems of both Riemann Liouville and Caputo forms.

2. **Differential and Integral Inequalities for Integer order**

In this section we recall the known results relative to ordinary differential equation with initial conditions of the form

$$x' = f(t,x), \quad x(t_0) = x_0 \tag{2.1}$$

where $f(t,x) \in C[[t_0,T] \times R^N, R^N]$, for $t \geq t_0$, and $\| f(t,x) \| \leq g(t, \| x \|)$.

Assuming the estimate on $\|f(t,x)\| \leq g(t,\|x\|)$ where $g(t,x) \in C[[t_0,T] \times R, R]$, and setting $m(t) = \|x\|$ we can obtain the following integral inequality,

$$m(t) \leq m(t_0) + \int_{t_0}^{t} g(s, m(s)ds, \quad t \geq t_0. \tag{2.2}$$

Consider the scalar equation with initial condition of the form:

$$\frac{du}{dt} = g(t,u), \quad u(t_0) = u_0, \tag{2.3}$$

where $g(t,u) \in C[[t_0,T] \times R, R]$. Further, if $g(t,u)$ is nondecreasing in u, we can prove that

$$m(t) \leq r(t,t_0,u_0), \tag{2.4}$$

whenever $m(t_0) \leq u_0$, where $r(t,t_0,u_0)$ is the maximal solution of (2.3), for $t \geq t_0$ or on the interval of existence.

Similarly, we can deduce the differential inequality

$$\frac{dm}{dt} \leq g(t, m(t)), \quad m(t_0) \leq u_0, \tag{2.5}$$

using the estimate

$$\|x + hf(t,x)\| \leq \|x\| + hg(t, \|x\|) + o(h), \tag{2.6}$$

in place of $\|f(t,x)\| \leq g(t, \|x\|)$. The advantage of (2.6) is that g need not be nondecreasing to make the conclusion as in (2.4). In fact, g need not be nonnegative either, and thus we get a better estimate for $\|x(t)\|$. See [5] for details. The integral inequalities, especially the Bellman-Gronwall type of inequalities, are very useful in obtaining the rate of convergence in the quasilinearization method and Generalized quasilinearization method. See [8].

Next, we obtain comparison results using integral and differential inequalities relative to the scalar differential equation of the form:

$$\frac{du}{dt} = f(t,u) + g(t,u), \quad u(0) = u_0, \tag{2.7}$$

where $f(t,x)$, and $g(t,x) \in C[[0,T] \times R, R]$. Initially, we prove the integral inequality result relative to (2.7).

Theorem 2.1. *Let $v, w \in C[J, \mathbb{R}]$ satisfy the following coupled integral inequalities:*

(i) $v(t) \leq v(0) + \int_0^t f(s, v(s))ds + \int_0^t g(s, w(s))ds,$

(ii) $w(t) \geq w(0) + \int_0^t f(s, w(s))ds + \int_0^t g(s, v(s))ds,$

(iii) *$f(t,u)$ is nondecreasing in u for each $t \in J = [0,T]$, and $f(t,u)$ satisfies the one sided Lipschitz condition of the form*

$$f(t, u_1) - f(t, u_2) \leq L(u_1 - u_2),$$

for some $L > 0$, whenever $u_1 \geq u_2$;

(iv) *$g(t,u)$ is nonincreasing in u for each $t \in J = [0,T]$, and $g(t,u)$ satisfies the one sided Lipschitz condition of the form*

$$g(t, u_1) - g(t, u_2) \geq -M(u_1 - u_2),$$

for some $M > 0$, whenever $u_1 \geq u_2$. Then $v(0) \leq w(0)$ implies that $v(t) \leq w(t)$, on J.

Proof. If $g(t,u) \equiv 0$, then this is the well known integral inequality result. Initially, we prove our result when one of the inequalities is strict and $v(0) < w(0)$. For that purpose, set $m(t) = v(t) - w(t)$. Certainly. $m(0) < 0$. If the conclusion $m(t) < 0$, that is $v(t) < w(t)$, on J is not true, then there exists a $t_1 > 0$, such that $v(t) < w(t)$ on $[0, t_1)$ and $v(t_1) = w(t_1)$. It follows from the hypotheses, that

$$0 = m(t_1) < \int_0^{t_1} [f(s, v(s)) - f(s, w(s))]ds + \int_0^{t_1} [g(s, w(s)) - g(s, v(s))]ds \leq 0,$$

which leads to a contradiction. This proves that $v(t) < w(t)$, on J.

In order to prove the result for non strict inequality we construct

$$v_\epsilon(t) = v(t) - \epsilon e^{2(L+M)t} \text{ and } w_\epsilon(t) = w(t) + \epsilon e^{2(L+M)t}$$

which satisfies strict inequalities. Consider

$$\begin{aligned} v_\epsilon(t) = v(t) - \epsilon e^{2(L+M)t} &\le v_\epsilon(0) + \epsilon + \int_0^t [f(s, v(s)) - f(s, v_\epsilon(s)) + f(s, v_\epsilon(s))]ds \\ &+ \int_0^t [g(s, w(s)) - g(s, w_\epsilon(s)) + g(s, w_\epsilon(s))]ds - \epsilon e^{2(L+M)t} \\ &\le v_\epsilon(0) + \int_0^t f(s, v_\epsilon(s))ds + \int_0^t g(s, w_\epsilon(s))ds + \frac{\epsilon}{2}(1 - e^{2(L+M)t}) \\ &< v_\epsilon(0) + \int_0^t f(s, v_\epsilon(s))ds + \int_0^t g(s, w_\epsilon(s)) \end{aligned}$$

for $t > 0$. Similarly, we can get a strict inequality for $w_\epsilon(t)$). Now using the strict inequality result, we get

$$v_\epsilon(t) < w_\epsilon(t)).$$

Taking the limit as $\epsilon \to 0$, the conclusion follows. □

Next, we can prove a comparison theorem for the differential inequality relative to (2.7). We merely state the theorem and indicate a brief proof.

Theorem 2.2. *Let $v, w \in C[J, \mathbb{R}]$ satisfy the following coupled integral inequalities*

(i) $v'(t) \le f(t, v(t)) + g(t, w(t))$, $v(0) \le u_0$;
(ii) $w'(t) \ge f(t, w(t)) + g(t, v(t))$, $w(0) \ge u_0$;
(iii) $f(t, u)$ *satisfies the one sided Lipschitz condition of the form*

$$f(t, u_1) - f(t, u_2) \le L(u_1 - u_2),$$

for some $L > 0$, whenever $u_1 \ge u_2$;

(iv) $g(t, u)$ *satisfies the one sided Lipschitz condition of the form*

$$g(t, u_1) - g(t, u_2) \ge -M(u_1 - u_2),$$

for some $M > 0$, whenever $u_1 \ge u_2$. Then $v(0) \le w(0)$ implies $v(t) \le w(t)$, on J.

Proof. Follows on the same lines as in [7] when $g = 0$. Even if $g \neq 0$, the proof follows on the same lines as if we have natural lower and upper solutions instead of coupled lower and upper solutions. □

3. Riemann Liouville Fractional Differential and Integral Inequalities

In this section, we develop some fractional differential and integral inequalities of Riemann Liouville type. For that purpose we recall some definitions, and some known results, and develop a few results which are useful in applications. Initially we consider the Riemann-Liouville (R-L) derivative of order q, $0 < q < 1$. Note, for simplicity we only consider results on the interval $J = (t_0, T]$, where $t > t_0$. Further, we will let $J_0 = [t_0, T]$, that is $J_0 = \overline{J}$.

Definition 3.1. Let $0 < q < 1$, $p = 1 - q$ and a function $\phi(t) \in C(J, R)$ is a C_p continuous function if $(t - t_0)^{1-q}\phi(t) \in C(J_0, R)$. The set of C_p functions is denoted $C_p(J, R)$. Further, given a function $\phi(t) \in C_p(J, R)$ we call the function $(t-t_0)^{1-q}\phi(t)$ the continuous extension of $\phi(t)$.

Riemann Liouville fractional derivative of order q is given by

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} u(s) ds,$$

where $0 < q < 1$. Similarly the Riemann Liouville right-fractional integral of order q is given by

$${}_{t_0}D^{-q} u(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} u(s) ds,$$

where $0 < q < 1$.

Consider the Riemann Liouville Volterra fractional integral equation given by

$$u(t) = \frac{u^0 (t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{(q-1)} (f(s, u(s)) + g(s, u(s)) ds, \quad t_0 < t \leq T, \tag{3.1}$$

where $u^0 = \Gamma(q)(t - t_0)^{1-q} u(t)|_{t=t_0}$. We need the following definition of the Mittag Leffler function, before we state our next result.

Definition 3.2. The Mittag Leffler function with parameters q and r is given by

$$E_{q,r}(\lambda((t-t_0)^q)) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^q)^k}{\Gamma(qk+r)},$$

which is entire for when $q, r > 0$. Also when $r = 1$, we get

$$E_{q,1}(\lambda((t-t_0)^q) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^q)^k}{\Gamma(qk+1)},$$

where $q > 0$.

It is easy to observe that $E_{1,1}(\lambda(t - t_0) = e^{\lambda(t-t_0)}$. Further, using the fact that $E_{q.q}$ is entire, we have that

$$D^q((t-t_0)^{q-1} E_{q,q}(\lambda(t-t_0)^q) = \lambda(t-t_0)^{q-1} E_{q,q}(\lambda(t-t_0)^q), \tag{3.2}$$

where λ is any constant. From the above relation it follows that

$$D^{-q}((t-t_0)^{q-1}E_{q,q}(\lambda(t-t_0)^q) = \frac{1}{\lambda}(t-t_0)^{q-1}E_{q,q}(\lambda(t-t_0)^q). \tag{3.3}$$

Theorem 3.3. *Let $v, w \in C_p[J, \mathbb{R}]$, $f, g, \in C[J_0 \times \mathbb{R}, \mathbb{R}]$ and satisfy the following coupled integral inequalities*

(i) $v(t) \leq \frac{v^0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{(q-1)}(f(s,v(s)) + g(s,w(s)))ds$,

(ii) $w(t) \geq \frac{w^0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{(q-1)}(f(s,w(s)) + g(s,v(s)))ds$,

(iii) *$f(t,u)$ is nondecreasing in u for each $t \in J_0$, and $f(t,u)$ satisfies the one sided Lipschitz condition of the form*

$$f(t,u_1) - f(t,u_2) \leq L(u_1 - u_2),$$

for some $L > 0$, whenever $u_1 \geq u_2$;

(iv) *$g(t,u)$ is non increasing in u for each $t \in J_0$, and $g(t,u)$ satisfies the one sided Lipschitz condition of the form*

$$g(t,u_1) - g(t,u_2) \geq -M(u_1 - u_2),$$

for some $M > 0$, whenever $u_1 \geq u_2$. Then $v^0 \leq w^0$ implies $v(t) \leq w(t)$, on J.

Proof. Initially, we prove the result when one of the inequalities is strict and $v^0 < w^0$. For that purpose, let $m(t) = v(t) - w(t)$. Certainly. $m^0 < 0$. If the conclusion $m(t) < 0$, that is $v(t) < w(t)$, on J is not true, then there exists a $t_1 > 0$, such that $v(t) < w(t)$ on $[t_0, t_1)$ and $v(t_1) = w(t_1)$. It follows from the hypotheses, that

$$\begin{aligned} 0 &= m(t_1) \\ &< \frac{1}{\Gamma(q)}\int_{t_0}^{t_1} (t_1 - s)^{q-1}[[f(s,v(s)) - f(s,w(s))] + [g(s,w(s)) - g(s,v(s))]]ds \leq 0, \end{aligned}$$

since $f(t,v(t)) - f(t,w(t)) + g(t,w(t)) - g(t,v(t)) \leq 0$, on $[0, t_1]$, using the increasing and decreasing nature of $f(t,u) and g(t,u)$ respectively. This leads to a contradiction. This proves that $v(t) < w(t)$, on J_0.

In order to prove the result for non strict inequality we construct $v_\epsilon(t)$ and $w_\epsilon(t)$:

$$v_\epsilon(t) = v(t) - \epsilon(t-t_0)^{q-1}E_{q,q}(2(L+M)(t-t_0)^q)$$

and

$$w_\epsilon(t) = w(t) + \epsilon(t-t_0)^{q-1}E_{q,q}(2(L+M)(t-t_0)^q).$$

It is easy to observe that $v_\epsilon^0 < v^0 \leq w^0 < w_\epsilon^0$. Now using the one sided Lipschitz conditions for $f(t,u)$ and $g(t,u)$ and (3.3) we can show that

$$v_\epsilon(t) < \frac{v_\epsilon^0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{(q-1)}(f(s,v_\epsilon(s) + g(s,w_\epsilon)(s)ds,$$

and

$$w_\epsilon(t) > \frac{w_\epsilon^0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{(q-1)}(f(s,w_\epsilon(s)+g(s,v_\epsilon(s))ds.$$

It easily follows from the strict inequality results that

$$v_\epsilon(t) \le w_\epsilon(t)$$

on J. Taking the limit as $\epsilon \to 0$, we get $v(t) \le w(t)$, on J_0. Next, in order to obtain a comparison result relative to the Riemann-Liouville fractional differential inequality, we recall the following result. □

Lemma 3.4. *Let $m(t) \in C_p[J,\mathbb{R}]$ (where $J_0 = [t_0,T]$) be such that for some $t_1 \in (t_0,T]$, $m(t_1) = 0$ and $m(t) \le 0$, on J, then $D^q m(t_1) \ge 0$.*

Proof. See [2, 13, 14] for details. However note that we have not assumed $m(t)$ to be Holder continuous as in [6]. In order to prove Lemma 3.4, without using the Holder continuity assumption, we have used the fact that $(t-t_0)^{1-q}m(t)$ is continuous on J_0, hence uniformly continuous. The above lemma is true for Caputo derivatives also, using the relation ${}^cD^q x(t) = D^q(x(t)-x(t_0))$ between the Caputo derivative and the Reimann-Liouville derivative. The next lemma states the Caputo derivative version, which we will be using in our next section. □

Lemma 3.5. *Let $m(t) \in C^1[J,\mathbb{R}]$ (where $J = [0,T]$) be such that $m(t) \le 0$ on J and for $t_1 > 0$, $m(t_1) = 0$, then ${}^cD^q m(t_1) \ge 0$.*

Now we, are in a position to prove the comparison result relative to the Riemann-Liouville fractional differential inequalities. For that purpose, consider the Riemann-Liouville fractional differential equation of the form:

$$D^q u(t) = f(t,u(t)) + g(t,u(t)), \quad \Gamma(q)(t-t_0)^{1-q}u(t)|_{t=t_0} = u^0, \tag{3.4}$$

where $f,g \in C[J_0 \times \mathbb{R},\mathbb{R}]$. The next result is the differential inequality result relative to the Riemann-Liouville fractional differential equation (3.4).

Theorem 3.6. *Let $v,w \in C_p[J,\mathbb{R}]$, $f,g,\in C[J_0 \times \mathbb{R},\mathbb{R}]$ and satisfy the following coupled differential inequalities*

(i) $D^q(v(t)) \le f(t,v(t)) + g(t,w(t))$, $\Gamma(q)(t-t_0)^{1-q}v(t)|_{t=t_0} \le u^0$;
(ii) $D^q(w(t)) \ge f(t,w(t)) + g(t,v(t))$, $\Gamma(q)(t-t_0)^{1-q}w(t)|_{t=t_0} \ge u^0$;
(iii) $f(t,u)$ *satisfies the one sided Lipschitz condition of the form*

$$f(t,u_1) - f(t,u_2) \le L(u_1-u_2),$$

for some $L > 0$, whenever $u_1 \ge u_2$;

(iv) *$g(t,u)$ satisfies the one sided Lipschitz condition of the form*

$$g(t,u_1) - g(t,u_2) \geq -M(u_1 - u_2),$$

for some $M > 0$, whenever $u_1 \geq u_2$. Then $v^0 \leq w^0$ implies $v(t) \leq w(t)$, on J.

Proof. We prove our result when one of the inequalities in (i) or (ii) of the hypotheses is strict. In this case set $m(t) = v(t) - w(t)$. Since $m(t)$ is $C_p[J,\mathbb{R}]$ and $v^0 < w^0$, if the conclusion does not hold with strict inequality, there exists a $t_1 > t_0$, where $m(t) < 0$ on (t_0, t_1) and $m(t_1) = 0$. Using Lemma 3.4 we get $D^q m(t_1) \geq 0$. This yields,

$$\begin{aligned} 0 \leq D^q m(t_1) &= D^q v(t_1) - D^q w(t_1) \\ &< [f(t_1, v(t_1)) + g(t_1, w(t_1))] - [f(t_1, w(t_1)) + g(t_1, v(t_1))] = 0 \end{aligned}$$

which is a contradiction. In order to prove the result for non strict inequalities, set

$$v_\epsilon(t) = v(t) - \epsilon(t-t_0)^{q-1} E_{q,q}(2(L+M)(t-t_0)^q)$$

and

$$w_\epsilon(t) = w(t) + \epsilon(t-t_0)^{q-1} E_{q,q}(2(L+M)(t-t_0)^q),$$

as in the proof of Theorem 3.3. From this it follows that $v_\epsilon^0 < v^0 \leq u^0 \leq w^0 < w_\epsilon^0$. Using Lipschitz conditions on $f(t,u)$ and $g(t,u)$ and the relation (3.2) we can obtain

$$D^q(v_\epsilon(t)) < f(t, v_\epsilon(t)) + g(t, w_\epsilon(t)),$$

and

$$D^q(w_\epsilon(t)) > f(t, w_\epsilon(t)) + g(t, v_\epsilon(t)).$$

From the strict inequality result we get $v_\epsilon(t) < w_\epsilon(t)$, on J. The conclusion follows by taking the limit as $\epsilon \to 0$. This completes the proof. □

4. **Caputo Fractional Differential and Integral Inequalities**

In this section we develop results relative to the Caputo fractional differential and integral inequalities.

The Caputo fractional derivative of order q is given by

$${}^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^t (t-s)^{-q} u'(s) ds,$$

where $0 < q < 1$. Similarly the Caputo right-fractional integral of order q is given by

$${}_{t_0}^{c}D^{-q} u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds,$$

where $0 < q < 1$.

Consider the Caputo fractional differential equation of order q where $0 < q < 1$, of the form:

$$(4.1) \qquad {}^cD^q u(t) = f(t, u(t)) + g(t, u(t), \quad u(t_0) = u_0,$$

where $f, g \in C[J_0 \times \mathbb{R}, \mathbb{R}]$.

The integral representation of (4.1) is given by equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}[f(s, u(s)) + g(s, u(s))]ds, \tag{4.2}$$

where $\Gamma(q)$ is the Gamma function.

The equivalence of (4.1) and (4.2) is established in [4]. See [4] for details. In order to compute the solution of the linear fractional differential equation with constant coefficients, we use the Mittag Leffler function. Also, consider the linear Caputo fractional differential equation

$$^cD^q u(t) = \lambda u(t) + f(t), \quad u(t_0) = u_0, \text{ on } J \tag{4.3}$$

where $J = [t_0, T]$, λ is a constant and $f(t) \in C[J, \mathbb{R}]$.

The solution of (4.3) exists and is unique. The explicit solution of (4.3) is given by

$$u(t) = u_0 E_{q,1}(\lambda(t-t_0)^q) + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(\lambda(s-t_0)^q) f(s)ds. \tag{4.4}$$

See [6] for details. In particular, if $\lambda = 0$, the solution $u(t)$ is given by

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s)ds, \tag{4.5}$$

where $\Gamma(q)$ is the Gamma function.

Note that if $^cD^q u(t) \leq \lambda u(t) + f(t)$, $u(t_0) = u_0$, on J in (4.3), then the conclusions in (4.4) and (4.5) will hold good with $\leq$ in place of equality. These inequalities will be useful in computing the rate of convergence of approximate solutions.

We recall the following definitions which are useful in proving the existence of solution and computation of the solution of (4.1).

Definition 4.1. The functions $v, w \in C^1([t_0, T], \mathbb{R})$ are called natural lower and upper solutions of (4.1) if:

$$^cD^q v(t) \leq f(t, v) + g(t, v), \quad v_0(t_0) \leq u_0,$$

and

$$^cD^q w_(t) \geq f(t, w) + g(t, w), \quad w(t_0) \geq u_0.$$

Definition 4.2. The functions $v, w \in C^1([0, T], \mathbb{R})$ are called coupled lower and upper solutions of type I of (4.1) if:

$$^cD^q v(t) \leq f(t, v) + g(t, w), \quad v(t_0) \leq u_0,$$

and

$$^cD^q w(t) \geq f(t, w) + g(t, v), \quad w(t_0) \geq u_0.$$

One can easily prove the existence of solution of (4.1) on the interval $[t_0, T]$ when we have natural lower and upper solution for (4.1), such that $v(t) \le w(t)$. Also, we can compute coupled minimal and maximal solution of (4.1) on the interval $[t_0, T]$, using generalized monotone method when we have coupled lower and upper solutions of type I of (4.1) with $v(t) \le w(t)$, without any extra assumption. In this section we merely state the differential inequality result for coupled lower and upper solutions of type I.

Theorem 4.3. *Let $v, w \in C^1[J, \mathbb{R}]$, $f, g, \in C[J \times \mathbb{R}, \mathbb{R}]$ are coupled lower and upper solutions of type I,such that*

(i) *$f(t, u)$ satisfies the one sided Lipschitz condition of the form*

$$f(t, u_1) - f(t, u_2) \le L(u_1 - u_2),$$

for some $L > 0$, whenever $u_1 \ge u_2$;

(ii) *$g(t, u)$ satisfies the one sided Lipschitz condition of the form*

$$g(t, u_1) - g(t, u_2) \ge -M(u_1 - u_2),$$

for some $M > 0$, whenever $u_1 \ge u_2$. Then $v(t_0) \le w(t_0)$ implies $v(t) \le w(t)$, on J.

The proof follows on the same lines as the proof of Theorem 3.6, and Lemma 3.5 with the Mittag-Leffler function $E_{q,1}(\lambda(t - t_0)^q)$, in place of $E_{q,q}(\lambda((t - t_0)^q))$.

Remark 4.4. We can develop Caputo integral inequality results similar to Theorem 3.3. The integral inequality results need an extra assumption which is not required for the corresponding scalar differential inequality results. Thus, the differential inequality results are useful in iterative methods like the monotone quasilinearization methods. The integral inequality results are useful in proving uniqueness, continuous dependency on the initial condition and also in finding the order of convergence.

5. **Conclusion**

In literature the integral inequality results require the increasing nature of the nonlinear term, whereas the differential inequality results do not require this assumption. In this work, we have developed results for Volterra type integral inequalities when the nonlinear function is the sum of an increasing and decreasing functions. We have extended this result to Riemann-Liouville type of Volterra integral inequalities. Earlier known Volterra integral inequalities can be obtained as a special case of our results. In addition we have developed differential inequality results with integer derivatives, Riemann-Lioville derivative and Caputo derivatives for coupled differential inequalities without any extra assumption which was required for integral inequalities. Both types of inequalities are useful in the qualitative study of ordinary and fractional integral and differential equations.

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